

94. $g(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

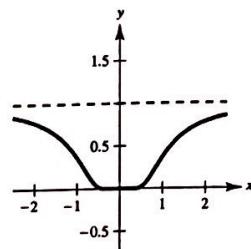
$$g'(0) = \lim_{x \rightarrow 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{e^{-1/x^2}}{x}$$

Let $y = \frac{e^{-1/x^2}}{x}$, then $\ln y = \ln\left(\frac{e^{-1/x^2}}{x}\right) = -\frac{1}{x^2} - \ln x = \frac{-1 - x^2 \ln x}{x^2}$. Since

$$\lim_{x \rightarrow 0} x^2 \ln x = \lim_{x \rightarrow 0} \frac{\ln x}{1/x^2} = \lim_{x \rightarrow 0} \frac{1/x}{-2/x^3} = \lim_{x \rightarrow 0} \left(-\frac{x^2}{2}\right) = 0$$

we have $\lim_{x \rightarrow 0} \left(\frac{-1 - x^2 \ln x}{x^2}\right) = -\infty$. Thus, $\lim_{x \rightarrow 0} y = e^{-\infty} = 0 \Rightarrow g'(0) = 0$.

Note: The graph appears to support this conclusion—the tangent line is horizontal at $(0, 0)$.



96. $\lim_{x \rightarrow a} f(x)^{g(x)}$

$$y = f(x)^{g(x)}$$

$$\ln y = g(x) \ln f(x)$$

$$\lim_{x \rightarrow a} g(x) \ln f(x) = (-\infty)(-\infty) = \infty$$

As $x \rightarrow a$, $\ln y \rightarrow \infty$, and hence $y = \infty$. Thus,

$$\lim_{x \rightarrow a} f(x)^{g(x)} = \infty.$$

98. $\lim_{x \rightarrow 0^+} x^{\ln 2/(1+\ln x)}$

Let $y = x^{\ln 2/(1+\ln x)}$, then:

$$\ln y = \frac{\ln 2}{1 + \ln x} \cdot \ln x = \frac{(\ln 2)(\ln x)}{1 + \ln x}$$

$$\begin{aligned} \lim_{x \rightarrow 0^+} \ln y &= \lim_{x \rightarrow 0^+} \frac{(\ln 2)(\ln x)}{1 + \ln x} = \lim_{x \rightarrow 0^+} \frac{(\ln 2)/x}{1/x} \\ &= \lim_{x \rightarrow 0^+} (\ln 2) = \ln 2 \end{aligned}$$

Thus, $\lim_{x \rightarrow \infty} y = e^{\ln 2} = 2$.

Section 7.8 Improper Integrals

2. Infinite discontinuity at $x = 3$.

$$\begin{aligned} \int_3^4 \frac{1}{(x-3)^{3/2}} dx &= \lim_{b \rightarrow 3^+} \int_b^4 (x-3)^{-3/2} dx \\ &= \lim_{b \rightarrow 3^+} \left[-2(x-3)^{-1/2} \right]_b^4 \\ &= \lim_{b \rightarrow 3^+} \left[-2 + \frac{2}{\sqrt{b-3}} \right] = \infty \end{aligned}$$

Diverges

4. Infinite discontinuity at $x = 1$.

$$\begin{aligned} \int_0^2 \frac{1}{(x-1)^{2/3}} dx &= \int_0^1 \frac{1}{(x-1)^{2/3}} dx + \int_1^2 \frac{1}{(x-1)^{2/3}} dx \\ &= \lim_{b \rightarrow 1^-} \int_0^b \frac{1}{(x-1)^{2/3}} dx + \lim_{c \rightarrow 1^+} \int_c^2 \frac{1}{(x-1)^{2/3}} dx \\ &= \lim_{b \rightarrow 1^-} \left[3\sqrt[3]{x-1} \right]_0^b + \lim_{c \rightarrow 1^+} \left[3\sqrt[3]{x-1} \right]_c^2 = (0+3) + (3-0) = 6 \end{aligned}$$

Converges

6. Infinite limit of integration.

$$\begin{aligned} \int_{-\infty}^0 e^{2x} dx &= \lim_{b \rightarrow -\infty} \int_b^0 e^{2x} dx \\ &= \lim_{b \rightarrow -\infty} \left[\frac{1}{2} e^{2x} \right]_b^0 = \frac{1}{2} - 0 = \frac{1}{2} \end{aligned}$$

