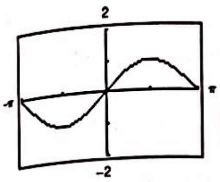
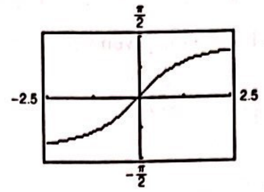
54. $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \sin x$

(See Exercise 47.)



56. $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \arctan x, \ -1 \le x \le 1$

(See Exercise 38 in Section 8.7.)



58. $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{(2n-1)!}$

Replace n with $n - 1$.

60. True; if

$$\sum_{n=0}^{\infty} a_n x^n$$

converges for $x = 2$, then we know that it must converge on $(-2, 2]$.

62. True

$$\int_0^1 f(x)\,dx = \int_0^1 \left(\sum_{n=0}^{\infty} a_n x^n \right) dx = \left[\sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1} \right]_0^1 = \sum_{n=0}^{\infty} \frac{a_n}{n+1}$$

Section 8.9 Representation of Functions by Power Series

2. (a) $f(x) = \frac{4}{5-x} = \frac{4/5}{1-x/5} = \frac{a}{1-r}$

$$= \sum_{n=0}^{\infty} \frac{4}{5} \left(\frac{x}{5} \right)^n = \sum_{n=0}^{\infty} \frac{4x^n}{5^{n+1}}$$

This series converges on $(-5, 5)$.

(b)

$$\begin{array}{r} \frac{4}{5} + \frac{4}{25}x + \frac{4}{125}x^2 + \frac{4x^3}{625} + \cdots \\ 5 - x \,\overline{)\, 4 \qquad\qquad\qquad\qquad\qquad} \\ \underline{4 - \frac{4}{5}x} \qquad\qquad\qquad \\ \frac{4}{5}x \qquad\qquad\qquad \\ \underline{\frac{4}{5}x - \frac{4}{25}x^2} \qquad\qquad \\ \frac{4}{25}x^2 \qquad\qquad \\ \underline{\frac{4}{25}x^2 - \frac{4x^3}{125}} \qquad \\ \frac{4x^3}{125} \qquad \\ \underline{\frac{4x^3}{125} - \frac{4x^4}{625}} \\ \vdots \qquad \end{array}$$

4. (a) $\frac{1}{1+x} = \frac{1}{1-(-x)} = \frac{a}{1-r}$

$$= \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$$

This series converges on $(-1, 1)$.

(b)

$$\begin{array}{r} 1 - x + x^2 - x^3 + \cdots \\ 1 + x \,\overline{)\, 1 \qquad\qquad\qquad\qquad} \\ \underline{1 + x} \qquad\qquad\qquad \\ -x \qquad\qquad\qquad \\ \underline{-x - x^2} \qquad\qquad \\ x^2 \qquad\qquad \\ \underline{x^2 + x^3} \qquad \\ -x^3 \qquad \\ \underline{-x^3 - x^4} \\ \vdots \qquad \end{array}$$

Scanned with CamScanner

6. Writing $f(x)$ in the form $\frac{a}{1-r}$, we have

$$\frac{4}{5-x} = \frac{4}{7-(x+2)} = \frac{4/7}{1-1/7(x+2)} = \frac{a}{1-r}.$$

Therefore, the power series for $f(x)$ is given by

$$\frac{4}{5-x} = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} \frac{4}{7}\left(\frac{1}{7}(x+2)\right)^n$$
$$= \sum_{n=0}^{\infty} \frac{4(x+2)^n}{7^{n+1}}.$$

$|x+2| < 7$ or $-5 < x < 9$

8. Writing $f(x)$ in the form $a/(1-r)$, we have

$$\frac{3}{2x-1} = \frac{3}{3+2(x-2)} = \frac{1}{1+(2/3)(x-2)} = \frac{a}{1-r}$$

which implies that $a = 1$ and $r = (-2/3)(x-2)$. Therefore, the power series for $f(x)$ is given by

$$\frac{3}{2x-1} = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty}\left[-\frac{2}{3}(x-2)\right]^n,$$
$$= \sum_{n=0}^{\infty} \frac{(-2)^n(x-2)^n}{3^n},$$

$|x-2| < \frac{3}{2}$ or $\frac{1}{2} < x < \frac{7}{2}$.

