

CHAPTER 7

Integration Techniques, L'Hôpital's Rule, and Improper Integrals

Section 7.1 Basic Integration Rules

Solutions to Odd-Numbered Exercises

1. (a) $\frac{d}{dx} [2\sqrt{x^2 + 1} + C] = 2\left(\frac{1}{2}\right)(x^2 + 1)^{-1/2}(2x) = \frac{2x}{\sqrt{x^2 + 1}}$

(b) $\frac{d}{dx} [\sqrt{x^2 + 1} + C] = \frac{1}{2}(x^2 + 1)^{-1/2}(2x) = \frac{x}{\sqrt{x^2 + 1}}$

(c) $\frac{d}{dx} \left[\frac{1}{2}\sqrt{x^2 + 1} + C \right] = \frac{1}{2}\left(\frac{1}{2}\right)(x^2 + 1)^{-1/2}(2x) = \frac{x}{2\sqrt{x^2 + 1}}$

(d) $\frac{d}{dx} [\ln(x^2 + 1) + C] = \frac{2x}{x^2 + 1}$

$\int \frac{x}{\sqrt{x^2 + 1}} dx$ matches (b).

3. (a) $\frac{d}{dx} [\ln\sqrt{x^2 + 1} + C] = \frac{1}{2}\left(\frac{2x}{x^2 + 1}\right) = \frac{x}{x^2 + 1}$

(b) $\frac{d}{dx} \left[\frac{2x}{(x^2 + 1)^2} + C \right] = \frac{(x^2 + 1)^2(2) - (2x)(2)(x^2 + 1)(2x)}{(x^2 + 1)^4} = \frac{2(1 - 3x^2)}{(x^2 + 1)^3}$

(c) $\frac{d}{dx} [\arctan x + C] = \frac{1}{1 + x^2}$

(d) $\frac{d}{dx} [\ln(x^2 + 1) + C] = \frac{2x}{x^2 + 1}$

$\int \frac{1}{x^2 + 1} dx$ matches (c).

5. $\int (3x - 2)^4 dx$

$u = 3x - 2, du = 3 dx, n = 4$

Use $\int u^n du$.

7. $\int \frac{1}{\sqrt{x}(1 - 2\sqrt{x})} dx$

$u = 1 - 2\sqrt{x}, du = -\frac{1}{\sqrt{x}} dx$

Use $\int \frac{du}{u}$.

9. $\int \frac{3}{\sqrt{1 - t^2}} dt$

$u = t, du = dt, a = 1$

Use $\int \frac{du}{\sqrt{a^2 - u^2}}$

11. $\int t \sin t^2 dt$

$u = t^2, du = 2t dt$

Use $\int \sin u du$.

13. $\int \cos x e^{\sin x} dx$

$u = \sin x, du = \cos x dx$

Use $\int e^u du$.

15. Let $u = -2x + 5, du = -2 dx$.

$$\begin{aligned}\int (-2x + 5)^{3/2} dx &= -\frac{1}{2} \int (-2x + 5)^{3/2}(-2) dx \\ &= -\frac{1}{5}(-2x + 5)^{5/2} + C\end{aligned}$$

19. Let $u = t^3 - 1, du = 3t^2 dt$.

$$\begin{aligned}\int t^2 \sqrt[3]{t^3 - 1} dt &= \frac{1}{3} \int (t^3 - 1)^{1/3}(3t^2) dt \\ &= \frac{1}{3} \frac{(t^3 - 1)^{4/3}}{4/3} + C \\ &= \frac{(t^3 - 1)^{4/3}}{4} + C\end{aligned}$$

23. Let $u = -t^3 + 9t + 1, du = (-3t^2 + 9) dt = -3(t^2 - 3) dt$.

$$\int \frac{t^2 - 3}{-t^3 + 9t + 1} dt = -\frac{1}{3} \int \frac{-3(t^2 - 3)}{-t^3 + 9t + 1} dt = -\frac{1}{3} \ln|-t^3 + 9t + 1| + C$$

$$\begin{aligned}25. \int \frac{x^2}{x-1} dx &= \int (x+1) dx + \int \frac{1}{x-1} dx \\ &= \frac{1}{2}x^2 + x + \ln|x-1| + C\end{aligned}$$

$$29. \int (1+2x^2)^2 dx = \int (4x^4 + 4x^2 + 1) dx = \frac{4}{5}x^5 + \frac{4}{3}x^3 + x + C = \frac{x}{15}(12x^4 + 20x^2 + 15) + C$$