Converges

$$\begin{aligned} 10. \int_1^\infty \frac{5}{x^3} dx &= \lim_{b \rightarrow \infty} \int_1^b \frac{5}{x^3} dx \\ &= \lim_{b \rightarrow \infty} \left[-\frac{5}{2} x^{-2} \right]_1^b = \frac{5}{2} \end{aligned}$$

$$\begin{aligned} 8. \int_0^\infty e^{-x} dx &\neq 0. \text{ You need to evaluate the limit.} \\ \lim_{b \rightarrow \infty} \int_0^b e^{-x} dx &= \lim_{b \rightarrow \infty} \left[-e^{-x} \right]_0^b \\ &= \lim_{b \rightarrow \infty} \left[-e^{-b} + 1 \right] = 1 \end{aligned}$$

$$14. \int_0^\infty x e^{-x/2} dx = \lim_{b \rightarrow \infty} \int_0^b x e^{-x/2} dx = \lim_{b \rightarrow \infty} \left[e^{-x/2}(-2x - 4) \right]_0^b = \lim_{b \rightarrow \infty} e^{-b/2}(-2b - 4) + 4 = 4$$

$$16. \int_0^\infty (x-1)e^{-x} dx = \lim_{b \rightarrow \infty} \int_0^b (x-1)e^{-x} dx = \lim_{b \rightarrow \infty} \left[-xe^{-x} \right]_0^b = \lim_{b \rightarrow \infty} \left(\frac{-b}{e^b} + 0 \right) = 0 \text{ by L'Hôpital's Rule.}$$

$$\begin{aligned} 18. \int_0^\infty e^{-ax} \sin bx dx &= \lim_{c \rightarrow \infty} \left[\frac{e^{-ax}(-a \sin bx - b \cos bx)}{a^2 + b^2} \right]_0^c \\ &= 0 - \frac{-b}{a^2 + b^2} = \frac{b}{a^2 + b^2} \end{aligned}$$

$$\begin{aligned} 20. \int_1^\infty \frac{\ln x}{x} dx &= \lim_{b \rightarrow \infty} \int_1^b \frac{\ln x}{x} dx \\ &= \lim_{b \rightarrow \infty} \left[\frac{(\ln x)^2}{2} \right]_1^b = \infty \text{ Diverges} \end{aligned}$$

$$\begin{aligned} 22. \int_0^\infty \frac{x^3}{(x^2 + 1)^2} dx &= \lim_{b \rightarrow \infty} \int_0^b \frac{x}{x^2 + 1} dx - \lim_{b \rightarrow \infty} \int_0^b \frac{x}{(x^2 + 1)^2} dx \\ &= \lim_{b \rightarrow \infty} \left[\frac{1}{2} \ln(x^2 + 1) + \frac{1}{2(x^2 + 1)} \right]_0^b \\ &= \infty - \frac{1}{2} \end{aligned}$$

$$24. \int_0^\infty \frac{e^x}{1 + e^x} dx = \lim_{b \rightarrow \infty} \left[\ln(1 + e^x) \right]_0^b = \infty - \ln 2$$

Diverges

Diverges

$$26. \int_0^\infty \sin \frac{x}{2} dx = \lim_{b \rightarrow \infty} \left[-2 \cos \frac{x}{2} \right]_0^b$$

Diverges since $\cos \frac{x}{2}$ does not approach a limit as $x \rightarrow \infty$.

$$28. \int_0^4 \frac{8}{x} dx = \lim_{b \rightarrow 0^+} \int_b^4 \frac{8}{x} dx = \lim_{b \rightarrow 0^+} \left[8 \ln x \right]_b^4 = \infty$$

Diverges

$$\begin{aligned} 30. \int_0^6 \frac{4}{\sqrt{6-x}} dx &= \lim_{b \rightarrow 6^-} \int_0^b 4(6-x)^{-1/2} dx \\ &= \lim_{b \rightarrow 6^-} \left[-8(6-x)^{1/2} \right]_0^b \\ &= -8(0) + 8\sqrt{6} \\ &= 8\sqrt{6} \end{aligned}$$

$$\begin{aligned} 32. \int_0^e \ln x^2 dx &= \lim_{b \rightarrow 0^+} \int_0^e 2 \ln x dx \\ &= \lim_{b \rightarrow 0^+} \left[2x \ln x - 2x \right]_b^e \\ &= \lim_{b \rightarrow 0^+} [(2e - 2e) - (2b \ln b - 2b)] \\ &= 0 \end{aligned}$$

$$34. \int_0^{\pi/2} \sec \theta d\theta = \lim_{b \rightarrow (\pi/2)} \left[\ln |\sec \theta + \tan \theta| \right]_0^b = \infty,$$

