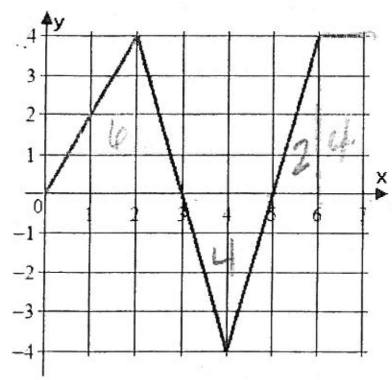
Definite Integrals and the Fundamental Theorem of Calculus

1. The given graph is f(x). Determine the following . . .



a) $\int_0^3 f(x)dx$ 6

b) $\int_3^7 4f(x)dx$ 8

c) $\int_0^7 f(x)dx$ 8

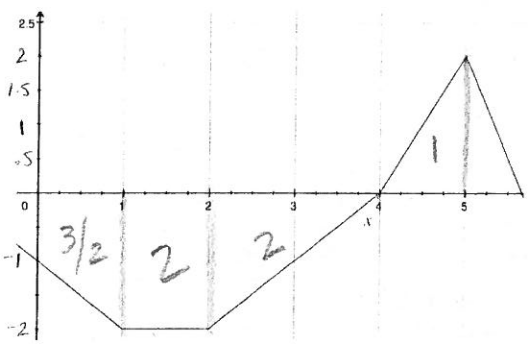
d) $\int_0^7 |f(x)|dx$ 16

e) $\int_5^7 f(x-2)dx$ $\int_3^5 f(x)dx$ -4

f) $\int_0^3 f(x)-4dx$ 6−12 = −6

g) $\int_3^0 f(x)dx$ −6

2. Suppose the given graph is f'(x) and f(4) = -3. Find the following . . .



$\frac{1}{2}(1)(1+2)$

$\frac{3}{2}$

$\int_0^4 f'dx = f(4)-f(0)$

$-5.5 = -3 - f(0)$

$f(0) = -3 + 5.5$

$= 2.5$

$\int_4^5 f'dx = f(5) - f(4)$

$1 = f(5) + 3$

a) f(5) −2

a) f(1) $-4 = f(4) - f(1)$

1 $-4 = -3 - f(1)$

b) Determine where f(x) has relative extrema on (0,5.75). Justify your answer.

min x=4 f' ch − to +

c) Find the absolute minimum and absolute maximum VALUES on [0, 5]. Show work.

$f(4) = -3$

$f(0) = 2.5$

$f(5) = -2$

*Check endpts and x=4 since f' ch. − to +

Abs min = −3

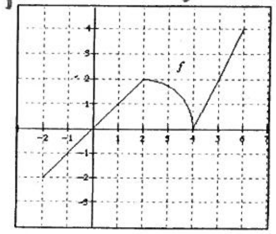
max = 2.5

Key

Worksheet 3. Graphical Analysis of $F(x)$ Using $F'(x)$

$f = F'(x)$

1. Let $F(x) = \int_2^x f(t)\,dt$. The graph of f on the interval $[-2,6]$ consists of two line segments and a quarter of a circle, as shown at right.



(a) Find $F(0)$ and $F(4)$.

−2 π

(b) Determine the interval where $F(x)$ is increasing. Justify your answer.

$(0,4)\cup(4,6)$ b/c $F' = f > 0$

(c) Find the critical numbers of $F(x)$ and determine if each corresponds to a relative minimum value, a relative maximum value, or neither. Justify your answers.

$F' = f = 0$ @ $x=0$ (min) $x=4$ (neither)

(d) Find the absolute extreme values of $F(x)$ and the x-values at which they occur. Justify your answers.

$F(-2)=0$ $F(6)=\pi+4$ $F(0)=-2$ $F(4)=\pi$

(e) Find the x-coordinates of the inflection points of $F(x)$. Justify your answer.

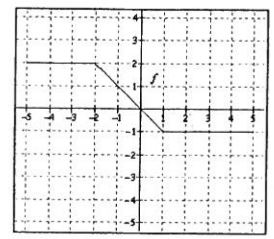
$x=2$ $x=4$ b/c $F'=f$ has local extrema

(f) Determine the intervals where the graph of $F(x)$ is concave down. Justify your answer.

