

# CHAPTER 7

## Integration Techniques, L'Hôpital's Rule, and Improper Integrals

### Section 7.1 Basic Integration Rules

Solutions to Even-Numbered Exercises

2. (a)  $\frac{d}{dx} [\ln \sqrt{x^2 + 1} + C] = \frac{1}{2} \left( \frac{2x}{x^2 + 1} \right) = \frac{x}{x^2 + 1}$

(b)  $\frac{d}{dx} \left[ \frac{2x}{(x^2 + 1)^2} + C \right] = \frac{(x^2 + 1)^2(2) - (2x)(2)(x^2 + 1)(2x)}{(x^2 + 1)^4} = \frac{2(1 - 3x^2)}{(x^2 + 1)^3}$

(c)  $\frac{d}{dx} [\arctan x + C] = \frac{1}{1 + x^2}$

(d)  $\frac{d}{dx} [\ln(x^2 + 1) + C] = \frac{2x}{x^2 + 1}$

$\int \frac{x}{x^2 + 1} dx$  matches (a).

4. (a)  $\frac{d}{dx} [2x \sin(x^2 + 1) + C] = 2x[\cos(x^2 + 1)(2x)] + 2 \sin(x^2 + 1) = 2[2x^2 \cos(x^2 + 1) + \sin(x^2 + 1)]$

(b)  $\frac{d}{dx} \left[ -\frac{1}{2} \sin(x^2 + 1) + C \right] = -\frac{1}{2} \cos(x^2 + 1)(2x) = -x \cos(x^2 + 1)$

(c)  $\frac{d}{dx} \left[ \frac{1}{2} \sin(x^2 + 1) + C \right] = \frac{1}{2} \cos(x^2 + 1)(2x) = x \cos(x^2 + 1)$

(d)  $\frac{d}{dx} [-2x \sin(x^2 + 1) + C] = -2x[\cos(x^2 + 1)(2x)] - 2 \sin(x^2 + 1) = -2[2x^2 \cos(x^2 + 1) + \sin(x^2 + 1)]$

$\int x \cos(x^2 + 1) dx$  matches (c).

6.  $\int \frac{2t - 1}{t^2 - t + 2} dt$

$u = t^2 - t + 2, du = (2t - 1) dt$

Use  $\int \frac{du}{u}$ .

8.  $\int \frac{2}{(2t - 1)^2 + 4} dt$

$u = 2t - 1, du = 2dt, a = 2$

Use  $\int \frac{du}{u^2 + a^2}$ .

10.  $\int \frac{-2x}{\sqrt{x^2 - 4}} dx$

$u = x^2 - 4, du = 2x dx, n = -\frac{1}{2}$

Use  $\int u^n du$ .

12.  $\int \sec 3x \tan 3x dx$

$u = 3x, du = 3 dx$

Use  $\int \sec u \tan u du$ .

14.  $\int \frac{1}{x\sqrt{x^2 - 4}} dx$

$u = x, du = dx, a = 2$

Use  $\int \frac{du}{u\sqrt{u^2 - a^2}}$ .

16. Let  $u = x - 4$ ,  $du = dx$ .

$$\begin{aligned}\int 6(x-4)^5 dx &= 6 \int (x-4)^5 dx = 6 \frac{(x-4)^6}{6} + C \\ &= (x-4)^6 + C\end{aligned}$$

18. Let  $u = t - 9$ ,  $du = dt$ .

$$\int \frac{2}{(t-9)^2} dt = 2 \int (t-9)^{-2} dt = \frac{-2}{t-9} + C$$

20. Let  $u = 4 - 2x^2$ ,  $du = -4x dx$ .

$$\begin{aligned}\int x \sqrt{4-2x^2} dx &= -\frac{1}{4} \int (4-2x^2)^{1/2} (-4x) dx \\ &= -\frac{1}{6} (4-2x^2)^{3/2} + C\end{aligned}$$

$$\begin{aligned}22. \int \left[ x - \frac{3}{(2x+3)^2} \right] dx &= \int x dx - \frac{3}{2} \int (2x+3)^{-2}(2) dx \\ &= \frac{x^2}{2} - \frac{3}{2} \frac{(2x+3)^{-1}}{-1} + C \\ &= \frac{x^2}{2} + \frac{3}{2(2x+3)} + C\end{aligned}$$