10. Writing $f(x)$ in the form $a/(1-r)$, we have

$$\frac{1}{2x-5} = \frac{1}{-5+2x} = \frac{-1/5}{1-(2/5)x} = \frac{a}{1-r}$$

which implies that $a = -1/5$ and $r = (2/5)x$. Therefore, the power series for $f(x)$ is given by

$$\frac{1}{2x-5} = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty}\left(-\frac{1}{5}\right)\left(\frac{2}{5}x\right)^n = -\sum_{n=0}^{\infty} \frac{2^n x^n}{5^{n+1}},$$

$|x| < \frac{5}{2}$ or $-\frac{5}{2} < x < \frac{5}{2}$.

12. Writing $f(x)$ in the form $a/(1-r)$, we have

$$\frac{4}{3x+2} = \frac{4}{8+3(x-2)} = \frac{1/2}{1+(3/8)(x-2)} = \frac{a}{1-r}$$

which implies that $a = 1/2$ and $r = (-3/8)(x-2)$. Therefore, the power series for $f(x)$ is given by

$$\frac{4}{3x+2} = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} \frac{1}{2}\left[-\frac{3}{8}(x-2)\right]^n$$
$$= \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-3)^n(x-2)^n}{8^n},$$

$|x-2| < \frac{8}{3}$ or $-\frac{2}{3} < x < \frac{14}{3}$.

14. $$\frac{4x-7}{2x^2+3x-2} = \frac{3}{x+2} - \frac{2}{2x-1} = \frac{3}{2+x} - \frac{2}{-1+2x} = \frac{3/2}{1+(1/2)x} + \frac{2}{1-2x}$$

Writing $f(x)$ as a sum of two geometric series, we have

$$\frac{4x-7}{2x^2+3x-2} = \sum_{n=0}^{\infty}\left(\frac{3}{2}\right)\left(-\frac{1}{2}x\right)^n + \sum_{n=0}^{\infty} 2(2x)^n = \sum_{n=0}^{\infty}\left[\frac{3(-1)^n}{2^{n+1}} + 2^{n+1}\right]x^n,\ |x| < \frac{1}{2} \text{ or } -\frac{1}{2} < x < \frac{1}{2}.$$

16. First finding the power series for $4/(4+x)$, we have

$$\frac{1}{1+(1/4)x} = \sum_{n=0}^{\infty}\left(-\frac{1}{4}x\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{4^n}$$

Now replace x with x^2.

$$\frac{4}{4+x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{4^n}.$$

The interval of convergence is $|x^2| < 4$ or $-2 < x < 2$ since

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2n+2}}{4^{n+1}} \cdot \frac{4^n}{(-1)^n x^{2n}}\right| = \left|-\frac{x^2}{4}\right| = \frac{|x^2|}{4}.$$

18. $$h(x) = \frac{x}{x^2-1} = \frac{1}{2(1+x)} - \frac{1}{2(1-x)} = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n x^n - \frac{1}{2}\sum_{n=0}^{\infty} x^n$$
$$= \frac{1}{2}\sum_{n=0}^{\infty}[(-1)^n - 1]x^n = \frac{1}{2}[0 - 2x + 0x^2 - 2x^3 + 0x^4 - 2x^5 + \cdots]$$
$$= \frac{1}{2}\sum_{n=0}^{\infty}(-2)x^{2n+1} = -\sum_{n=0}^{\infty} x^{2n+1},\ 1 < x < 1$$

