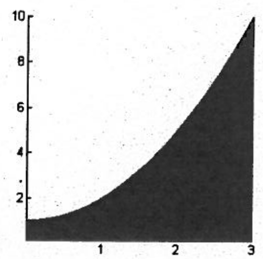
Rectangular Approximation Method

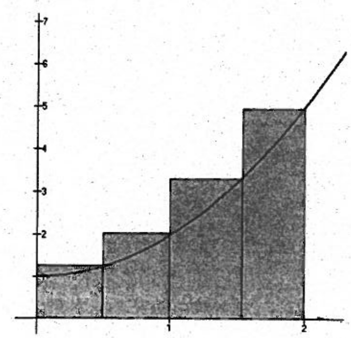
Suppose we wish to find the area under the graph of some function $f(x)$. There are many reasons in mathematics, science, engineering, statistics, etc. why we need to do this. Most have NOTHING to do with geometric area. As we will see, the area under a curve represents some other phycial concept. Thus, we often need to compute the area under a curve. We will first see how we can APPROXIMATE this area. Later, we will find the EXACT area under many curves.

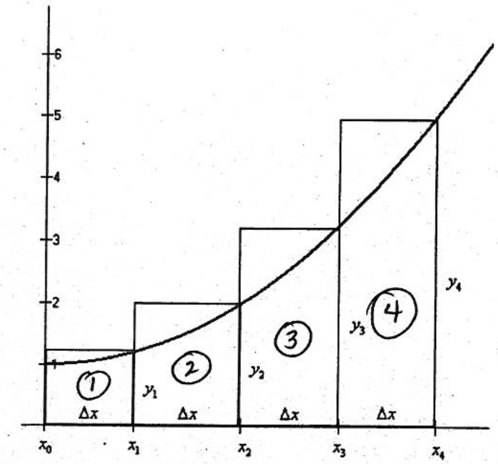


The first method we will use to approximate this area is known as the **RECTANGULAR APPROXIMATION METHOD (RAM)**
There are 3 Rectangular Approximation Methods: RIGHT RAM, LEFT RAM and MIDPOINT RAM.

Example: RIGHT RAM

In the example at the right we wish to find the area under the graph of $f(x)=x^2+1$ from $x=0$ to $x=2$. We will estimate this area by using 4rectangles where the RIGHT hand side of the rectangle touches the curve.





width of rectangles $= \Delta x$
height of rectangle $= y$ value

In the example above, we made each rectangle of equal width. This is not necessary but we often do it to simplify calculations. Here, $\Delta x=\frac{2-0}{4}=\frac{1}{2}$. Thus,

$x_0=0 \quad x_1=\frac{1}{2} \quad x_2=1 \quad x_3=\frac{3}{2} \quad x_4=2.$

To find the corresponding values of y we use the fact that $y_i=f(x_i)$. * Substitute into $f(x)=x^2+1$

$y_0=f(0)=1 \quad y_1=f\left(\frac{1}{2}\right)=\frac{5}{4} \quad y_2=f(1)=2 \quad y_3=f\left(\frac{3}{2}\right)=\frac{13}{4} \quad y_4=f(2)=5$

Now we can find the area of each rectangle: Since doing right rect. method, start with y_1.

Rectangle 1 $=\frac{1}{2}\left(\frac{5}{4}\right)=\frac{5}{8}$ Rectangle 2 $=\frac{1}{2}(2)=1=\frac{8}{8}$

Rectangle 3 $=\frac{1}{2}\left(\frac{13}{4}\right)=\frac{13}{8}$ Rectangle 4 $=\frac{1}{2}(5)=\frac{5}{2}=\frac{20}{8}$

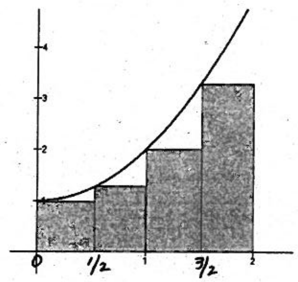
Total Area $=\frac{5}{8}+\frac{8}{8}+\frac{13}{8}+\frac{20}{8}=\frac{46}{8}=\boxed{\frac{23}{4}}$ → Add up all rectangles

Notice that in each case the rectangle contained the curve. Thus, the appoximation we got is GREATER than the actual area. But in this case we have an UPPER bound on the area.