31. Let $u = 2\pi x^2, du = 4\pi x dx$.

$$\begin{aligned}\int x(\cos 2\pi x^2) dx &= \frac{1}{4\pi} \int (\cos 2\pi x^2)(4\pi x) dx \\ &= \frac{1}{4\pi} \sin 2\pi x^2 + C\end{aligned}$$

33. Let $u = \pi x, du = \pi dx$.

$$\int \csc(\pi x) \cot(\pi x) dx = \frac{1}{\pi} \int \csc(\pi x) \cot(\pi x) \pi dx = -\frac{1}{\pi} \csc(\pi x) + C$$

35. Let $u = 5x, du = 5 dx$.

$$\int e^{5x} dx = \frac{1}{5} \int e^{5x} (5) dx = \frac{1}{5} e^{5x} + C$$

17. Let $u = z - 4, du = dz$

$$\begin{aligned}\int \frac{5}{(z-4)^5} dz &= 5 \int (z-4)^{-5} dx = 5 \frac{(z-4)^{-4}}{-4} + C \\ &= \frac{-5}{4(z-4)^4} + C\end{aligned}$$

$$\begin{aligned}21. \int \left[v + \frac{1}{(3v-1)^3} \right] dv &= \int v dv + \frac{1}{3} \int (3v-1)^{-3} (3) dv \\ &= \frac{1}{2}v^2 - \frac{1}{6(3v-1)^2} + C\end{aligned}$$

27. Let $u = 1 + e^x, du = e^x dx$.

$$\int \frac{e^x}{1+e^x} dx = \ln(1+e^x) + C$$

37. Let $u = 1 + e^x, du = e^x dx$.

$$\begin{aligned}\int \frac{2}{e^{-x}+1} dx &= 2 \int \left(\frac{1}{e^{-x}+1} \right) \left(\frac{e^x}{e^x} \right) dx \\ &= 2 \int \frac{e^x}{1+e^x} dx = 2 \ln(1+e^x) + C\end{aligned}$$

39. $\int \frac{\ln x^2}{x} dx = 2 \int (\ln x) \frac{1}{x} dx = 2 \frac{(\ln x)^2}{2} + C = (\ln x)^2 + C$

41. $\int \frac{1 + \sin x}{\cos x} dx = \int (\sec x + \tan x) dx = \ln|\sec x + \tan x| + \ln|\sec x| + C = \ln|\sec x(\sec x + \tan x)| + C$

43. $\frac{1}{\cos \theta - 1} = \frac{1}{\cos \theta - 1} \cdot \frac{\cos \theta + 1}{\cos \theta + 1} = \frac{\cos \theta + 1}{\cos^2 \theta - 1} = \frac{\cos \theta + 1}{-\sin^2 \theta}$
 $= -\csc \theta \cdot \cot \theta - \csc^2 \theta$

$$\begin{aligned}\int \frac{1}{\cos \theta - 1} d\theta &= \int (-\csc \theta \cot \theta - \csc^2 \theta) d\theta \\&= -\csc \theta + \cot \theta + C \\&= \frac{1}{\sin \theta} + \frac{\cos \theta}{\sin \theta} + C \\&= \frac{1 + \cos \theta}{\sin \theta} + C\end{aligned}$$

45. $\int \frac{3z+2}{z^2+9} dz = \frac{3}{2} \int \frac{2z}{z^2+9} dz + 2 \int \frac{dz}{z^2+9}$
 $= \frac{3}{2} \ln(z^2+9) + \frac{2}{3} \arctan\left(\frac{z}{3}\right) + C$

47. Let $u = 2t - 1$, $du = 2 dt$.

$$\begin{aligned}\int \frac{-1}{\sqrt{1-(2t-1)^2}} dt &= -\frac{1}{2} \int \frac{2}{\sqrt{1-(2t-1)^2}} dt \\&= -\frac{1}{2} \arcsin(2t-1) + C\end{aligned}$$

