The Taylor series about $x = 5$ for a certain function f converges to $f(x)$ for all x in the interval of convergence. The nth derivative of f at $x = 5$ is given by $f^{(n)}(5) = \dfrac{(-1)^n n!}{2^n (n+2)}$, and $f(5) = \dfrac{1}{2}$.

(a) Write the third-degree Taylor polynomial for f about $x = 5$.

(b) Find the radius of convergence of the Taylor series for f about $x = 5$.

(c) Show that the sixth-degree Taylor polynomial for f about $x = 5$ approximates $f(6)$ with error less than $\dfrac{1}{1000}$.

(a) $f'(5) = \dfrac{-1!}{2(3)}$, $f''(5) = \dfrac{2!}{4(4)}$, $f'''(5) = \dfrac{-3!}{8(5)}$

$$P_3(f,5)(x) = \frac{1}{2} - \frac{1}{6}(x-5) + \frac{1}{16}(x-5)^2 - \frac{1}{40}(x-5)^3$$

3 : $P_3(f,5)(x)$
<−1> each error or missing term

Note: <−1> max for improper use of extra terms, equality or +...

(b) $a_n = \dfrac{f^{(n)}(5)}{n!} = \dfrac{(-1)^n}{2^n (n+2)}$

$$\lim_{n\to\infty} \left| \frac{\dfrac{(-1)^{n+1}(x-5)^{n+1}}{2^{n+1}(n+3)}}{\dfrac{(-1)^n (x-5)^n}{2^n (n+2)}} \right| = \lim_{n\to\infty} \frac{1}{2}\left(\frac{n+2}{n+3}\right)|x-5|$$

$$= \frac{|x-5|}{2} < 1$$

The radius of convergence is 2.

4 :
- 1 : general term
- 1 : sets up ratio test
- 1 : computes the limit
- 1 : applies ratio test to get radius of convergence

(c) The Taylor series about $x = 5$ for the function f, when evaluated at $x = 6$, is an alternating series with absolute value of terms decreasing to 0. The error in approximating $f(6)$ with the 6th degree Taylor polynomial at $x = 6$ is less than the first omitted term in the series.

$$\left| f(6) - P_6(f,5)(6) \right| \le \frac{1}{2^7(9)} = \frac{1}{1152} < \frac{1}{1000}$$

2 :
- 1 : error bound $< \dfrac{1}{1000}$
- 1 : refers to an alternating series and indicates the error bound is found from the next term

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Question 6

The Maclaurin series for $\ln\left(\frac{1}{1-x}\right)$ is $\sum_{n=1}^{\infty} \frac{x^n}{n}$ with interval of convergence $-1 \le x < 1$.

(a) Find the Maclaurin series for $\ln\left(\frac{1}{1+3x}\right)$ and determine the interval of convergence.

(b) Find the value of $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.

(c) Give a value of p such that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converges, but $\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ diverges. Give reasons why your value of p is correct.

(d) Give a value of p such that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges, but $\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ converges. Give reasons why your value of p is correct.

(a) $\ln\left(\frac{1}{1+3x}\right) = \ln\left(\frac{1}{1-(-3x)}\right)$

$$= \sum_{n=1}^{\infty} \frac{(-3x)^n}{n} \text{ or } \sum_{n=1}^{\infty} (-1)^n \frac{3^n}{n} x^n$$

We must have $-1 \le -3x < 1$, so interval of convergence is $-\frac{1}{3} < x \le \frac{1}{3}$.

$2 \begin{cases} 1 : \text{series} \\ 1 : \text{interval of convergence} \end{cases}$

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = \ln\left(\frac{1}{1-(-1)}\right) = \ln\left(\frac{1}{2}\right)$

1 : answer

(c) Some p such that $0 < p \le \frac{1}{2}$ because $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converges by AST, but the p-series $\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ diverges for $2p \le 1$.

$3 \begin{cases} 1 : \text{correct } p \\ 1 : \text{reason why } \sum \frac{(-1)^n}{n^p} \text{ converges} \\ 1 : \text{reason why } \sum \frac{1}{n^{2p}} \text{ diverges} \end{cases}$

(d) Some p such that $\frac{1}{2} < p \le 1$ because the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges for $p \le 1$ and the p-series $\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ converges for $2p > 1$.

$3 \begin{cases} 1 : \text{correct } p \\ 1 : \text{reason why } \sum \frac{1}{n^p} \text{ diverges} \\ 1 : \text{reason why } \sum \frac{1}{n^{2p}} \text{ converges} \end{cases}$

Question 6

A function f is defined by

$$f(x) = \frac{1}{3} + \frac{2}{3^2}x + \frac{3}{3^3}x^2 + \cdots + \frac{n+1}{3^{n+1}}x^n + \cdots$$

for all x in the interval of convergence of the given power series.