LEFT RAM

Example: Estimate the area under $f(x) = x^2 + 1$ from $x = 0$ to $x = 2$ using 4 LEFT rectangles.

so heights are y-values at left endpts of each interval.



$x_0 = 0$	$y_0 = 1$
$x_1 = \frac{1}{2}$	$y_1 = \frac{5}{4}$
$x_2 = 1$	$y_2 = 2$
$x_3 = \frac{3}{2}$	$y_3 = \frac{13}{4}$
$x_4 = 2$	$y_4 = 5$

Total Area $= \frac{1}{2}(1) + \frac{1}{2}\left(\frac{5}{4}\right) + \frac{1}{2}(2) + \frac{1}{2}\left(\frac{13}{4}\right) = \boxed{\frac{19}{4}}$

Since each of these rectangles is BELOW the curve we now have a LOWER bound on the actual area. Using the result from Example 1 we get:

$\frac{19}{4} < Area < \frac{23}{4}$.

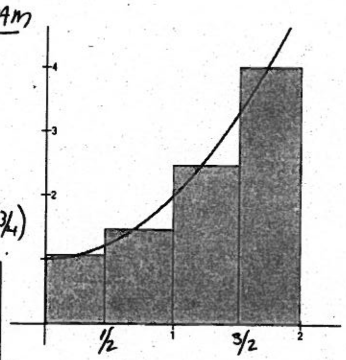
MIDPOINT RAM * NOT $\frac{LRAM+RRAM}{2}$

Example: Estimate the area under $f(x) = x^2 + 1$ from $x = 0$ to $x = 2$ using 4 MIDPOINT rectangles. Here we need to use the midpoints of each interval.

* If interval is $[0, \frac{1}{2}]$ use $f(\frac{1}{4})$ but $\Delta x = \frac{1}{2}$. If $[\frac{1}{2}, 1]$ use $f(\frac{3}{4})$ etc.

x	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{5}{4}$	$\frac{7}{4}$
y	$\frac{17}{16}$	$\frac{25}{16}$	$\frac{41}{16}$	$\frac{65}{16}$

A B



C D

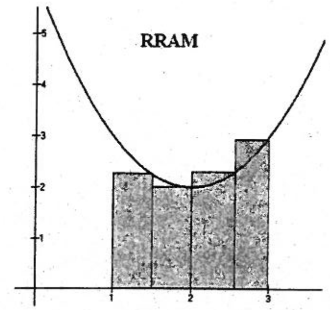
Total Area $\approx \frac{1}{2}\left(\frac{17}{16}\right) + \frac{1}{2}\left(\frac{25}{16}\right) + \frac{1}{2}\left(\frac{41}{16}\right) + \frac{1}{2}\left(\frac{65}{16}\right)$

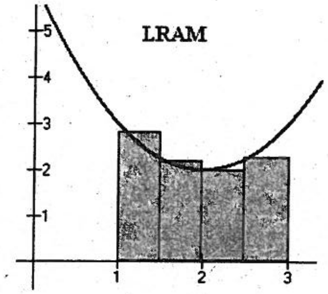
$\approx \frac{1}{2}A + \frac{1}{2}B + \frac{1}{2}C + \frac{1}{2}D = \boxed{\frac{148}{32} = \frac{37}{8} = 4.625}$

Here we are not sure how this compares with the exact answer but generally it is VERY close.

In fact, the exact answer is $\frac{14}{3} = 4.6666667$ Thus, we are of by only .04!!

Example: Use 4 rectangles to estimate the area under $f(x) = x^2 - 4x + 6$ from $x = 1$ to $x = 3$.





x	1	1.5	2	2.5	3
y	3	2.25	2	2.25	3
	A	B	C	D	E

RRAM

The **Total Area** $\approx \frac{1}{2}(2.25) + \frac{1}{2}(2) + \frac{1}{2}(2.25) + \frac{1}{2}(3) =$

$\approx \frac{1}{2}B + \frac{1}{2}C + \frac{1}{2}D + \frac{1}{2}E = \boxed{4.75}$

In this case, some rectangles are ABOVE the curve and some are BELOW. Thus, we have no way of knowing if our estimate is too high or too low. We only hope that it is fairly accurate!