Scanned with CamScanner

20. By taking the second derivative, we have $\frac{d^2}{dx^2}\left[\frac{1}{x+1}\right] = \frac{2}{(x+1)^3}$. Therefore,

$$\frac{2}{(x+1)^3} = \frac{d^2}{dx^2}\left[\sum_{n=0}^{\infty}(-1)^n x^n\right]$$

$$= \frac{d}{dx}\left[\sum_{n=1}^{\infty}(-1)^n n x^{n-1}\right] = \sum_{n=2}^{\infty}(-1)^n n(n-1)x^{n-2} = \sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^n, \; -1 < x < 1.$$

22. By integrating, we have

$$\int \frac{1}{1+x}\,dx = \ln(1+x) + C_1 \text{ and } \int \frac{1}{1-x}\,dx = -\ln(1-x) + C_2.$$

$f(x) = \ln(1-x^2) = \ln(1+x) - [-\ln(1-x)]$. Therefore,

$$\ln(1-x^2) = \int \frac{1}{1+x}\,dx - \int \frac{1}{1-x}\,dx$$

$$= \int\left[\sum_{n=0}^{\infty}(-1)^n x^n\right]dx - \int\left[\sum_{n=0}^{\infty} x^n\right]dx = \left[C_1 + \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1}\right] - \left[C_2 + \sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}\right]$$

$$= C + \sum_{n=0}^{\infty}\frac{[(-1)^n - 1]x^{n+1}}{n+1} = C + \sum_{n=0}^{\infty}\frac{-2x^{2n+2}}{2n+2} = C + \sum_{n=0}^{\infty}\frac{(-1)x^{2n+2}}{n+1}$$

To solve for C, let $x = 0$ and conclude that $C = 0$. Therefore,

$$\ln(1-x^2) = -\sum_{n=0}^{\infty}\frac{x^{2n+2}}{n+1}, \; -1 < x < 1$$

24. $\frac{2x}{x^2+1} = 2x\sum_{n=0}^{\infty}(-1)^n x^{2n}$ (See Exercise 23.)

$$= \sum_{n=0}^{\infty}(-1)^n 2x^{2n+1}$$

Since $\frac{d}{dx}(\ln(x^2+1)) = \frac{2x}{x^2+1}$, we have

$$\ln(x^2+1) = \int\left[\sum_{n=0}^{\infty}(-1)^n 2x^{2n+1}\right]dx = C + \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{n+1}, \; -1 \le x \le 1.$$

To solve for C, let $x = 0$ and conclude that $C = 0$. Therefore,

$$\ln(x^2+1) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{n+1}, \; -1 \le x \le 1.$$

26. Since $\int \frac{1}{4x^2+1}\,dx = \frac{1}{2}\arctan(2x)$, we can use the result of Exercise 25 to obtain

$$\arctan(2x) = 2\int \frac{1}{4x^2+1}\,dx = 2\int\left[\sum_{n=0}^{\infty}(-1)^n 4^n x^{2n}\right]dx = C + 2\sum_{n=0}^{\infty}\frac{(-1)^n 4^n x^{2n+1}}{2n+1}, \; -\frac{1}{2} < x \le \frac{1}{2}.$$

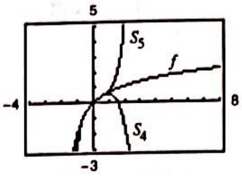
To solve for C, let $x = 0$ and conclude that $C = 0$. Therefore,

$$\arctan(2x) = 2\sum_{n=0}^{\infty}\frac{(-1)^n 4^n x^{2n+1}}{2n+1}, \; -\frac{1}{2} < x \le \frac{1}{2}.$$

Scanned with CamScanner

28. $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \le \ln(x+1)$

$\le x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$



x	0.0	0.2	0.4	0.6	0.8	1.0
$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$	0.0	0.18227	0.33493	0.45960	0.54827	0.58333
$\ln(x+1)$	0.0	0.18232	0.33647	0.47000	0.58779	0.69315
$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$	0.0	0.18233	0.33698	0.47515	0.61380	0.78333

In Exercise 35–38, $\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$.

