Area Practice - SWOK

3-4 → Set up an integral to find area of shaded region

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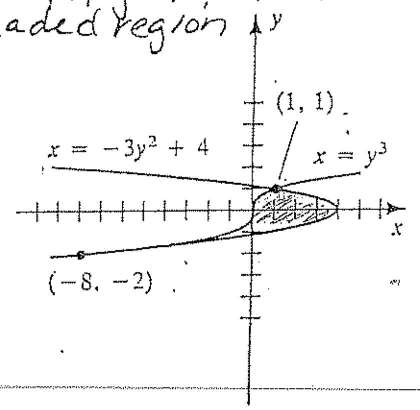

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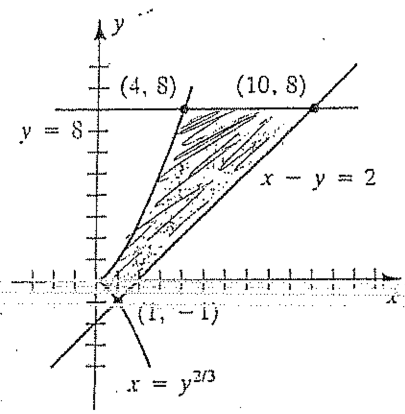

Exer. 5–22: Sketch the region bounded by the graphs of the equations and find its area.

5 $y = x^2$; $y = 4x$

6 $x + y = 3$; $y + x^2 = 3$

7 $y = x^2 + 1$; $y = 5$

8 $y = 4 - x^2$; $y = -4$

9 $y = 1/x^2$; $y = -x^2$; $x = 1$; $x = 2$

10 $y = x^3$; $y = x^2$

11 $y^2 = -x$; $x - y = 4$; $y = -1$; $y = 2$

12 $x = y^2$; $y - x = 2$; $y = -2$; $y = 3$

13 $y^2 = 4 + x$; $y^2 + x = 2$

14 $x = y^2$; $x - y = 2$

15 $x = 4y - y^3$; $x = 0$

16 $x = y^{2/3}$; $x = y^2$

17 $y = x^3 - x$; $y = 0$

18 $y = x^3 - x^2 - 6x$; $y = 0$

19 $x = y^3 + 2y^2 - 3y$; $x = 0$

20 $x = 9y - y^3$; $x = 0$

21 $y = x\sqrt{4 - x^2}$; $y = 0$

22 $y = x\sqrt{x^2 - 9}$; $y = 0$; $x = 5$

Exer. 23–24: Find the area of the region between the graphs of the two equations from $x = 0$ to $x = \pi$.

23 $y = \sin 4x$; $y = 1 + \cos \frac{1}{3}x$

24 $y = 4 + \cos 2x$; $y = 3 \sin \frac{1}{2}x$

Exer. 25–26: Set up sums of integrals that can be used to find the area of the shaded region by integrating with respect to (a) x and (b) y.

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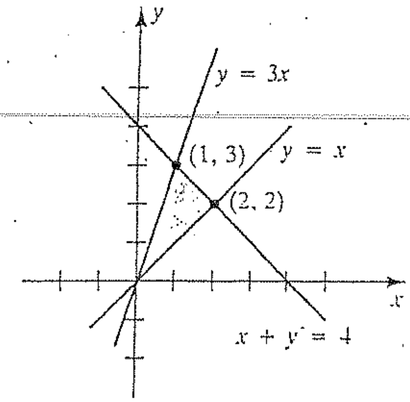

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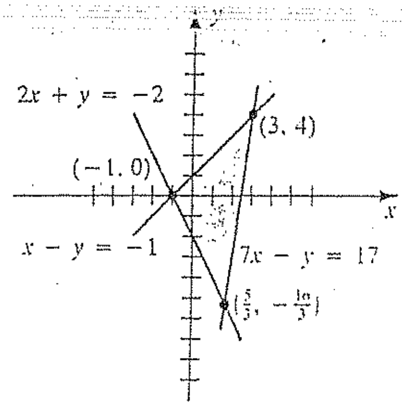

Exer. 27–30: Set up sums of integrals that can be used to find the area of the region bounded by the graphs of the equations by integrating with respect to (a) x and (b) y.