Diverges

$$36. \int_0^2 \frac{1}{\sqrt{4-x^2}} dx = \lim_{b \rightarrow 2^-} \left[\arcsin \left(\frac{x}{2} \right) \right]_0^b = \frac{\pi}{2}$$

38. $\int_0^2 \frac{1}{4-x^2} dx = \lim_{b \rightarrow 2^-} \int_0^b \frac{1}{4} \left(\frac{1}{2+x} + \frac{1}{2-x} \right) dx = \lim_{b \rightarrow 2^-} \left[\frac{1}{4} \ln \left| \frac{2+x}{2-x} \right| \right]_0^b = \infty - 0$
Diverges

40. $\int_1^3 \frac{2}{(x-2)^{8/3}} dx = \int_1^2 2(x-2)^{-8/3} dx + \int_2^3 2(x-2)^{-8/3} dx$
 $= \lim_{b \rightarrow 2^-} \int_1^b 2(x-2)^{-8/3} dx + \lim_{c \rightarrow 2^+} \int_c^3 2(x-2)^{-8/3} dx$
 $= \lim_{b \rightarrow 2^-} \left[-\frac{6}{5}(x-2)^{-5/3} \right]_1^b + \lim_{c \rightarrow 2^+} \left[-\frac{6}{5}(x-2)^{-5/3} \right]_c^3 = \infty$
Diverges

42. $\int \frac{1}{x \ln x} dx = \ln |\ln |x|| + C$

Thus,

$$\begin{aligned} \int_1^\infty \frac{1}{x \ln x} dx &= \int_1^e \frac{1}{x \ln x} dx + \int_e^\infty \frac{1}{x \ln x} dx \\ &= \lim_{b \rightarrow 1^+} \left[\ln(\ln x) \right]_1^e + \lim_{c \rightarrow \infty} \left[\ln(\ln x) \right]_e^\infty. \end{aligned}$$

Diverges

44. If $p = 1$, $\int_0^1 \frac{1}{x} dx = \lim_{a \rightarrow 0^+} \ln x \Big|_a^1 = \lim_{a \rightarrow 0^+} -\ln a = \infty$.

Diverges. If $p \neq 1$,

$$\int_0^1 \frac{1}{x^p} dx = \lim_{a \rightarrow 0^+} \left[\frac{x^{1-p}}{1-p} \right]_a^1 = \lim_{a \rightarrow 0^+} \left[\frac{1}{1-p} - \frac{a^{1-p}}{1-p} \right].$$

This converges to $\frac{1}{1-p}$ if $1-p > 0$ or $p < 1$.

46. (a) Assume $\int_a^\infty g(x) dx = L$ (converges).

Since $0 \leq f(x) \leq g(x)$ on $[a, \infty)$, $0 \leq \int_a^\infty f(x) dx \leq \int_a^\infty g(x) dx = L$ and $\int_a^\infty f(x) dx$ converges.

(b) $\int_a^\infty g(x) dx$ diverges, because otherwise, by part (a), if $\int_a^\infty g(x) dx$ converges, then so does $\int_a^\infty f(x) dx$.

48. $\int_0^1 \frac{1}{\sqrt[3]{x}} dx = \frac{1}{1-(1/3)} = \frac{3}{2}$ converges.
(See Exercise 44, $p = \frac{1}{3}$.)

50. $\int_0^\infty x^4 e^{-x} dx$ converges.
(See Exercise 45.)

52. Since $\frac{1}{\sqrt{x-1}} \geq \frac{1}{x}$ on $[2, \infty)$ and $\int_2^\infty \frac{1}{x} dx$ diverges by Exercise 43, $\int_2^\infty \frac{1}{\sqrt{x-1}} dx$ diverges.

