Section 8.9 Representation of Functions by Power Series

1. (a) $$\frac{1}{2-x} = \frac{1/2}{1-(x/2)} = \frac{a}{1-r}$$

$$= \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}}$$

This series converges on $(-2, 2)$.

(b) $$\begin{array}{r l} & \frac{1}{2} + \frac{x}{4} + \frac{x^2}{8} + \frac{x^3}{16} + \cdots \\ 2 - x \,\big) & 1 \\ & \underline{1 - \frac{x}{2}} \\ & \frac{x}{2} \\ & \underline{\frac{x}{2} - \frac{x^2}{4}} \\ & \frac{x^2}{4} \\ & \underline{\frac{x^2}{4} - \frac{x^3}{8}} \\ & \frac{x^3}{8} \\ & \underline{\frac{x^3}{8} - \frac{x^4}{16}} \\ & \vdots \end{array}$$

3. (a) $$\frac{1}{2+x} = \frac{1/2}{1-(-x/2)} = \frac{a}{1-r}$$

$$= \sum_{n=0}^{\infty} \frac{1}{2}\left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^{n+1}}$$

This series converges on $(-2, 2)$.

(b) $$\begin{array}{r l} & \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16} + \cdots \\ 2 + x \,\big) & 1 \\ & \underline{1 + \frac{x}{2}} \\ & -\frac{x}{2} \\ & \underline{-\frac{x}{2} - \frac{x^2}{4}} \\ & \frac{x^2}{4} \\ & \underline{\frac{x^2}{4} + \frac{x^3}{8}} \\ & -\frac{x^3}{8} \\ & \underline{-\frac{x^3}{8} - \frac{x^4}{16}} \\ & \vdots \end{array}$$

5. Writing $f(x)$ in the form $a/(1-r)$, we have

$$\frac{1}{2-x} = \frac{1}{-3-(x-5)} = \frac{-1/3}{1+(1/3)(x-5)}$$

which implies that $a = -1/3$ and $r = (-1/3)(x-5)$.

Therefore, the power series for $f(x)$ is given by

$$\frac{1}{2-x} = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} -\frac{1}{3}\left[-\frac{1}{3}(x-5)\right]^n$$

$$= \sum_{n=0}^{\infty} \frac{(x-5)^n}{(-3)^{n+1}}, \; |x-5| < 3 \text{ or } 2 < x < 8.$$

7. Writing $f(x)$ in the form $a/(1-r)$, we have

$$\frac{3}{2x-1} = \frac{-3}{1-2x} = \frac{a}{1-r}$$

which implies that $a = -3$ and $r = 2x$.

Therefore, the power series for $f(x)$ is given by

$$\frac{3}{2x-1} = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} (-3)(2x)^n$$

$$= -3\sum_{n=0}^{\infty} (2x)^n, \; |2x| < 1 \text{ or } -\frac{1}{2} < x < \frac{1}{2}.$$

9. Writing $f(x)$ in the form $a/(1-r)$, we have

$$\frac{1}{2x-5} = \frac{-1}{11-2(x+3)}$$

$$= \frac{-1/11}{1-(2/11)(x+3)} = \frac{a}{1-r}$$

which implies that $a = -1/11$ and $r = (2/11)(x+3)$. Therefore, the power series for $f(x)$ is given by

$$\frac{1}{2x-5} = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty}\left(-\frac{1}{11}\right)\left[\frac{2}{11}(x+3)\right]^n$$

$$= -\sum_{n=0}^{\infty} \frac{2^n(x+3)^n}{11^{n+1}},$$

$$|x+3| < \frac{11}{2} \text{ or } -\frac{17}{2} < x < \frac{5}{2}.$$

11. Writing $f(x)$ in the form $a/(1-r)$, we have

$$\frac{3}{x+2} = \frac{3}{2+x} = \frac{3/2}{1+(1/2)x} = \frac{a}{1-r}$$

which implies that $a = 3/2$ and $r = (-1/2)x$. Therefore, the power series for $f(x)$ is given by

$$\frac{3}{x+2} = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} \frac{3}{2}\left(-\frac{1}{2}x\right)^n$$

$$= 3\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^{n+1}} = \frac{3}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n,$$

$|x| < 2$ or $-2 < x < 2$.

13. $\dfrac{3x}{x^2+x-2} = \dfrac{2}{x+2} + \dfrac{1}{x-1} = \dfrac{2}{2+x} + \dfrac{1}{-1+x} = \dfrac{1}{1+(1/2)x} + \dfrac{-1}{1-x}$

Writing $f(x)$ as a sum of two geometric series, we have

$$\frac{3x}{x^2+x-2} = \sum_{n=0}^{\infty}\left(-\frac{1}{2}x\right)^n + \sum_{n=0}^{\infty}(-1)(x)^n = \sum_{n=0}^{\infty}\left[\frac{1}{(-2)^n} - 1\right]x^n.$$

The interval of convergence is $-1 < x < 1$ since

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(1-(-2)^{n+1})x^{n+1}}{(-2)^{n+1}} \cdot \frac{(-2)^n}{(1-(-2)^n)x^n}\right| = \lim_{n\to\infty}\left|\frac{(1-(-2)^{n+1})x}{-2-(-2)^{n+1}}\right| = |x|.$$

15. $\dfrac{2}{1-x^2} = \dfrac{1}{1-x} + \dfrac{1}{1+x}$

Writing $f(x)$ as a sum of two geometric series, we have

$$\frac{2}{1-x^2} = \sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(1+(-1)^n)x^n = \sum_{n=0}^{\infty} 2x^{2n}.$$

The interval of convergence is $|x^2| < 1$ or $-1 < x < 1$ since $\displaystyle\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{2x^{2n+2}}{2x^2}\right| = |x^2|$.

17. $\dfrac{1}{1+x} = \displaystyle\sum_{n=0}^{\infty}(-1)^n x^n$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty}(-1)^n(-x)^n = \sum_{n=0}^{\infty}(-1)^{2n}x^n = \sum_{n=0}^{\infty} x^n$$

$$h(x) = \frac{-2}{x^2-1} = \frac{1}{1+x} + \frac{1}{1-x} = \sum_{n=0}^{\infty}(-1)^n x^n + \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty}[(-1)^n + 1]x^n$$

$$= 2 + 0x + 2x^2 + 0x^3 + 2x^4 + 0x^5 + 2x^6 + \cdots = \sum_{n=0}^{\infty} 2x^{2n},\ -1 < x < 1 \text{ (See Exercise 15.)}$$

19. By taking the first derivative, we have $\dfrac{d}{dx}\left[\dfrac{1}{x+1}\right] = \dfrac{-1}{(x+1)^2}$. Therefore,

$$\frac{-1}{(x+1)^2} = \frac{d}{dx}\left[\sum_{n=0}^{\infty}(-1)^n x^n\right] = \sum_{n=1}^{\infty}(-1)^n nx^{n-1}$$

$$= \sum_{n=0}^{\infty}(-1)^{n+1}(n+1)x^n,\ -1 < x < 1.$$

21. By integrating, we have $\displaystyle\int \frac{1}{x+1}\,dx = \ln(x+1)$. Therefore,

$$\ln(x+1) = \int\left[\sum_{n=0}^{\infty}(-1)^n x^n\right]dx = C + \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1},\ -1 < x \le 1.$$

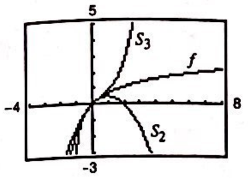
To solve for C, let $x = 0$ and conclude that $C = 0$. Therefore,

$$\ln(x+1) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1},\ -1 < x \le 1.$$

23. $\dfrac{1}{x^2+1} = \displaystyle\sum_{n=0}^{\infty}(-1)^n(x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n},\ -1 < x < 1$

25. Since, $\dfrac{1}{x+1} = \displaystyle\sum_{n=0}^{\infty}(-1)^n x^n$, we have $\dfrac{1}{4x^2+1} = \displaystyle\sum_{n=0}^{\infty}(-1)^n(4x^2)^n = \sum_{n=0}^{\infty}(-1)^n 4^n x^{2n} = \sum_{n=0}^{\infty}(-1)^n(2x)^{2n},\ -\frac{1}{2} < x < \frac{1}{2}.$

27. $x - \frac{x^2}{2} \le \ln(x+1) \le x - \frac{x^2}{2} + \frac{x^3}{3}$



x	0.0	0.2	0.4	0.6	0.8	1.0
$x - \frac{x^2}{2}$	0.000	0.180	0.320	0.420	0.480	0.500
$\ln(x+1)$	0.000	0.180	0.336	0.470	0.588	0.693
$x - \frac{x^2}{2} + \frac{x^3}{3}$	0.000	0.183	0.341	0.492	0.651	0.833

29. $g(x) = x$, line, Matches (c)

31. $g(x) = x - \frac{x^3}{3} + \frac{x^5}{5}$, Matches (a)

33. $f(x) = \arctan x$ is an odd function (symmetric to the origin)

In Exercises 35 and 37, $\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$.

35. $\arctan \frac{1}{4} = \sum_{n=0}^{\infty} (-1)^n \frac{(1/4)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)4^{2n+1}} = \frac{1}{4} - \frac{1}{192} + \frac{1}{5120} + \cdots$

Since $\frac{1}{5120} < 0.001$, we can approximate the series by its first two terms: $\arctan \frac{1}{4} \approx \frac{1}{4} - \frac{1}{192} \approx 0.245$.

37. $\frac{\arctan x^2}{x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{2n+1}$

$$\int \frac{\arctan x^2}{x}\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(4n+2)(2n+1)}$$

$$\int_0^{1/2} \frac{\arctan x^2}{x}\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(4n+2)(2n+1)2^{4n+2}} = \frac{1}{8} - \frac{1}{1152} + \cdots$$

Since $\frac{1}{1152} < 0.001$, we can approximate the series by its first term: $\int_0^{1/2} \frac{\arctan x^2}{x}\,dx \approx 0.125$

In Exercises 39 and 41, use $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, $|x| < 1$.

39. (a) $\frac{1}{(1-x)^2} = \frac{d}{dx}\left[\frac{1}{1-x}\right] = \frac{d}{dx}\left[\sum_{n=0}^{\infty} x^n\right] = \sum_{n=1}^{\infty} nx^{n-1}$, $|x| < 1$

(b) $\frac{x}{(1-x)^2} = x\sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=1}^{\infty} nx^n$, $|x| < 1$

(c) $\frac{1+x}{(1-x)^2} = \frac{1}{(1-x)^2} + \frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} n(x^{n-1} + x^n)$, $|x| < 1$

$= \sum_{n=0}^{\infty} (2n+1)x^n$, $|x| < 1$

(d) $\frac{x(1+x)}{(1-x)^2} = x\sum_{n=0}^{\infty} (2n+1)x^n = \sum_{n=0}^{\infty} (2n+1)x^{n+1}$, $|x| < 1$

41. $P(n) = \left(\frac{1}{2}\right)^n$

$$E(n) = \sum_{n=1}^{\infty} nP(n) = \sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n = \frac{1}{2}\sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^{n-1}$$

$$= \frac{1}{2}\frac{1}{[1-(1/2)]^2} = 2$$

Since the probability of obtaining a head on a single toss is $\frac{1}{2}$, it is expected that, on average, a head will be obtained in two tosses.

43. Replace x with $(-x)$.

45. Replace x with $(-x)$ and multiply the series by 5.

47. Let $\arctan x + \arctan y = \theta$. Then,

$$\tan(\arctan x + \arctan y) = \tan\theta$$

$$\frac{\tan(\arctan x) + \tan(\arctan y)}{1 - \tan(\arctan x)\tan(\arctan y)} = \tan\theta$$

$$\frac{x+y}{1-xy} = \tan\theta$$

$$\arctan\left(\frac{x+y}{1-xy}\right) = \theta. \text{ Therefore, } \arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right) \text{ for } xy \neq 1.$$

49. (a) $2\arctan\frac{1}{2} = \arctan\frac{1}{2} + \arctan\frac{1}{2} = \arctan\left[\frac{2(1/2)}{1-(1/2)^2}\right] = \arctan\frac{4}{3}$

$$2\arctan\frac{1}{2} - \arctan\frac{1}{7} = \arctan\frac{4}{3} + \arctan\left(-\frac{1}{7}\right) = \arctan\left[\frac{(4/3)-(1/7)}{1+(4/3)(1/7)}\right] = \arctan\frac{25}{25} = \arctan 1 = \frac{\pi}{4}$$

(b) $\pi = 8\arctan\frac{1}{2} - 4\arctan\frac{1}{7} \approx 8\left[\frac{1}{2} - \frac{(0.5)^3}{3} + \frac{(0.5)^5}{5} - \frac{(0.5)^7}{7}\right] - 4\left[\frac{1}{7} - \frac{(1/7)^3}{3} + \frac{(1/7)^5}{5} - \frac{(1/7)^7}{7}\right] \approx 3.14$

51. From Exercise 21, we have

$$\ln(x+1) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n}$$

$$= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^n}{n}.$$

Thus, $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{2^n n} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}(1/2)^n}{n}$

$$= \ln\left(\frac{1}{2}+1\right) = \ln\frac{3}{2} \approx 0.4055$$

53. From Exercise 51, we have

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{5^n n} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}(2/5)^n}{n}$$

$$= \ln\left(\frac{2}{5}+1\right) = \ln\frac{7}{5} \approx 0.3365.$$

55. From Exercise 54, we have

$$\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^{2n+1}(2n+1)} = \sum_{n=0}^{\infty}(-1)^n\frac{(1/2)^{2n+1}}{2n+1} = \arctan\frac{1}{2} \approx 0.4636.$$

57. The series in Exercise 54 converges to its sum at a slower rate because its terms approach 0 at a much slower rate.

59. $f(x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^n}{n}, \quad 0 < x \leq 2$

$$f(0.5) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(-0.5)^n}{n} = \sum_{n=1}^{\infty} -\frac{(1/2)^n}{n}$$

$$\sum_{n=1}^{\infty} -\frac{(1/2)^n}{n} = -0.6931$$

Section 8.10 Taylor and Maclaurin Series

1. For $c = 0$, we have:

$$f(x) = e^{2x}$$

$$f^{(n)}(x) = 2^n e^{2x} \Longrightarrow f^{(n)}(0) = 2^n$$

$$e^{2x} = 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \frac{16x^4}{4!} + \cdots = \sum_{n=0}^{\infty}\frac{(2x)^n}{n!}$$