27 $y = \sqrt{x}$; $y = -x$; $x = 1$; $x = 4$

28 $y = 1 - x^2$; $y = x - 1$

29 $y = x + 3$; $x = -y^2 + 3$

30 $x = y^2$; $x = 2y^3 - 4$

Exer. 31–36: Find the area of the region between the graphs of f and g if x is restricted to the given interval.

31 $f(x) = 6 - 3x^2$; $g(x) = 3x$; $[0, 2]$

32 $f(x) = x^3 - 4$; $g(x) = x + 2$; $[1, 4]$

33 $f(x) = x^3 - 4x + 2$; $g(x) = 2$; $[-1, 3]$

34 $f(x) = x^2$; $g(x) = x^3$; $[-1, 2]$

Exercises 6.1

1 The shaded region is an R_x region. $A=\int_{-2}^{2}\left[(x^2+1)-(x-2)\right]dx$.

2 The shaded region is an R_x region. $A=\int_{1}^{4}\left[(-x+6)-\sqrt{x}\right]dx$.

3 The shaded region is an R_y region. $A=\int_{-2}^{1}\left[(-3y^2+4)-y^2\right]dy$.

4 The shaded region is an R_y region. $A=\int_{-1}^{8}\left[(y+2)-y^{2/3}\right]dy$.

5 $x^2=4x \Rightarrow x=0,4$. $4x\geq x^2$ on $[0,4] \Rightarrow$

$$A=\int_0^4(4x-x^2)\,dx=\left[2x^2-\tfrac{1}{3}x^3\right]_0^4=32-\tfrac{64}{3}=\tfrac{32}{3}.$$

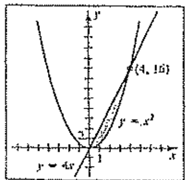

Figure 5

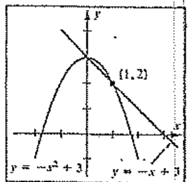

Figure 6

6 $3-x=3-x^2 \Rightarrow x=0,1$. $3-x^2\geq 3-x$ on $[0,1] \Rightarrow$

$$A=\int_0^1\left[(3-x^2)-(3-x)\right]dx=\left[-\tfrac{1}{3}x^3+\tfrac{1}{2}x^2\right]_0^1=-\tfrac{1}{3}+\tfrac{1}{2}=\tfrac{1}{6}.$$

Note: Symmetry is used without mention whenever possible in the solutions and answers.

7 $x^2+1=5 \Rightarrow x=\pm 2$. $5\geq(x^2+1)$ on $[-2,2] \Rightarrow$

$$A=2\int_0^2\left[5-(x^2+1)\right]dx=2\left[4x-\tfrac{1}{3}x^3\right]_0^2=2\left(\tfrac{16}{3}\right)=\tfrac{32}{3}.$$

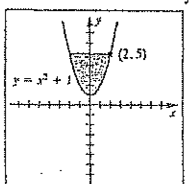

Figure 7

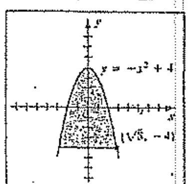

Figure 8

8 $4-x^2=-4 \Rightarrow x=\pm\sqrt{8}$. $4-x^2\geq -4$ on $[-\sqrt{8},\sqrt{8}] \Rightarrow$

$$A=2\int_0^{\sqrt{8}}\left[(4-x^2)-(-4)\right]dx=2\left[8x-\tfrac{1}{3}x^3\right]_0^{\sqrt{8}}=2\left(\tfrac{16\sqrt{8}}{3}\right)=\tfrac{64\sqrt{2}}{3}.$$

9 $1/x^2>-x^2 \Rightarrow A=\int_1^2\left[x^{-2}-(-x^2)\right]dx=\left[-\tfrac{1}{x}+\tfrac{1}{3}x^3\right]_1^2=\tfrac{13}{6}-\left(-\tfrac{2}{3}\right)=\tfrac{17}{6}.$

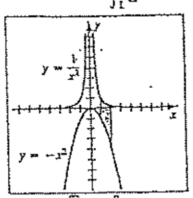

Figure 9

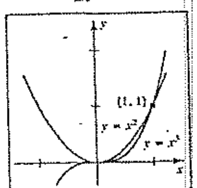

Figure 10

10 $x^3=x^2 \Rightarrow x=0,1$. $x^2\geq x^3$ on $[0,1] \Rightarrow$

$$A=\int_0^1(x^2-x^3)\,dx=\left[\tfrac{1}{3}x^3-\tfrac{1}{4}x^4\right]_0^1=\tfrac{1}{3}-\tfrac{1}{4}=\tfrac{1}{12}.$$