$F'=f$ dec. $(2,4)$

omit

2. Let $H(x) = \int_0^{x+2} f(t)\,dt$, where f is defined on the interval $[-5,5]$ and the graph of f consists of three line segments, as shown at right.

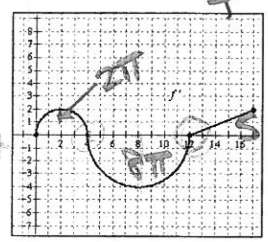


(a) Determine the domain of $H(x)$.

(b) Determine the range of $H(x)$.

(c) Determine the x-coordinates of the relative extrema of $H(x)$. Justify your answer.

3. The graph below is of the function $f'(x)$, the derivative of the function $f(x)$, on the interval $0 \le x \le 17$. The graph consists of two semicircles and one line segment. Horizontal tangents are located at $x = 2$ and $x = 8$, and a vertical tangent is located at $x = 4$.



(a) On what intervals is $f(x)$ increasing? Justify your answer.

(0,4) ∪ (12, 17) f' > 0

(b) For what values of x does $f(x)$ have a relative minimum value? Justify.

x = 12 b/c f' ch - to +

(c) On what intervals, for $0 < x < 17$, is the graph of $f(x)$ concave up? Justify.

f' incr. (0, 2) ∪ (8, 17)

(d) For what values of x, for $0 < x < 17$, is $f''(x)$ undefined?

x = 4 & x = 12

(e) Identify the x-coordinates of all points of inflection of $f(x)$. Justify.

x = 2 x = 8 f' local extrema

(f) For what value of x does $f(x)$ reach its absolute maximum value? Justify.

f(4)

(g) If $f(4) = 3$, find $f(12)$.

$\int_4^{12} f'\,dx = f(12) - f(4)$

$-8\pi = f(12) - 3$

$f(12) = 3 - 8\pi$

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See separate sheet for answers

1.

t (minutes)	0	2	5	7	11	12
$r'(t)$ (feet per minute)	5.7	4.0	2.0	1.2	0.6	0.5

The volume of a spherical hot air balloon expands as the air inside the balloon is heated. The radius of the balloon, in feet, is modeled by a twice-differentiable function r of time t, where t is measured in minutes. For $0 < t < 12$, the graph of r is concave down. The table above gives selected values of the rate of change, $r'(t)$, of the radius of the balloon over the time interval $0 \le t \le 12$. The radius of the balloon is 30 feet when $t = 5$. (Note: The volume of a sphere of radius r is given by $V = \frac{4}{3}\pi r^3$.)

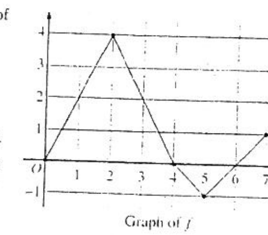
(a) Estimate the radius of the balloon when $t = 5.4$ using the tangent line approximation at $t = 5$. Is your estimate greater than or less than the true value? Give a reason for your answer.

(b) Find the rate of change of the volume of the balloon with respect to time when $t = 5$. Indicate units of measure.

(c) Use a right Riemann sum with the five subintervals indicated by the data in the table to approximate $\int_0^{12} r'(t)\,dt$. Using correct units, explain the meaning of $\int_0^{12} r'(t)\,dt$ in terms of the radius of the balloon.

(d) Is your approximation in part (c) greater than or less than $\int_0^{12} r'(t)\,dt$? Give a reason for your answer.

2. Let f be a function defined on the closed interval $[0, 7]$. The graph of f, consisting of four line segments, is shown above. Let g be the function given by $g(x) = \int_2^x f(t)\,dt$.



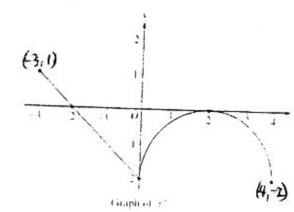
(a) Find $g(3)$, $g'(3)$, and $g''(3)$.

(b) Find the average rate of change of g on the interval $0 \le x \le 3$.

(c) For how many values c, where $0 < c < 3$, is $g'(c)$ equal to the average rate found in part (b)? Explain your reasoning.

(d) Find the x-coordinate of each point of inflection of the graph of g on the interval $0 < x < 7$. Justify your answer.

3. Let f be a function defined on the closed interval $-3 \le x \le 4$ with $f(0) = 3$. The graph of f', the derivative of f, consists of one line segment and a semicircle, as shown above.