49. Let $u = \cos\left(\frac{2}{t}\right)$, $du = \frac{2 \sin(2/t)}{t^2} dt$.

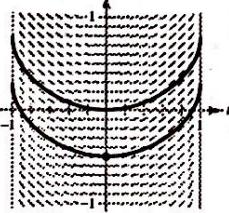
$$\begin{aligned}\int \frac{\tan(2/t)}{t^2} dt &= \frac{1}{2} \int \frac{1}{\cos(2/t)} \left[\frac{2 \sin(2/t)}{t^2} \right] dt \\&= \frac{1}{2} \ln \left| \cos\left(\frac{2}{t}\right) \right| + C\end{aligned}$$

51. $\int \frac{3}{\sqrt{6x-x^2}} dx = 3 \int \frac{1}{\sqrt{9-(x-3)^2}} dx = 3 \arcsin\left(\frac{x-3}{3}\right) + C$

53. $\int \frac{4}{4x^2+4x+65} dx = \int \frac{1}{[x+(1/2)]^2+16} dx = \frac{1}{4} \arctan\left[\frac{x+(1/2)}{4}\right] + C = \frac{1}{4} \arctan\left(\frac{2x+1}{8}\right) + C$

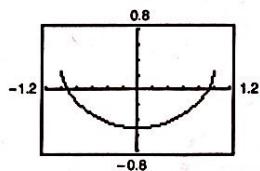
55. $\frac{ds}{dt} = \frac{t}{\sqrt{1-t^4}}, \left(0, -\frac{1}{2}\right)$

(a)

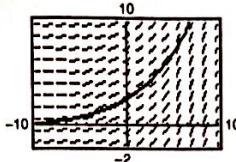


(b) $u = t^2, du = 2t dt$

$$\begin{aligned}\int \frac{t}{\sqrt{1-t^4}} dt &= \frac{1}{2} \int \frac{2t}{\sqrt{1-(t^2)^2}} dt = \frac{1}{2} \arcsin t^2 + C \\(0, -\frac{1}{2}): -\frac{1}{2} &= \frac{1}{2} \arcsin 0 + C \Rightarrow C = -\frac{1}{2} \\s &= \frac{1}{2} \arcsin t^2 - \frac{1}{2}\end{aligned}$$



57.



$$y = 3e^{0.2x}$$

$$61. \frac{dy}{dx} = \frac{\sec^2 x}{4 + \tan^2 x}$$

Let $u = \tan x, du = \sec^2 x dx$.

$$y = \int \frac{\sec^2 x}{4 + \tan^2 x} dx = \frac{1}{2} \arctan\left(\frac{\tan x}{2}\right) + C$$

65. Let $u = -x^2, du = -2x dx$.

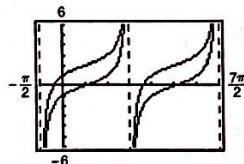
$$\begin{aligned} \int_0^1 xe^{-x^2} dx &= -\frac{1}{2} \int_0^1 e^{-x^2} (-2x) dx = \left[-\frac{1}{2} e^{-x^2} \right]_0^1 \\ &= \frac{1}{2}(1 - e^{-1}) \approx 0.316 \end{aligned}$$

69. Let $u = 3x, du = 3 dx$.

$$\begin{aligned} \int_0^{2/\sqrt{3}} \frac{1}{4 + 9x^2} dx &= \frac{1}{3} \int_0^{2/\sqrt{3}} \frac{3}{4 + (3x)^2} dx \\ &= \left[\frac{1}{6} \arctan\left(\frac{3x}{2}\right) \right]_0^{2/\sqrt{3}} \\ &= \frac{\pi}{18} \approx 0.175 \end{aligned}$$

$$73. \int \frac{1}{1 + \sin \theta} d\theta = \tan \theta - \sec \theta + C \quad (\text{or } \frac{-2}{1 + \tan(\theta/2)})$$

The antiderivatives are vertical translations of each other.