11 Equivalent equations are $x=-y^2$ and $x=y+4$. $y+4>-y^2$ on $[-1,2] \Rightarrow$

$$A=\int_{-1}^{2}\left[(4+y)-(-y^2)\right]dy=\left[4y+\tfrac{1}{2}y^2+\tfrac{1}{3}y^3\right]_{-1}^{2}=\tfrac{38}{3}-\left(-\tfrac{23}{6}\right)=\tfrac{33}{2}.$$

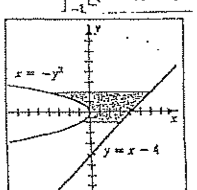

Figure 11

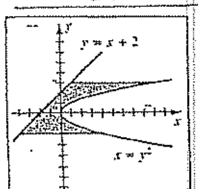

Figure 12

12 $y^2>y-2$ on $[-2,3] \Rightarrow$

$$A=\int_{-2}^{3}\left[y^2-(y-2)\right]dy=\left[\tfrac{1}{3}y^3-\tfrac{1}{2}y^2+2y\right]_{-2}^{3}=\tfrac{21}{2}-\left(-\tfrac{26}{3}\right)=\tfrac{115}{6}.$$

13 $y^2-4=2-y^2 \Rightarrow y=\pm\sqrt{3}$. $2-y^2\geq y^2-4$ on $[-\sqrt{3},\sqrt{3}] \Rightarrow$

$$A=2\int_0^{\sqrt{3}}\left[(2-y^2)-(y^2-4)\right]dy=2\left[6y-\tfrac{2}{3}y^3\right]_0^{\sqrt{3}}=2(4\sqrt{3})=8\sqrt{3}.$$

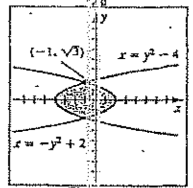

Figure 13

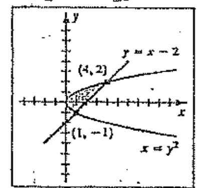

Figure 14

14 $y^2=y+2 \Rightarrow y=-1,2$. $y+2\geq y^2$ on $[-1,2] \Rightarrow$

$$A=\int_{-1}^{2}\left[(y+2)-y^2\right]dy=\left[\tfrac{1}{2}y^2+2y-\tfrac{1}{3}y^3\right]_{-1}^{2}=\tfrac{10}{3}-\left(-\tfrac{7}{6}\right)=\tfrac{9}{2}.$$

15 $x=y(2-y)(2+y)\geq 0$ on $[0,2]$ and $x\leq 0$ on $[-2,0]$.

$$A=2\int_0^2\left[(4y-y^3)-0\right]dy=2\left[2y^2-\tfrac{1}{4}y^4\right]_0^2=2(4)=8.$$

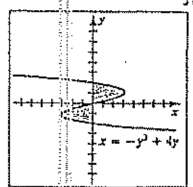

Figure 15

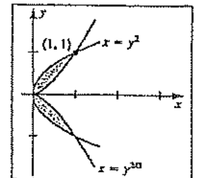

Figure 16

16 $y^{2/3}=y^2 \Rightarrow y=0,\pm 1$. $y^{2/3}\geq y^2$ on $[-1,1] \Rightarrow$

$$A=2\int_0^1(y^{2/3}-y^2)\,dy=2\left[\tfrac{3}{5}y^{5/3}-\tfrac{1}{3}y^3\right]_0^1=2\left(\tfrac{4}{15}\right)=\tfrac{8}{15}.$$

17 $x^3-x=0 \Rightarrow x=0,\pm 1$. $x^3-x\geq 0$ on $[-1,0]$ and $x^3-x\leq 0$ on $[0,1]$.

$$A=2\int_0^1\left[0-(x^3-x)\right]dx=2\left[-\tfrac{1}{4}x^4+\tfrac{1}{2}x^2\right]_0^1=2\left(\tfrac{1}{4}\right)=\tfrac{1}{2}.$$

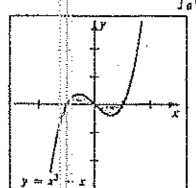

Figure 17

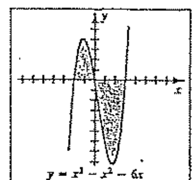

Figure 18

18 $y=x(x-3)(x+2)\geq 0$ on $[-2,0]$ and $y\leq 0$ on $[0,3]$.