30. $g(x) = x - \frac{x^3}{3}$, cubic with 3 zeros.

Matches (d)

32. $g(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}$,

Matches (b)

34. The approximations of degree 3, 7, 11, . . . $(4n - 1, n = 1, 2, \ldots)$ have relative extrema.

In Exercises 36 and 38, $\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$.

36.

$$\arctan x^2 = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{2n+1}$$

$$\int \arctan x^2\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{(4n+3)(2n+1)} + C, C = 0$$

$$\int_0^{3/4} \arctan x^2\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{(3/4)^{4n+3}}{(4n+3)(2n+1)}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{3^{4n+3}}{(4n+3)(2n+1)4^{4n+3}}$$

$$= \frac{27}{192} - \frac{2187}{344{,}064} + \frac{177{,}147}{230{,}686{,}720}$$

Since $177{,}147/230{,}686{,}720 < 0.001$, we can approximate the series by its first two terms: 0.13427

38.

$$x^2 \arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+3}}{2n+1}$$

$$\int x^2 \arctan x\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+4}}{(2n+4)(2n+1)}$$

$$\int_0^{1/2} x^2 \arctan x\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+4)(2n+1)2^{2n+4}} = \frac{1}{64} - \frac{1}{1152} + \cdots$$

Since $\frac{1}{1152} < 0.001$, we can approximate the series by its first term: $\int_0^{1/2} x^2 \arctan x\,dx \approx 0.015625$.

Scanned with CamScanner

In Exercises 40 and 42, $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, $|x|1$.

40. Replace n with $n+1$.

$$\sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty} (n+1)x^n$$

42. (a) $\frac{1}{3}\sum_{n=1}^{\infty} n\left(\frac{2}{3}\right)^n = \frac{2}{9}\sum_{n=1}^{\infty} n\left(\frac{2}{3}\right)^{n-1} = \frac{2}{9}\frac{1}{[1-(2/3)]^2} = 2$

(b) $\frac{1}{10}\sum_{n=1}^{\infty} n\left(\frac{9}{10}\right)^n = \frac{9}{100}\sum_{n=1}^{\infty} n\left(\frac{9}{10}\right)^{n-1}$

$$= \frac{9}{100}\cdot\frac{1}{[1-(9/10)]^2} = 9$$

44. Replace x with x^2.

46. Integrate the series and multiply by (-1).

48. (a) From Exercise 47, we have

$$\arctan\frac{120}{119} - \arctan\frac{1}{239} = \arctan\frac{120}{119} + \arctan\left(-\frac{1}{239}\right)$$

$$= \arctan\left[\frac{(120/119)+(-1/239)}{1-(120/119)(-1/239)}\right] = \arctan\left(\frac{28{,}561}{28{,}561}\right) = \arctan 1 = \frac{\pi}{4}$$

(b) $2\arctan\frac{1}{5} = \arctan\frac{1}{5} + \arctan\frac{1}{5} = \arctan\left[\frac{2(1/5)}{1-(1/5)^2}\right] = \arctan\frac{10}{24} = \arctan\frac{5}{12}$

$$4\arctan\frac{1}{5} = 2\arctan\frac{1}{5} + 2\arctan\frac{1}{5} = \arctan\frac{5}{12} + \arctan\frac{5}{12} = \arctan\left[\frac{2(5/12)}{1-(5/12)^2}\right] = \arctan\frac{120}{119}$$

$$4\arctan\frac{1}{5} - \arctan\frac{1}{239} = \arctan\frac{120}{119} - \arctan\frac{1}{239} = \frac{\pi}{4} \text{ (see part (a).)}$$

50. (a) $\arctan\frac{1}{2} + \arctan\frac{1}{3} = \arctan\left[\frac{(1/2)+(1/3)}{1-(1/2)(1/3)}\right] = \arctan\left(\frac{5/6}{5/6}\right) = \frac{\pi}{4}$