(a) On what intervals, if any, is f increasing? Justify your answer.

(b) Find the x-coordinate of each point of inflection of the graph of f on the open interval $-3 < x < 4$. Justify your answer.

(c) Find an equation for the line tangent to the graph of f at the point (0,3).

(d) Find $f(-3)$ and $f(4)$. Show the work that leads to your answers.

1a) At $t=5$, $r'(5)=2$ & $r(5)=30$

$$r-30=2(t-5) \Rightarrow r=2(t-5)+30$$

$$r(5.4) \approx 2(5.4-5)+30 = 30.8 \text{ ft.}$$

This estimate is greater than the true value b/c $r(t)$ is concave down so the tangent line is above the curve.

b) $V=\frac{4}{3}\pi r^3$ $\quad \frac{dV}{dt}=4\pi r^2\frac{dr}{dt}=4\pi(30)^2(2)$

$$=7200\pi \text{ ft}^3/\text{min}$$

c) $\int_0^{12} r'(t)\,dt \approx 2(4)+3(2)+2(1.2)+4(0.6)+1(0.5)$

$= r(12)-r(0)$ $\quad = 19.3$ ft.

The radius of the balloon increases by 19.3 ft from 0 to 12 minutes.

d) The approx. is less than the actual value because $r'(t)$ is decreasing so the rectangles are below the curve.

x . 7

2

(a) $g(3) = \int_2^3 f(t)\,dt = \frac{1}{2}(4+2) = 3$

$g'(3) = f(3) = 2$

$g''(3) = f'(3) = \frac{0-4}{4-2} = -2$

$3 : \begin{cases} 1 : g(3) \\ 1 : g'(3) \\ 1 : g''(3) \end{cases}$

(b) $\frac{g(3)-g(0)}{3} = \frac{1}{3}\int_0^3 f(t)\,dt$

$= \frac{1}{3}\left(\frac{1}{2}(2)(4) + \frac{1}{2}(4+2)\right) = \frac{7}{3}$

$2 : \begin{cases} 1 : g(3) - g(0) = \int_0^3 f(t)\,dt \\ 1 : \text{answer} \end{cases}$

(c) There are two values of c.

We need $\frac{7}{3} = g'(c) = f(c)$

The graph of f intersects the line $y = \frac{7}{3}$ at two places between 0 and 3.

$2 : \begin{cases} 1 : \text{answer of 2} \\ 1 : \text{reason} \end{cases}$

Note: 1/2 if answer is 1 by MVT

(d) $x = 2$ and $x = 5$

because $g' = f$ changes from increasing to decreasing at $x = 2$, and from decreasing to increasing at $x = 5$.

$2 : \begin{cases} 1 : x = 2 \text{ and } x = 5 \text{ only} \\ 1 : \text{justification} \\ \quad \text{(ignore discussion at } x = 4) \end{cases}$

3

(a) The function f is increasing on $[-3,-2)$ since $f' > 0$ for $-3 \le x < -2$.

$2 : \begin{cases} 1 : \text{interval} \\ 1 : \text{reason} \end{cases}$

(b) $x = 0$ and $x = 2$

f' changes from decreasing to increasing at $x = 0$ and from increasing to decreasing at $x = 2$

$2 : \begin{cases} 1 : x = 0 \text{ and } x = 2 \text{ only} \\ 1 : \text{justification} \end{cases}$

(c) $f'(0) = -2$

Tangent line is $y = -2x + 3$.

1 : equation

(d) $f(0) - f(-3) = \int_{-3}^0 f'(t)\,dt$

$= \frac{1}{2}(1)(1) - \frac{1}{2}(2)(2) = -\frac{3}{2}$

$f(-3) = f(0) + \frac{3}{2} = \frac{9}{2}$

$f(4) - f(0) = \int_0^4 f'(t)\,dt$

$= -\left(8 - \frac{1}{2}(2)^2\pi\right) = -8 + 2\pi$

$f(4) = f(0) - 8 + 2\pi = -5 + 2\pi$

4 :

- 1 : $\pm\left(\frac{1}{2} - 2\right)$ (difference of areas of triangles)
- 1 : answer for $f(-3)$ using FTC
- 1 : $\pm\left(8 - \frac{1}{2}(2)^2\pi\right)$ (area of rectangle − area of semicircle)
- 1 : answer for $f(4)$ using FTC