Section 8.6 The Ratio and Root Tests

2. $\dfrac{(2k-2)!}{(2k)!} = \dfrac{(2k-2)!}{(2k)(2k-1)(2k-2)!} = \dfrac{1}{(2k)(2k-1)}$

4. Use the Principle of Mathematical Induction. When $k = 3$, the formula is valid since $\dfrac{1}{1} = \dfrac{2^3 3!(3)(5)}{6!} = 1$. Assume that

$$\frac{1}{1 \cdot 3 \cdot 5 \cdots (2n-5)} = \frac{2^n n!(2n-3)(2n-1)}{(2n)!}$$

and show that

$$\frac{1}{1 \cdot 3 \cdot 5 \cdots (2n-5)(2n-3)} = \frac{2^{n+1}(n+1)!(2n-1)(2n+1)}{(2n+2)!}.$$

To do this, note that:

$$\begin{aligned}
\frac{1}{1 \cdot 3 \cdot 5 \cdots (2n-5)(2n-3)} &= \frac{1}{1 \cdot 3 \cdot 5 \cdots (2n-5)} \cdot \frac{1}{(2n-3)} \\
&= \frac{2^n n!(2n-3)(2n-1)}{(2n)!} \cdot \frac{1}{(2n-3)} \\
&= \frac{2^n n!(2n-1)}{(2n)!} \cdot \frac{(2n+1)(2n+2)}{(2n+1)(2n+2)} \\
&= \frac{2^n (2)(n+1)n!(2n-1)(2n+1)}{(2n)!(2n+1)(2n+2)} \\
&= \frac{2^{n+1}(n+1)!(2n-1)(2n+1)}{(2n+2)!}
\end{aligned}$$

The formula is valid for all $n \geq 3$.

6. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n \left(\frac{1}{n!}\right) = \frac{3}{4} + \frac{9}{16}\left(\frac{1}{2}\right) + \cdots$

$S_1 = \frac{3}{4}, S_2 \approx 1.03$

Matches (c)

8. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}4}{(2n)!} = \frac{4}{2} - \frac{4}{24} + \cdots$

$S_1 = 2$

Matches (b)

10. $\displaystyle\sum_{n=0}^{\infty} 4e^{-n} = 4 + \frac{4}{e} + \cdots$

$S_1 = 4$

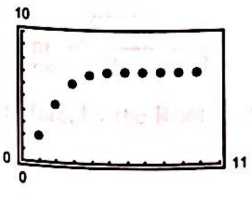
Matches (e)

12. (a) Ratio Test: $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\dfrac{(n+1)^2+1}{(n+1)!}}{\dfrac{n^2+1}{n!}} = \lim_{n\to\infty} \left(\frac{n^2+2n+2}{n^2+1}\right)\left(\frac{1}{n+1}\right) = 0 < 1$. Converges

(b)

n	5	10	15	20	25
S_n	7.0917	7.1548	7.1548	7.1548	7.1548

(c)



(d) The sum is approximately 7.15485

(e) The more rapidly the terms of the series approach 0, the more rapidly the sequence of the partial sums approaches the sum of the series.

14. $\sum_{n=0}^{\infty} \frac{3^n}{n!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3^{n+1}}{(n+1)!}\cdot\frac{n!}{3^n}\right|$$

$$= \lim_{n\to\infty}\frac{3}{n+1} = 0$$

Therefore, by the Ratio Test, the series converges.

16. $\sum_{n=1}^{\infty} n\left(\frac{3}{2}\right)^n$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)3^{n+1}}{2^{n+1}}\cdot\frac{2^n}{n3^n}\right|$$

$$= \lim_{n\to\infty}\frac{3(n+1)}{2n} = \frac{3}{2}$$

Therefore, by the Ratio Test, the series diverges.