54. Since $\frac{1}{\sqrt{x}(1+x)} \leq \frac{1}{x^{3/2}}$ on $[1, \infty)$ and $\int_1^\infty \frac{1}{x^{3/2}} dx$ converges by Exercise 43, $\int_1^\infty \frac{1}{\sqrt{x}(1+x)} dx$ converges.

56. $\frac{1}{\sqrt{x} \ln x} \geq \frac{1}{x}$ since $\sqrt{x} \ln x < x$ on $[2, \infty)$. Since $\int_2^\infty \frac{1}{x} dx$ diverges by Exercise 43, $\int_2^\infty \frac{1}{\sqrt{x} \ln x} dx$ diverges.

58. See the definitions, pages 540, 543.

60. Answers will vary.

(a) $\int_{-\infty}^\infty \frac{e^x}{1+e^{2x}} dx$
Converges (Example 4)

(b) $\int_{-\infty}^\infty x dx$
Diverges

62. $f(t) = t$

$$\begin{aligned} F(s) &= \int_0^\infty te^{-st} dt = \lim_{b \rightarrow \infty} \left[\frac{1}{s^2} (-st - 1)e^{-st} \right]_0^b \\ &= \frac{1}{s^2}, s > 0 \end{aligned}$$

64. $f(t) = e^{at}$

$$\begin{aligned} F(s) &= \int_0^\infty e^{at} e^{-st} dt = \int_0^\infty e^{t(a-s)} dt \\ &= \lim_{b \rightarrow \infty} \left[\frac{1}{a-s} e^{t(a-s)} \right]_0^b \\ &= 0 - \frac{1}{a-s} = \frac{1}{s-a}, s > a \end{aligned}$$

66. $f(t) = \sin at$

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st} \sin at dt \\ &= \lim_{b \rightarrow \infty} \left[\frac{e^{-st}}{s^2 + a^2} (-s \sin at - a \cos at) \right]_0^b \\ &= 0 + \frac{a}{s^2 + a^2} = \frac{a}{s^2 + a^2}, s > 0 \end{aligned}$$

68. $f(t) = \sinh at$

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st} \sinh at dt = \int_0^\infty e^{-st} \left(\frac{e^{at} - e^{-at}}{2} \right) dt = \frac{1}{2} \int_0^\infty [e^{t(-s+a)} - e^{t(-s-a)}] dt \\ &= \lim_{b \rightarrow \infty} \frac{1}{2} \left[\frac{1}{(-s+a)} e^{t(-s+a)} - \frac{1}{(-s-a)} e^{t(-s-a)} \right]_0^b = 0 - \frac{1}{2} \left[\frac{1}{(-s+a)} - \frac{1}{(-s-a)} \right] \\ &= \frac{-1}{2} \left[\frac{1}{(-s+a)} - \frac{1}{(-s-a)} \right] = \frac{a}{s^2 - a^2}, s > |a| \end{aligned}$$

70. (a) $A = \int_1^\infty \frac{1}{x^2} dx = \left[-\frac{1}{x} \right]_1^\infty = 1$

(b) Disk:

$$V = \pi \int_1^\infty \frac{1}{x^4} dx = \lim_{b \rightarrow \infty} \left[-\frac{\pi}{3x^3} \right]_1^b = \frac{\pi}{3}$$

(c) Shell:

$$V = 2\pi \int_1^\infty x \left(\frac{1}{x^2} \right) dx = \lim_{b \rightarrow \infty} \left[2\pi(\ln x) \right]_1^b = \infty$$

Diverges

72. $(x-2)^2 + y^2 = 1$

$$2(x-2) + 2yy' = 0$$

$$y' = \frac{-(x-2)}{y}$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + [(x-2)^2/y^2]} = \frac{1}{y} \text{ (Assume } y > 0\text{.)}$$

$$\begin{aligned} S &= 4\pi \int_1^3 \frac{x}{y} dx = 4\pi \int_1^3 \frac{x}{\sqrt{1 - (x-2)^2}} dx = 4\pi \int_1^3 \left[\frac{x-2}{\sqrt{1 - (x-2)^2}} + \frac{2}{\sqrt{1 - (x-2)^2}} \right] dx \\ &= \lim_{\substack{a \rightarrow 1^+ \\ b \rightarrow 3^-}} \left\{ 4\pi \left[-\sqrt{1 - (x-2)^2} + 2 \arcsin(x-2) \right] \right\}_a^b = 4\pi [0 + 2 \arcsin(1) - 2 \arcsin(-1)] = 8\pi^2 \end{aligned}$$