24. Let  $u = x^2 + 2x - 4$ ,  $du = 2(x+1) dx$ .

$$\begin{aligned}\int \frac{x+1}{\sqrt{x^2+2x-4}} dx &= \frac{1}{2} \int (x^2+2x-4)^{-1/2} (2)(x+1) dx \\ &= \sqrt{x^2+2x-4} + C\end{aligned}$$

$$26. \int \frac{2x}{x-4} dx = \int 2 dx + \int \frac{8}{x-4} dx = 2x + 8 \ln|x-4| + C$$

$$\begin{aligned}28. \int \left( \frac{1}{3x-1} - \frac{1}{3x+1} \right) dx &= \frac{1}{3} \int \frac{1}{3x-1} (3) dx - \frac{1}{3} \int \frac{1}{3x+1} (3) dx \\ &= \frac{1}{3} \ln|3x-1| - \frac{1}{3} \ln|3x+1| + C = \frac{1}{3} \ln \left| \frac{3x-1}{3x+1} \right| + C\end{aligned}$$

$$30. \int x \left( 1 + \frac{1}{x} \right)^3 dx = \int x \left( 1 + \frac{3}{x} + \frac{3}{x^2} + \frac{1}{x^3} \right) dx = \int \left( x + 3 + \frac{3}{x} + \frac{1}{x^2} \right) dx = \frac{1}{2}x^2 + 3x + 3 \ln|x| - \frac{1}{x} + C$$

$$\begin{aligned}32. \int \sec 4x dx &= \frac{1}{4} \int \sec(4x)(4) dx \\ &= \frac{1}{4} \ln|\sec 4x + \tan 4x| + C\end{aligned}$$

34. Let  $u = \cos x$ ,  $du = -\sin x dx$ .

$$\begin{aligned}\int \frac{\sin x}{\sqrt{\cos x}} dx &= - \int (\cos x)^{-1/2} (-\sin x) dx \\ &= -2\sqrt{\cos x} + C\end{aligned}$$

36. Let  $u = \cot x$ ,  $du = -\csc^2 x dx$ .

$$\int \csc^2 x e^{\cot x} dx = - \int e^{\cot x} (-\csc^2 x) dx = -e^{\cot x} + C$$

$$\begin{aligned}38. \int \frac{5}{3e^x - 2} dx &= 5 \int \left( \frac{1}{3e^x - 2} \right) \left( \frac{e^{-x}}{e^{-x}} \right) dx \\ &= 5 \int \frac{e^{-x}}{3 - 2e^{-x}} dx \\ &= \frac{5}{2} \int \frac{1}{3 - 2e^{-x}} (2e^{-x}) dx \\ &= \frac{5}{2} \ln|3 - 2e^{-x}| + C\end{aligned}$$

40. Let  $u = \ln(\cos x)$ ,  $du = \frac{-\sin x}{\cos x} dx$   
 $= -\tan x dx$

$$\begin{aligned}\int (\tan x)(\ln \cos x) dx &= - \int (\ln \cos x)(-\tan x) dx \\ &= \frac{-[\ln(\cos x)]^2}{2} + C\end{aligned}$$

$$\begin{aligned}44. \int \frac{2}{3(\sec x - 1)} dx &= \frac{2}{3} \int \frac{1}{\sec x - 1} \cdot \left( \frac{\sec x + 1}{\sec x + 1} \right) dx \\ &= \frac{2}{3} \int \frac{\sec x + 1}{\tan^2 x} dx \\ &= \frac{2}{3} \int \frac{\sec x}{\tan^2 x} dx + \frac{2}{3} \int \cot^2 x dx \\ &= \frac{2}{3} \int \frac{\cos x}{\sin^2 x} dx + \frac{2}{3} \int (\csc^2 x - 1) dx \\ &= \frac{2}{3} \left( -\frac{1}{\sin x} \right) - \frac{2}{3} \cot x - \frac{2}{3}x + C \\ &= -\frac{2}{3} [\csc x + \cot x + x] + C\end{aligned}$$

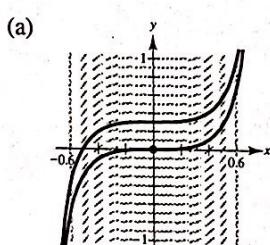
48. Let  $u = \sqrt{3}x$ ,  $du = \sqrt{3}dx$ .

$$\begin{aligned}\int \frac{1}{4 + 3x^2} dx &= \frac{1}{\sqrt{3}} \int \frac{\sqrt{3}}{4 + (\sqrt{3}x)^2} dx \\ &= \frac{1}{2\sqrt{3}} \arctan\left(\frac{\sqrt{3}x}{2}\right) + C\end{aligned}$$