18. $\sum_{n=1}^{\infty} \frac{n^3}{2^n}$

$$\lim_{n\to\infty}\left|\frac{a_n+1}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^3/2^{n+1}}{n^3/2^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)^3}{2n^3}\right| = \frac{1}{2}$$

Therefore, by the Ratio Test, the series converges.

20. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+2)}{n(n+1)}$

$$a_{n+1} = \frac{n+3}{(n+1)(n+2)} \le \frac{n+2}{n(n+1)} = a_n$$

$$\lim_{n\to\infty}\frac{n+2}{n(n+1)} = 0$$

Therefore, by Theorem 8.14, the series converges.

Note: The Ratio Test is inconclusive since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1.$

The series converges conditionally.

22. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(3/2)^n}{n^2}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(3/2)^{n+1}}{n^2+2n+1}\cdot\frac{n^2}{(3/2)^n}\right|$$

$$= \lim_{n\to\infty}\frac{3n^2}{2(n^2+2n+1)} = \frac{3}{2} > 1$$

Therefore, by the Ratio Test, the series diverges.

24. $\sum_{n=1}^{\infty} \frac{(2n)!}{n^5}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(2n+2)!}{(n+1)^5}\cdot\frac{n^5}{(2n)!}\right|$$

$$= \lim_{n\to\infty}\frac{(2n+2)(2n+1)n^5}{(n+1)^5} = \infty$$

Therefore, by the Ratio Test, the series diverges.

26. $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^{n+1}}{(n+1)!}\cdot\frac{n!}{n^n}\right|$$

$$= \lim_{n\to\infty}\frac{(n+1)(n+1)^n n!}{(n+1)n!n^n}$$

$$= \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = e > 1$$

Therefore, by the Ratio Test, the series diverges.

28. $\sum_{n=0}^{\infty} \frac{(n!)^2}{(3n)!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{[(n+1)!]^2}{(3n+3)!}\cdot\frac{(3n)!}{(n!)^2}\right|$$

$$= \lim_{n\to\infty}\frac{(n+1)^2}{(3n+3)(3n+2)(3n+1)} = 0$$

Therefore, by the Ratio Test, the series converges.

30. $\sum_{n=0}^{\infty} \frac{(-1)^n 2^{4n}}{(2n+1)!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{4n+4}}{(2n+3)!}\cdot\frac{(2n+1)!}{2^{4n}}\right| = \lim_{n\to\infty}\frac{2^4}{(2n+3)(2n+2)} = 0$$

Therefore, by the Ratio Test, the series converges.

32. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 2 \cdot 4 \cdot 6 \cdots 2n}{2 \cdot 5 \cdot 8 \cdots (3n-1)}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2 \cdot 4 \cdots 2n(2n+2)}{2 \cdot 5 \cdots (3n-1)(3n+2)} \cdot \frac{2 \cdot 5 \cdots (3n-1)}{2 \cdot 4 \cdots 2n}\right| = \lim_{n\to\infty}\frac{2n+2}{3n+2} = \frac{2}{3}$$

Therefore, by the Ratio Test, the series converges.

Note: The first few terms of this series are $-\frac{2}{2} + \frac{2 \cdot 4}{2 \cdot 5} - \frac{2 \cdot 4 \cdot 6}{2 \cdot 5 \cdot 8} + \cdots$

34. (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1}{(n+1)^4} \cdot \frac{n^4}{1}\right| = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^4 = 1$$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1}{(n+1)^p} \cdot \frac{n^p}{1}\right| = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^p = 1$$

36. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n}{n+1}\right)^n$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{2n}{n+1}\right)^n}$$

$$= \lim_{n\to\infty} \frac{2n}{n+1} = 2$$

Therefore, by the Root Test, the series diverges.

38. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{-3n}{2n+1}\right)^{3n}$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{-3n}{2n+1}\right)^{3n}\right|}$$

$$= \lim_{n\to\infty}\left(\frac{3n}{2n+1}\right)^3 = \left(\frac{3}{2}\right)^3 = \frac{27}{8}$$

Therefore, by the Root Test, the series diverges.