74. (a) $F(x) = \frac{K}{x^2}, 5 = \frac{K}{(4000)^2}, K = 80,000,000$

$$W = \int_{4000}^\infty \frac{80,000,000}{x^2} dx = \lim_{b \rightarrow \infty} \left[\frac{-80,000,000}{x} \right]_{4000}^b = 20,000 \text{ mi-ton}$$

$$(b) \quad \frac{W}{2} = 10,000 = \left[\frac{-80,000,000}{x} \right]_{4000}^b = \frac{-80,000,000}{b} + 20,000$$

$$\frac{80,000,000}{b} = 10,000$$

$$b = 8000$$

Therefore, 4000 miles above the earth's surface.

76. (a) $\int_{-\infty}^{\infty} \frac{2}{5} e^{-2t/5} dt = \int_0^{\infty} \frac{2}{5} e^{-2t/5} dt = \lim_{b \rightarrow \infty} \left[-e^{-2t/5} \right]_0^b = 1$

(b) $\int_0^4 \frac{2}{5} e^{-2t/5} dt = \left[-e^{-2t/5} \right]_0^4 = -e^{-8/5} + 1$
 $\approx 0.7981 = 79.81\%$

(c) $\int_0^{\infty} t \left[\frac{2}{5} e^{-2t/5} \right] dt = \lim_{b \rightarrow \infty} \left[-te^{2t/5} - \frac{5}{2} e^{-2t/5} \right]_0^b = \frac{5}{2}$

78. (a) $C = 650,000 + \int_0^5 25,000(1 + 0.08t)e^{-0.06t} dt$

$$= 650,000 + 25,000 \left[-\frac{1}{0.06} e^{-0.06t} - 0.08 \left(\frac{t}{0.06} e^{-0.06t} + \frac{1}{(0.06)^2} e^{-0.06t} \right) \right]_0^5 \approx \$778,512.58$$

(b) $C = 650,000 + \int_0^{10} 25,000(1 + 0.08t)e^{-0.06t} dt$

$$= 650,000 + 25,000 \left[-\frac{1}{0.06} e^{-0.06t} - 0.08 \left(\frac{t}{0.06} e^{-0.06t} + \frac{1}{(0.06)^2} e^{-0.06t} \right) \right]_0^{10} \approx \$905,718.14$$

(c) $C = 650,000 + \int_0^{\infty} 25,000(1 + 0.08t)e^{-0.06t} dt$

$$= 650,000 + 25,000 \lim_{b \rightarrow \infty} \left[-\frac{t}{0.06} e^{-0.06t} - 0.08 \left(\frac{t}{0.06} e^{-0.06t} + \frac{1}{(0.06)^2} e^{-0.06t} \right) \right]_0^b \approx \$1,622,222.22$$

80. (a) $\int_1^{\infty} \frac{1}{x} dx = \lim_{b \rightarrow \infty} \left[\ln|x| \right]_1^b = \infty$

$$\int_1^{\infty} \frac{1}{x^2} dx = \lim_{b \rightarrow \infty} \left[-\frac{1}{x} \right]_1^b = 1$$

$\int_1^{\infty} \frac{1}{x^n} dx$ will converge if $n > 1$ and will diverge if $n \leq 1$.

(c) Let $dv = \sin x dx \Rightarrow v = -\cos x$

$$u = \frac{1}{x} \Rightarrow du = -\frac{1}{x^2} dx$$

$$\begin{aligned} \int_1^{\infty} \frac{\sin x}{x} dx &= \lim_{b \rightarrow 0} \left[-\frac{\cos x}{x} \right]_1^b - \int_1^{\infty} \frac{\cos x}{x^2} dx \\ &= \cos 1 - \int_1^{\infty} \frac{\cos x}{x^2} dx \end{aligned}$$

Converges

82. (a) Yes, the integral is not defined at $x = \pi/2$.

(b) As $n \rightarrow \infty$, the integral approaches $4(\pi/4) = \pi$.

$$(d) I_n = \int_0^{\pi/2} \frac{4}{1 + (\tan x)^n} dx$$

$$I_2 \approx 3.14159$$

$$I_4 \approx 3.14159$$

$$I_8 \approx 3.14159$$

$$I_{12} \approx 3.14159$$

(b) It would appear to converge.