(b) $\pi = 4\left[\arctan\frac{1}{2} + \arctan\frac{1}{3}\right]$

$$= 4\left[\frac{1}{2} - \frac{(1/2)^3}{3} + \frac{(1/2)^5}{5} - \frac{(1/2)^7}{7}\right] + 4\left[\frac{1}{3} - \frac{(1/3)^3}{3} + \frac{(1/3)^5}{5} - \frac{(1/3)^7}{7}\right]$$

$$\approx 4(0.4635) + 4(0.3217) \approx 3.14$$

52. From Exercise 51, we have

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{3^n n} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}(1/3)^n}{n}$$

$$= \ln\left(\frac{1}{3}+1\right) = \ln\frac{4}{3} \approx 0.2877.$$

54. From Example 5, we have $\arctan x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$.

$$\sum_{n=0}^{\infty}(-1)^n\frac{1}{2n+1} = \sum_{n=0}^{\infty}(-1)^n\frac{(1)^{2n+1}}{2n+1}$$

$$= \arctan 1 = \frac{\pi}{4} \approx 0.7854$$

56. From Exercise 54, we have

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{3^{2n-1}(2n-1)} = \sum_{n=0}^{\infty}(-1)^n\frac{1}{3^{2n+1}(2n+1)}$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{(1/3)^{2n+1}}{2n+1}$$

$$= \arctan\frac{1}{3} \approx 0.3218.$$

Scanned with CamScanner

58. From Example 5, we have $\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$.

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{3^n(2n+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(\sqrt{3})^{2n}(2n+1)} \cdot \frac{\sqrt{3}}{\sqrt{3}}$$
$$= \sqrt{3} \sum_{n=0}^{\infty} \frac{(-1)^n(1/\sqrt{3})^{2n+1}}{2n+1}$$
$$= \sqrt{3} \arctan \frac{1}{\sqrt{3}}$$
$$= \sqrt{3}\left(\frac{\pi}{6}\right) = \frac{\pi}{2\sqrt{3}}$$

Section 8.10 Taylor and Maclaurin Series

2. For $c = 0$, we have

$$f(x) = e^{3x}$$
$$f^{(n)}(x) = 3^n e^{3x} \Rightarrow f^{(n)}(0) = 3^n$$
$$e^{3x} = 1 + 3x + \frac{9x^2}{2!} + \frac{27x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{(3x)^n}{n!}$$

4. For $c = \pi/4$, we have:

$$f(x) = \sin x \qquad f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$
$$f'(x) = \cos x \qquad f'\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$
$$f''(x) = -\sin x \qquad f''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$
$$f'''(x) = -\cos x \qquad f'''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$
$$f^{(4)}(x) = \sin x \qquad f^{(4)}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

and so on. Therefore we have:

$$\sin x = \sum_{n=0}^{\infty} \frac{f^{(n)}(\pi/4)[x - (\pi/4)]^n}{n!}$$
$$= \frac{\sqrt{2}}{2}\left[1 + \left(x - \frac{\pi}{4}\right) - \frac{[x - (\pi/4)]^2}{2!} - \frac{[x - (\pi/4)]^3}{3!} + \frac{[x - (\pi/4)]^4}{4!} + \cdots\right]$$
$$= \frac{\sqrt{2}}{2}\left\{\sum_{n=0}^{\infty} \frac{(-1)^{n(n+1)/2}[x - (\pi/4)]^{n+1}}{(n+1)!} + 1\right\}$$

6. For $c = 1$, we have:

$$f(x) = e^x$$
$$f^{(n)}(x) = e^x \Rightarrow f^{(n)}(1) = e$$
$$e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)(x-1)^n}{n!} = e\left[1 + (x-1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \frac{(x-1)^4}{4!} + \cdots\right] = e\sum_{n=0}^{\infty} \frac{(x-1)^n}{n!}$$

Scanned with CamScanner