40. $\displaystyle\sum_{n=0}^{\infty} e^{-n}$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{e^n}} = \frac{1}{e}$$

Therefore, by the Root Test, the series converges.

42. $\displaystyle\sum_{n=0}^{\infty} \frac{n+1}{3^n}$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{n+1}{3^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{n+1}}{3}$$

Let $y = \lim_{n\to\infty} \sqrt[x]{x+1}$

$$\ln y = \lim_{n\to\infty} \left(\ln \sqrt[x]{x+1}\right)$$

$$= \lim_{n\to\infty} \frac{1}{x}\ln(x+1)$$

$$= \lim_{n\to\infty} \frac{\ln(x+1)}{x} = \frac{1}{x+1} = 0.$$

Since $\ln y = 0$, $y = e^0 = 1$, so

$$\lim_{n\to\infty} \frac{\sqrt{n+1}}{3} = \frac{1}{3}.$$

Therefore, by the Root Test, the series converges.

44. $\displaystyle\sum_{n=1}^{\infty} \frac{5}{n} = 5\sum_{n=1}^{\infty} \frac{1}{n}$

This is the divergent harmonic series.

46. $\sum_{n=1}^{\infty}\left(\frac{\pi}{4}\right)^n$

Since $\pi/4 < 1$, this is convergent geometric series.

48. $\sum_{n=1}^{\infty}\frac{n}{2n^2+1}$

$$\lim_{n\to\infty}\frac{n/(2n^2+1)}{1/n} = \lim_{n\to\infty}\frac{n^2}{2n^2+1} = \frac{1}{2} > 0$$

This series diverges by limit comparison to the divergent harmonic series

$$\sum_{n=1}^{\infty}\frac{1}{n}.$$

50. $\sum_{n=1}^{\infty}\frac{10}{3\sqrt{n^3}}$

$$\lim_{n\to\infty}\frac{10/3n^{3/2}}{1/n^{3/2}} = \frac{10}{3}$$

Therefore, the series converges by a limit comparison test with the p-series

$$\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}.$$

52. $\sum_{n=1}^{\infty}\frac{2^n}{4n^2-1}$

$$\lim_{n\to\infty}\frac{2^n}{4n^2-1} = \lim_{n\to\infty}\frac{(\ln 2)2^n}{8n} = \lim_{n\to\infty}\frac{(\ln 2)^2 2^n}{8} = \infty$$

Therefore, the series diverges by the nth Term Test for Divergence.

54. $\sum_{n=2}^{\infty}\frac{(-1)^n}{n\ln(n)}$

$$a_{n+1} = \frac{1}{(n+1)\ln(n+1)} \le \frac{1}{n\ln(n)} = a_n$$

$$\lim_{n\to\infty}\frac{1}{n\ln(n)} = 0$$

Therefore, by the Alternating Series Test, the series converges.

56. $\sum_{n=1}^{\infty}\frac{\ln(n)}{n^2}$

$$\frac{\ln(n)}{n^2} \le \frac{1}{n^{3/2}}$$

Therefore, the series converges by comparison with the p-series

$$\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}.$$

58. $\sum_{n=1}^{\infty}\frac{(-1)^n 3^n}{n2^n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3^{n+1}}{(n+1)2^{n+1}}\cdot\frac{n2^n}{3^n}\right| = \lim_{n\to\infty}\frac{3n}{2(n+1)} = \frac{3}{2}$$

Therefore, by the Ratio Test, the series diverges.

60. $\sum_{n=1}^{\infty}\frac{3\cdot 5\cdot 7\cdots(2n+1)}{18^n(2n-1)n!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3\cdot 5\cdot 7\cdots(2n+1)(2n+3)}{18^{n+1}(2n+1)(2n-1)n!}\cdot\frac{18^n(2n-1)n!}{3\cdot 5\cdot 7\cdots(2n+1)}\right| = \lim_{n\to\infty}\frac{(2n+3)(2n-1)}{18(2n+1)(2n-1)} = \frac{2}{18} = \frac{1}{9}$$

Therefore, by the Ratio Test, the series converge.