$$77. \text{Log Rule: } \int \frac{du}{u} = \ln|u| + C, u = x^2 + 1.$$

$$\begin{aligned} 59. y &= \int (1 + e^x)^2 dx = \int (e^{2x} + 2e^x + 1) dx \\ &= \frac{1}{2} e^{2x} + 2e^x + x + C \end{aligned}$$

63. Let $u = 2x, du = 2 dx$.

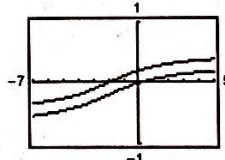
$$\begin{aligned} \int_0^{\pi/4} \cos 2x dx &= \frac{1}{2} \int_0^{\pi/4} \cos 2(2x) dx \\ &= \left[\frac{1}{2} \sin 2x \right]_0^{\pi/4} = \frac{1}{2} \end{aligned}$$

67. Let $u = x^2 + 9, du = 2x dx$.

$$\begin{aligned} \int_0^4 \frac{2x}{\sqrt{x^2 + 9}} dx &= \int_0^4 (x^2 + 9)^{-1/2} (2x) dx \\ &= \left[2\sqrt{x^2 + 9} \right]_0^4 = 4 \end{aligned}$$

$$71. \int \frac{1}{x^2 + 4x + 13} dx = \frac{1}{3} \arctan\left(\frac{x+2}{3}\right) + C$$

The antiderivatives are vertical translations of each other.



$$75. \text{Power Rule: } \int u^n du = \frac{u^{n+1}}{n+1} + C, n \neq -1.$$

$$u = x^2 + 1, n = 3$$

79. The are equivalent because

$$e^{x+C_1} = e^x \cdot e^{C_1} = Ce^x, C = e^{C_1}$$

81. $\sin x + \cos x = a \sin(x + b)$

$$\sin x + \cos x = a \sin x \cos b + a \cos x \sin b$$

$$\sin x + \cos x = (a \cos b) \sin x + (a \sin b) \cos x$$

Equate coefficients of like terms to obtain the following.

$$1 = a \cos b \quad \text{and} \quad 1 = a \sin b$$

Thus, $a = 1/\cos b$. Now, substitute for a in $1 = a \sin b$.

$$1 = \left(\frac{1}{\cos b}\right) \sin b$$

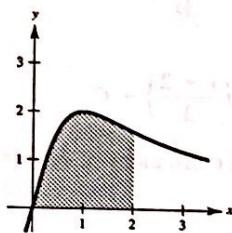
$$1 = \tan b \Rightarrow b = \frac{\pi}{4}$$

Since $b = \frac{\pi}{4}$, $a = \frac{1}{\cos(\pi/4)} = \sqrt{2}$. Thus, $\sin x + \cos x = \sqrt{2} \sin\left(x + \frac{\pi}{4}\right)$.

$$\int \frac{dx}{\sin x + \cos x} = \int \frac{dx}{\sqrt{2} \sin(x + (\pi/4))} = \frac{1}{\sqrt{2}} \int \csc\left(x + \frac{\pi}{4}\right) dx = -\frac{1}{\sqrt{2}} \ln \left| \csc\left(x + \frac{\pi}{4}\right) + \cot\left(x + \frac{\pi}{4}\right) \right| + C$$

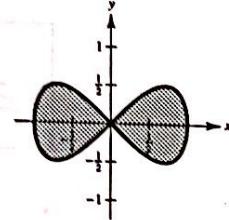
83. $\int_0^2 \frac{4x}{x^2 + 1} dx \approx 3$

Matches (a).



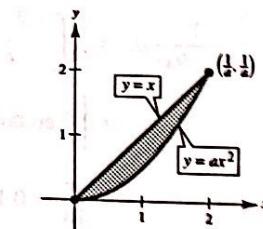
85. Let $u = 1 - x^2$, $du = -2x dx$.

$$\begin{aligned} A &= 4 \int_0^1 x \sqrt{1 - x^2} dx \\ &= -2 \int_0^1 (1 - x^2)^{1/2} (-2x) dx \\ &= \left[-\frac{4}{3}(1 - x^2)^{3/2} \right]_0^1 = \frac{4}{3} \end{aligned}$$



87. $\int_0^{1/a} (x - ax^2) dx = \left[\frac{1}{2}x^2 - \frac{a}{3}x^3 \right]_0^{1/a} = \frac{1}{6a^2}$

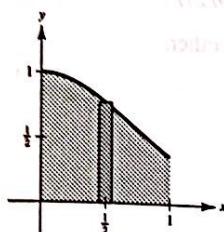
Let $\frac{1}{6a^2} = \frac{2}{3}$, $12a^2 = 3$, $a = \frac{1}{2}$.