$$A=\int_{-2}^{0}\left[(x^3-x^2-6x)-0\right]dx+\int_0^3\left[0-(x^3-x^2-6x)\right]dx=\tfrac{16}{3}+\tfrac{63}{4}=\tfrac{253}{12}.$$

19 $x=y(y+3)(y-1)\geq 0$ on $[-3,0]$ and $x\leq 0$ on $[0,1]$. $A=$

$$\int_{-3}^{0}\left[(y^3+2y^2-3y)-0\right]dy+\int_0^1\left[0-(y^3+2y^2-3y)\right]dy=\tfrac{45}{4}+\tfrac{7}{12}=\tfrac{71}{6}.$$

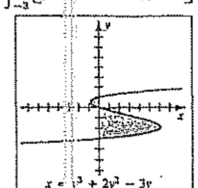

Figure 19

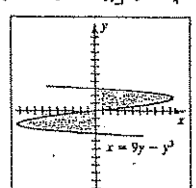

Figure 20

21 $y=x\sqrt{4-x^2}\leq 0$ on $[-2,0]$ and $y\geq 0$ on $[0,2]$.

$A=2\int_0^2 x\sqrt{4-x^2}\,dx$. $u=4-x^2 \Rightarrow -du=2x\,dx$. $x=0,2 \Rightarrow u=4,0$.

$$A=-\int_4^0 u^{1/2}\,du=-\left[\tfrac{2}{3}u^{3/2}\right]_4^0=-\tfrac{2}{3}(0-8)=\tfrac{16}{3}.$$

23 $\sin 4x\leq 1+\cos\frac{1}{3}x \Rightarrow$

$$A=\int_0^{\pi}\left[(1+\cos\tfrac{1}{3}x)-\sin 4x\right]dx$$
$$=\left[x+3\sin\tfrac{1}{3}x+\tfrac{1}{4}\cos 4x\right]_0^{\pi}$$
$$=\left(\pi+\tfrac{3}{2}\sqrt{3}+\tfrac{1}{4}\right)-\left(\tfrac{1}{4}\right)=\pi+\tfrac{3}{2}\sqrt{3}\approx 5.74.$$

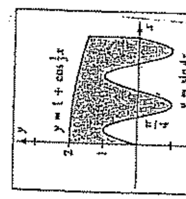

Figure 23

24 $3\sin\frac{1}{2}x\leq 4+\cos 2x \Rightarrow A=\int_0^{\pi}\left[(4+\cos 2x)-3\sin\frac{1}{2}x\right]dx=$

$$\left[4x+\tfrac{1}{2}\sin 2x+6\cos\tfrac{1}{2}x\right]_0^{\pi}=4\pi-6\approx 6.57.$$

25 (a) On $[0,1]$, $3x\geq x$ and on $[1,2]$, $(4-x)\geq x$.

$$A=\int_0^1(3x-x)\,dx+\int_1^2\left[(4-x)-x\right]dx. = 2$$

(b) On $[0,2]$, $y\geq\frac{1}{3}y$ and on $[2,3]$, $(4-y)\geq\frac{1}{3}y$.

$$A=\int_0^2\left(y-\tfrac{1}{3}y\right)dy+\int_2^3\left[(4-y)-\tfrac{1}{3}y\right]dy. = 2$$

26 (a) On $[-1,\frac{5}{3}]$, $(x+1)\geq(-2x-2)$ and on $[\frac{5}{3},3]$, $(x+1)\geq(7x-17)$.

$$A=\int_{-1}^{5/3}\left[(x+1)-(-2x-2)\right]dx+\int_{5/3}^{3}\left[(x+1)-(7x-17)\right]dx.$$

27. a) $\int_1^4[\sqrt{x}-(-x)]dx$

b) $\int_{-4}^{-1}[4-(-y)]dy+\int_{-1}^{1}(4-1)dy+\int_1^2(4-y^2)dy$

29. a) $\int_{-6}^{-1}[(x+3)-(-\sqrt{3-x})]dx+2\int_{-1}^{3}\sqrt{3-x}\,dx$

b) $\int_{-3}^{2}[(3-y^2)-(y-3)]dy$