62. (b) and (c)

$$\sum_{n=0}^{\infty}(n+1)\left(\frac{3}{4}\right)^n = \sum_{n=1}^{\infty}n\left(\frac{3}{4}\right)^{n-1}$$

$$= 1 + 2\left(\frac{3}{4}\right) + 3\left(\frac{3}{4}\right)^2 + 4\left(\frac{3}{4}\right)^3 + \cdots$$

64. (a) and (b) are the same.

$$\sum_{n=2}^{\infty}\frac{(-1)^n}{(n-1)2^{n-1}} = \frac{1}{2} - \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3} - \cdots$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n2^n} = \frac{1}{2} - \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3} - \cdots$$

66. Replace n with $n + 2$.

$$\sum_{n=2}^{\infty} \frac{2^n}{(n-2)!} = \sum_{n=0}^{\infty} \frac{2^{n+2}}{n!}$$

68. $$\sum_{k=0}^{\infty} \frac{(-3)^k}{1 \cdot 3 \cdot 5 \ldots (2k+1)} = \sum_{k=0}^{\infty} \frac{(-3)^k 2^k k!}{(2k)!(2k+1)}$$

$$= \sum_{k=0}^{\infty} \frac{(-6)^k k!}{(2k+1)!}$$

$$\approx 0.40967$$

(See Exercise 3 and use 10 terms, $k = 9$.)

70. See Theorem 8.18.

72. One example is $\displaystyle\sum_{n=1}^{\infty} \left(-100 + \frac{1}{n}\right)$.

74. Assume that

$\lim_{n\to\infty} |a_{n+1}/a_n| = L > 1$ or that $\lim_{n\to\infty} |a_{n+1}/a_n| = \infty$.

Then there exists $N > 0$ such that $|a_{n+1}/a_n| > 1$ for all $n > N$. Therefore,

$$|a_{n+1}| > |a_n|, \quad n > N \Rightarrow \lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum a_n \text{ diverges}$$

76. The differentiation test states that if

$$\sum_{n=1}^{\infty} U_n$$

is an infinite series with real terms and $f(x)$ is a real function such that $f(1/n) = U_n$ for all positive integers n and d^2f/dx^2 exists at $x = 0$, then

$$\sum_{n=1}^{\infty} U_n$$

converges absolutely if $f(0) = f'(0) = 0$ and diverges otherwise. Below are some examples.

Convergent Series	Divergent Series
$\sum \frac{1}{n^3}, f(x) = x^3$	$\sum \frac{1}{n}, f(x) = x$
$\sum\left(1 - \cos\frac{1}{n}\right), f(x) = 1 - \cos x$	$\sum \sin\frac{1}{n}, f(x) = \sin x$

Section 8.7 Taylor Polynomials and Approximations

2. $y = \frac{1}{8}x^4 - \frac{1}{2}x^2 + 1$

y-axis symmetry

Three relative extrema

Matches (c)

4. $y = e^{-1/2}\left[\frac{1}{3}(x-1)^3 - (x-1) + 1\right]$

Cubic

Matches (b)

6. $f(x) = \dfrac{4}{\sqrt[3]{x}} = 4x^{-1/3}$ $\qquad f(8) = 2$

$f'(x) = -\dfrac{4}{3}x^{-4/3}$ $\qquad f'(8) = -\dfrac{1}{12}$

$P_1(x) = f(8) + f'(8)(x - 8)$

$= 2 + \left(-\dfrac{1}{12}\right)(x - 8)$

$P_1(x) = -\dfrac{1}{12}x + \dfrac{8}{3}$