(a) Find the interval of convergence for this power series. Show the work that leads to your answer.

(b) Find $\lim_{x \to 0} \dfrac{f(x) - \frac{1}{3}}{x}$.

(c) Write the first three nonzero terms and the general term for an infinite series that represents $\int_0^1 f(x)\,dx$.

(d) Find the sum of the series determined in part (c).

(a) $$\lim_{n \to \infty} \left| \frac{\frac{(n+2)x^{n+1}}{3^{n+2}}}{\frac{(n+1)x^n}{3^{n+1}}} \right| = \lim_{n \to \infty} \left| \frac{(n+2)}{(n+1)} \frac{x}{3} \right| = \left| \frac{x}{3} \right| < 1$$

At $x = -3$, the series is $\sum_{n=0}^{\infty} (-1)^n \frac{n+1}{3}$, which diverges.

At $x = 3$, the series is $\sum_{n=0}^{\infty} \frac{n+1}{3}$, which diverges.

Therefore, the interval of convergence is $-3 < x < 3$.

4 :
- 1 : sets up ratio test
- 1 : computes limit
- 1 : conclusion of ratio test
- 1 : endpoint conclusion

(b) $$\lim_{x \to 0} \frac{f(x) - \frac{1}{3}}{x} = \lim_{x \to 0} \left(\frac{2}{3^2} + \frac{3}{3^3}x + \frac{4}{3^4}x^2 + \cdots \right) = \frac{2}{9}$$

1 : answer

(c) $$\int_0^1 f(x)\,dx = \int_0^1 \left(\frac{1}{3} + \frac{2}{3^2}x + \frac{3}{3^3}x^2 + \cdots + \frac{n+1}{3^{n+1}}x^n + \cdots \right) dx$$

$$= \left(\frac{1}{3}x + \frac{1}{3^2}x^2 + \frac{1}{3^3}x^3 + \cdots + \frac{1}{3^{n+1}}x^{n+1} + \cdots \right) \Bigg|_{x=0}^{x=1}$$

$$= \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^{n+1}} + \cdots$$

3 :
- 1 : antidifferentiation of series
- 1 : first three terms for definite integral series
- 1 : general term

(d) The series representing $\int_0^1 f(x)\,dx$ is a geometric series.

Therefore, $\int_0^1 f(x)\,dx = \dfrac{\frac{1}{3}}{1 - \frac{1}{3}} = \dfrac{1}{2}$.

1 : answer

4. The function f has derivatives of all orders for all real numbers x. Assume $f(2) = -3$, $f'(2) = 5$, $f''(2) = 3$, and $f'''(2) = -8$.
   (a) Write the third–degree Taylor polynomial for f about $x = 2$ and use it to approximate $f(1.5)$.
   (b) The fourth derivative of f satisfies the inequality $|f^{(4)}(x)| \le 3$ for all x in the closed interval $[1.5, 2]$. Use the Lagrange error bound on the approximation to $f(1.5)$ found in part (a) to explain why $f(1.5) \ne -5$.
   (c) Write the fourth–degree Taylor polynomial, $P(x)$, for $g(x) = f(x^2 + 2)$ about $x = 0$. Use P to explain why g must have a relative minimum at $x = 0$.

(a) $T_3(f,2)(x) = -3 + 5(x-2) + \frac{3}{2}(x-2)^2 - \frac{8}{6}(x-2)^3$

$f(1.5) \approx T_3(f,2)(1.5)$

$= -3 + 5(-0.5) + \frac{3}{2}(-0.5)^2 - \frac{4}{3}(-0.5)^3$

$= -4.958\overline{3} = -4.958$

4 { 3: $T_3(f,2)(x)$
<−1> each error
1: approximation of $f(1.5)$ }

(b) Lagrange Error Bound $= \frac{3}{4!}|1.5-2|^4 = 0.0078125$

$f(1.5) > -4.958\overline{3} - 0.0078125 = -4.966 > -5$

Therefore, $f(1.5) \ne -5$.

2 { 1: value of Lagrange Error Bound
1: explanation }

(c) $P(x) = T_4(g,0)(x)$

$= T_2(f,2)(x^2+2) = -3 + 5x^2 + \frac{3}{2}x^4$

The coefficient of x in $P(x)$ is $g'(0)$. This coefficient is 0, so $g'(0) = 0$.

The coefficient of x^2 in $P(x)$ is $\frac{g''(0)}{2!}$. This coefficient is 5, so $g''(0) = 10$ which is greater than 0.

Therefore, g has a relative minimum at $x = 0$.

3 { 2: $T_4(g,0)(x)$
<−1> each incorrect, missing, or extra term
1: explanation }

Note:
<−1> max for improper use of $+\ldots$ or equality