LRAM

The **Total Area** $\approx \frac{1}{2}(3) + \frac{1}{2}(2.25) + \frac{1}{2}(2) + \frac{1}{2}(2.25)$

$\approx \frac{1}{2}A + \frac{1}{2}B + \frac{1}{2}C + \frac{1}{2}D = \boxed{4.75}$

It is a fluke that both answers are the same in this case. Thus, a very good answer is 4.75

MRAM

	[1,1.5]	[1.5,2]	[2,2.5]	[2.5,3]
x	5/4	1.75	2.25	2.75
y	2.5625	2.0625	2.0625	2.5625

Total Area $\approx \frac{1}{2}(2.5625) + \frac{1}{2}(2.0625) + \frac{1}{2}(2.0625) + \frac{1}{2}(2.5625)$

$= 4.625$

"Real-world" Example using data:

You and a companion are driving along a twisty stretch of dirt road in a car whose speedometer works but whose odometer is broken. To find out how long this stretch of road is, you record the car's speed at 10 second intervals (after converted to ft/sec for convenience). Estimate the length of the road using a) left rectangular approximations and b) right rectangular approximations.

Time (sec)	Velocity (ft/sec)	Time (sec)	Velocity (ft/sec)
0	0	70	15
10	44	80	22
20	15	90	35
30	35	100	44
40	30	110	30
50	44	120	35
60	35		

"How far" = (velocity)(time)
(ft/s)(sec) = feet

$\Delta t = 10$ sec 12 subintervals $[0,120]$

a) LRAM (starts @ left-endpt $v(0) \to v(110)$)
$= 10(0)+10(44)+10(15)+10(35)+10(30)+10(44)+10(35)+10(15)+10(22)+10(35)+10(44)+10(30) = 3490$ ft.

b) RRAM (starts @ rt-endpt of each interval $\Rightarrow v(10) \to v(120)$)
$= 10(44)+10(15)+10(35)+10(30)+10(44)+10(35)+10(15)+10(22)+10(35)+10(44)+10(30)+10(35) = 3840$ ft.

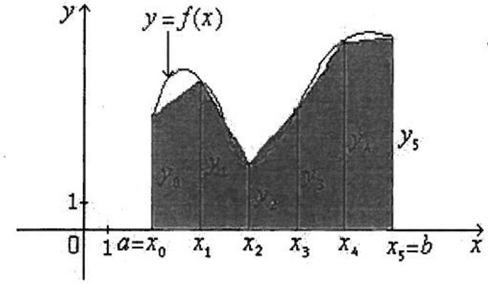
* If 12 subintervals should have 12 sums

* Can't do MRAM with 12 subintervals b/c don't have data needed. Would need $v(5)$, $v(15)$, etc.

Could do MRAM with 6 subintervals though...
then $\Delta t = 20$ sec
$MRAM = 20(v(10))+20(v(30))+20(v(50))+20(v(70))+20(v(90))+20(v(110))$
$= 20[44+35+44+15+35+30] = 4060$ ft.

* To get best approx, want as many "rectangles" as possible. Actual area would have infinite rectangles $(n\to\infty)$ with $\Delta x \to 0$.

Trapezoidal Rule – Approx areas using trapezoids.



Remember,
$A_{trap} = \frac{1}{2}h(b_1+b_2)$
Here, $h = \Delta x$ and
$b_1 = y_1$, $b_2 = y_2$
etc.

Approx area under $f(x)$ on $[a,b]$ using 5 trap.

$\approx \frac{1}{2}\Delta x\left(y(a)+y(x_1)\right) + \frac{1}{2}\Delta x\left(y(x_1)+y(x_2)\right) + \frac{1}{2}\Delta x\left(y(x_2)+y(x_3)\right) + \frac{1}{2}\Delta x\left(y(x_3)+y(x_4)\right) + \frac{1}{2}\Delta x\left(y(x_4)+y(b)\right)$

* If Δx is equal, then could factor out...

$\approx \frac{1}{2}\Delta x\left[y(a)+2y(x_1)+2y(x_2)+2y(x_3)+2y(x_4)+y(b)\right]$

Ex. Approx area under $f(x) = x^2+1$ from $[0,2]$ using 4 equal subintervals with Trapezoidal Rule.

$\Delta x = 1/2$

x	0	1/2	1	3/2	2
y	1	5/4	2	13/4	5

$\approx \frac{1}{2}\left(\frac{1}{2}\right)\left(1+\frac{5}{4}\right) + \frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{5}{4}+2\right) + \frac{1}{2}\left(\frac{1}{2}\right)\left(2+\frac{13}{4}\right) + \frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{13}{4}+5\right)$

$= \frac{1}{2}\left(\frac{1}{2}\right)\left[1+2\left(\frac{5}{4}\right)+2(2)+2\left(\frac{13}{4}\right)+5\right] = 4.75$

Ex.

t (sec)	0	1	4	6	10
		$\Delta t=1$	$\Delta t=3$	$\Delta t=2$	$\Delta t=4$
v(t) (ft/s)	3	12	18	21	15

> Unequal subintervals. Approx dist. traveled
* Must do individual subint.

$*(\Delta t)(v(t))$

a) LRAM : $1(3)+3(12)+2(18)+4(21) \approx 159$ ft

b) RRAM : $1(12)+3(18)+2(21)+4(15) = 168$ ft

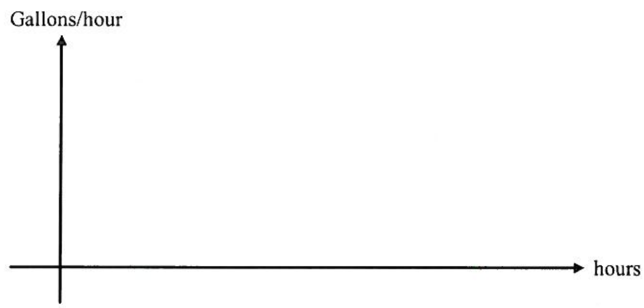
c) Trap Rule : $\frac{1}{2}(1)(3+12)+\frac{1}{2}(3)(12+18)+\frac{1}{2}(2)(18+21)+\frac{1}{2}(4)(21+15) = 163.5$

Rectangular Approx Methods wkst

All work must be shown in this course for full credit. Unsupported answers may receive NO credit.

1. Suppose an oil pump is producing 800 gallons per hour for the first 5 hours of operation. For the next 4 hours, the pumps production is increased to 900 gallons per hour, and then for the next 3 hours, the production is cut to 600 gallons per hour.

a) Make a graph modeling this situation.



b) The term "area under a graph" is the area between the graph and the horizontal axis. Find the area under the graph from 0 to 5 hours. ***What does this value represent***?

c) Find the total area under the graph for the entire 12 hours. What does this value represent?

2. The function f is continuous on the closed interval [2, 8] and has values that are given in the table below.

x	2	5	7	8
$f(x)$	10	30	40	20

Using the subintervals [2, 5], [5, 7], and [7, 8], what are the following approximations of the area under the curve?

a) LRAM

b) RRAM

c) Trapezoid Approximation

d) Write an algebraic expression (you don't have enough information to simplify it) that would give an approximation of the area under the curve using MRAM. Answer will be in terms of f(~).

3. Let R be the region enclosed between the graphs of $y = 2x - x^2$ and the x-axis for $0 \le x \le 2$.

a) Sketch the region R.

b) Partition [0, 2] into 4 subintervals and find the following:

i) LRAM

ii) RRAM

iii) MRAM

iv) Trapezoidal Approximation

3. Oil is leaking out of a tanker damaged at sea. The damage to the tanker is worsening as evidenced by the increased leakage each hour, recorded in the table.

a) Find an estimate using a Midpoint Sum for the total quantity of oil that has escaped in the first 8 hours using 4 intervals of equal width.

Time (h)	Leakage (gal/h)
0	50
1	70
2	97
3	136
4	190
5	265
6	369
7	516
8	720

b) Without calculating them, will LRAM or RRAM yield a "higher" estimate in this case? Why?

4. