8.9 Continued ⇒ Manipulations + error

$$f(x)=\frac{1}{1+x^2}=\sum_{n=0}^{\infty}(-1)^n x^{2n}=1-x^2+x^4-\cdots$$

$$\arctan x=\int_0^x \frac{1}{1+x^2}\,dx=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}=x-\frac{1}{3}x^3+\frac{1}{5}x^5-\cdots$$

must rewrite - can't leave x in denom!

$$\frac{\arctan x}{x}=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)x}=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2n+1}=1-\frac{1}{3}x^2+\frac{1}{5}x^4-\cdots$$

(divide by x)

$$\frac{\arctan x}{x}-1=\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{2n+1}=-\frac{1}{3}x^2+\frac{1}{5}x^4-\cdots$$

(subtract 1, so 1st term goes away. Series same but now n=1)

must rewrite so only x^m

$$x^2\arctan x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}\cdot x^2=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+3}}{2n+1}=x^3-\frac{1}{3}x^5+\frac{1}{5}x^7-\cdots$$

(mult all terms by x^2)

$$\arctan(x^2)=\sum_{n=0}^{\infty}\frac{(-1)^n (x^2)^{2n+1}}{2n+1}=\sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+2}}{2n+1}=x^2-\frac{1}{3}x^6+\frac{1}{5}x^{10}-\cdots$$

(substitute x^2 for all x's)

$f(x)=\frac{1}{x}$ centered @ 1

$$\frac{1}{x}=\frac{1}{1+(x-1)}=1-(x-1)+(x-1)^2-\cdots=\sum_{n=0}^{\infty}(-1)^n(x-1)^n$$

$\ln x$ centered @ 1

$$\ln x=\int_1^x \frac{1}{x}\,dx=(x-1)-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3\cdots=\sum_{n=0}^{\infty}\frac{(-1)^n(x-1)^{n+1}}{n+1}$$

$$\int_1^x \ln x\,dx=\frac{1}{2}(x-1)^2-\frac{1}{6}(x-1)^3+\frac{1}{12}(x-1)^4\cdots=\sum_{n=0}^{\infty}\frac{(-1)^n(x-1)^{n+2}}{(n+1)(n+2)}$$

Alternating Series remainder (error) in approx

If conv. alt series that is decreasing, then the absolute value of remainder R_n in approx sum S_n is less than or equal to the 1st omitted term.

$$\left| S_{actual} - S_n \right| = |R_n| \le a_{n+1}$$

$$\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots$$

Approx $\ln(1.1)$ using 1st 3 terms + find error in approx.

$$\ln(1.1) \approx (1.1-1) - \frac{1}{2}(1.1-1)^2 + \frac{1}{3}(1.1-1)^3 = .095\overline{3}$$

B/c alt + decr, error $= R_3 \le \left| \frac{1}{4}(1.1-1)^4 \right| = .000025$

So actual value = approx ± error

$$.095\overline{3} - .000025 \le \ln(1.1) \le .095\overline{3} + .000025$$

Sometimes you come up w/ # of terms to use:

Approx $\int_1^{1.2} \ln x\,dx$ with error $\le .001$.

$$\int_1^{1.2} \ln x\,dx = \int_1^{1.2} (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots = \frac{1}{2}(x-1)^2 - \frac{1}{6}(x-1)^3 + \frac{1}{12}(x-1)^4 - \cdots \Big|_1^{1.2}$$

$$= \underbrace{\frac{1}{2}(.2)^2}_{.02} - \underbrace{\frac{1}{6}(.2)^3}_{.0013} + \underbrace{\frac{1}{12}(.2)^4}_{.00013} - \cdots$$

1st one $\le .001$ so this is error

$$\int_1^{1.2} \ln x\,dx \approx \frac{1}{2}(.2)^2 - \frac{1}{6}(.2)^3 = .0187$$

$$\text{error} = R_3 \le \frac{1}{12}(.2)^4 = .00013$$

matches degree used for approx