89. (a) Shell Method:

Let $u = -x^2$, $du = -2x dx$.

$$\begin{aligned} V &= 2\pi \int_0^1 xe^{-x^2} dx \\ &= -\pi \int_0^1 e^{-x^2} (-2x) dx \\ &= \left[-\pi e^{-x^2} \right]_0^1 \\ &= \pi(1 - e^{-1}) \approx 1.986 \end{aligned}$$



(b) Shell Method:

$$\begin{aligned} V &= 2\pi \int_0^b xe^{-x^2} dx \\ &= \left[-\pi e^{-x^2} \right]_0^b \\ &= \pi(1 - e^{-b^2}) = \frac{4}{3} \end{aligned}$$

$$\begin{aligned} e^{-b^2} &= \frac{3\pi - 4}{3\pi} \\ b &= \sqrt{\ln\left(\frac{3\pi}{3\pi - 4}\right)} \\ &\approx 0.743 \end{aligned}$$

$$91. A = \int_0^4 \frac{5}{\sqrt{25 - x^2}} dx = \left[5 \arcsin \frac{x}{5} \right]_0^4 = 5 \arcsin \frac{4}{5}$$

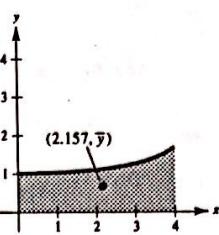
$$\bar{x} = \frac{1}{A} \int_0^4 x \left(\frac{5}{\sqrt{25 - x^2}} \right) dx$$

$$= \frac{1}{5 \arcsin(4/5)} \left(-\frac{5}{2} \right) \int_0^4 (25 - x^2)^{-1/2} (-2x) dx$$

$$= \frac{1}{5 \arcsin(4/5)} (-5) \left[(25 - x^2)^{1/2} \right]_0^4$$

$$= -\frac{1}{\arcsin(4/5)} [3 - 5]$$

$$= \frac{2}{\arcsin(4/5)} \approx 2.157$$



$$93. y = \tan(\pi x)$$

$$y' = \pi \sec^2(\pi x)$$

$$1 + (y')^2 = 1 + \pi^2 \sec^4(\pi x)$$

$$s = \int_0^{1/4} \sqrt{1 + \pi^2 \sec^4(\pi x)} dx \\ \approx 1.0320$$

Section 7.2 Integration by Parts

$$1. \frac{d}{dx} [\sin x - x \cos x] = \cos x - (-x \sin x + \cos x) = x \sin x. \text{ Matches (b)}$$

$$3. \frac{d}{dx} [x^2 e^x - 2xe^x + 2e^x] = x^2 e^x + 2xe^x - 2xe^x - 2e^x + 2e^x = x^2 e^x. \text{ Matches (c)}$$

$$5. \int xe^{2x} dx$$

$$u = x, dv = e^{2x} dx$$

$$7. \int (\ln x)^2 dx$$

$$u = (\ln x)^2, dv = dx$$

$$9. \int x \sec^2 x dx$$

$$u = x, dv = \sec^2 x dx$$

$$11. dv = e^{-2x} dx \Rightarrow v = \int e^{-2x} dx = -\frac{1}{2} e^{-2x}$$

$$u = x \Rightarrow du = dx$$

$$\begin{aligned} \int xe^{-2x} dx &= -\frac{1}{2} xe^{-2x} - \int -\frac{1}{2} e^{-2x} dx \\ &= -\frac{1}{2} xe^{-2x} - \frac{1}{4} e^{-2x} + C = \frac{-1}{4e^{2x}}(2x + 1) + C \end{aligned}$$