52.  $\int \frac{1}{(x-1)\sqrt{4x^2 - 8x + 3}} dx = \int \frac{2}{[2(x-1)]\sqrt{[2(x-1)]^2 - 1}} dx = \text{arcsec}|2(x-1)| + C$

54.  $\int \frac{1}{\sqrt{1 - 4x - x^2}} dx = \int \frac{1}{\sqrt{5 - (x^2 + 4x + 4)}} dx = \int \frac{1}{\sqrt{5 - (x+2)^2}} dx = \arcsin\left(\frac{x+2}{\sqrt{5}}\right) + C \quad (a = \sqrt{5})$

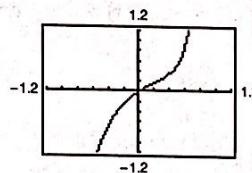
56.  $\frac{dy}{dx} = \tan^2(2x)$ ,  $(0, 0)$



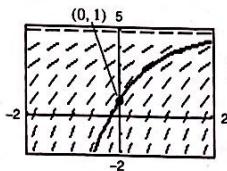
(b)  $\int \tan^2(2x) dx = \int (\sec^2(2x) - 1) dx = \frac{1}{2} \tan(2x) - x + C$

$(0, 0): 0 = C$

$$y = \frac{1}{2} \tan(2x) - x$$



58.



$$\begin{aligned}60. r &= \int \frac{(1 + e^t)^2}{e^t} dt = \int \frac{1 + 2e^t + e^{2t}}{e^t} dt \\ &= \int (e^{-t} + 2 + e^t) dt = -e^{-t} + 2t + e^t + C\end{aligned}$$

62. Let  $u = 2x$ ,  $du = 2 dx$ .

$$\begin{aligned} y &= \int \frac{1}{x\sqrt{4x^2 - 1}} dx = \int \frac{2}{2x\sqrt{(2x)^2 - 1}} dx \\ &= \text{arcsec}|2x| + C \end{aligned}$$

66. Let  $u = 1 - \ln x$ ,  $du = \frac{-1}{x} dx$ .

$$\begin{aligned} \int_1^e \frac{1 - \ln x}{x} dx &= - \int_1^e (1 - \ln x) \left( \frac{-1}{x} \right) dx \\ &= \left[ -\frac{1}{2}(1 - \ln x)^2 \right]_1^e = \frac{1}{2} \end{aligned}$$

70.  $\int_0^4 \frac{1}{\sqrt{25 - x^2}} dx = \left[ \arcsin \frac{x}{5} \right]_0^4 = \arcsin \frac{4}{5} \approx 0.927$

72.  $\int \frac{x - 2}{x^2 + 4x + 13} dx = \frac{1}{2} \ln(x^2 + 4x + 13) - \frac{4}{3} \arctan\left(\frac{x+2}{3}\right) + C$

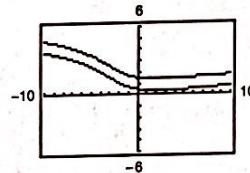
The antiderivatives are vertical translations of each other.

64. Let  $u = \sin t$ ,  $du = \cos t dt$ .

$$\int_0^\pi \sin^2 t \cos t dt = \left[ \frac{1}{3} \sin^3 t \right]_0^\pi = 0$$

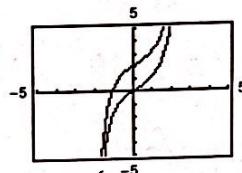
68.  $\int_1^2 \frac{x-2}{x} dx = \int_1^2 \left(1 - \frac{2}{x}\right) dx$

$$= \left[ x - 2 \ln x \right]_1^2 = 1 - 2 \ln 4 \approx -0.386$$



74.  $\int \left( \frac{e^x + e^{-x}}{2} \right)^3 dx = \frac{1}{24} [e^{3x} + 9e^x - 9e^{-x} - e^{-3x}] + C$

The antiderivatives are vertical translations of each other.



78. Arctan Rule:  $\int \frac{du}{a^2 + u^2} = \frac{1}{a} \arctan\left(\frac{u}{a}\right) + C$

76.  $\int \sec u \tan u du = \sec u + C$

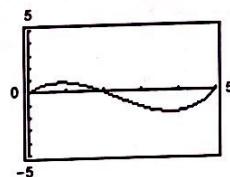
80. They differ by a constant:

$$\sec^2 x + C_1 = (\tan^2 x + 1) + C_1 = \tan^2 x + C$$

82.  $f(x) = \frac{1}{5}(x^3 - 7x^2 + 10x)$

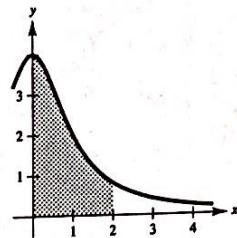
$$\int_0^5 f(x) dx < 0 \text{ because}$$

more area is below the  $x$ -axis than above.



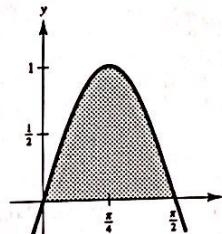
84.  $\int_0^2 \frac{4}{x^2 + 1} dx \approx 4$

Matches (d).



86.  $A = \int_0^{\pi/2} \sin 2x dx$

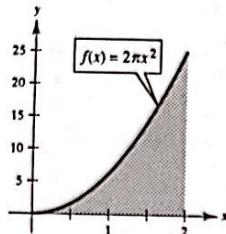
$$= \left[ -\frac{1}{2} \cos 2x \right]_0^{\pi/2} = 1$$



88.  $\int_0^2 2\pi x^2 dx$

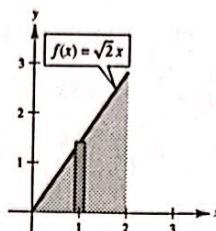
- (a) Let  $f(x) = 2\pi x^2$  over the interval  $[0, 2]$ .

$$A = \int_0^2 (2\pi x^2) dx$$



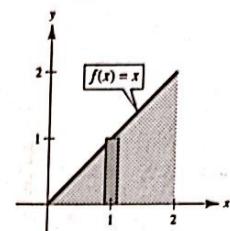
- (b) Let  $f(x) = \sqrt{2}x$  over the interval  $[0, 2]$ . Revolve this region about the  $x$ -axis.

$$V = \pi \int_0^2 (\sqrt{2}x)^2 dx \\ = \int_0^2 2\pi x^2 dx$$



- (c) Let  $f(x) = x$  over the interval  $[0, 2]$ . Revolve this region about the  $y$ -axis.

$$V = 2\pi \int_0^2 x(x) dx \\ = \int_0^2 2\pi x^2 dx$$



90. (a)  $\frac{1}{(\pi/n) - 0} \int_0^{\pi/n} \sin(nx) dx = \frac{1}{\pi} \int_0^{\pi/n} \sin(nx)(n) dx = \left[ -\frac{1}{\pi} \cos(nx) \right]_0^{\pi/n} = \frac{2}{\pi}$

(b)  $\frac{1}{3 - (-3)} \int_{-3}^3 \frac{1}{1 + x^2} dx = \left[ \frac{1}{6} \arctan x \right]_{-3}^3 = \frac{1}{3} \arctan 3$

92.  $y = 2\sqrt{x}$

$$y' = \frac{1}{\sqrt{x}}$$

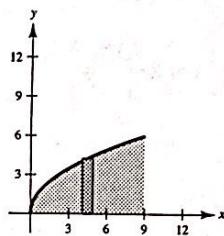
$$1 + (y')^2 = 1 + \frac{1}{x} = \frac{x+1}{x}$$

$$S = 2\pi \int_0^9 2\sqrt{x} \sqrt{\frac{x+1}{x}} dx$$

$$= 2\pi \int_0^9 2\sqrt{x+1} dx$$

$$= \left[ 4\pi \left( \frac{2}{3} \right) (x+1)^{3/2} \right]_0^9$$

$$= \frac{8\pi}{3} (10\sqrt{10} - 1) \approx 256.545$$



94.  $y = x^{2/3}$

$$y' = \frac{2}{3x^{1/3}}$$

$$1 + (y')^2 = 1 + \frac{4}{9x^{2/3}}$$

$$s = \int_1^8 \sqrt{1 + \frac{4}{9x^{2/3}}} dx \approx 7.6337$$

## Section 7.2 Integration by Parts

2.  $\frac{d}{dx} [x^2 \sin x + 2x \cos x - 2 \sin x] = x^2 \cos x + 2x \sin x - 2x \sin x + 2 \cos x - 2 \cos x = x^2 \cos x$ . Matches (d)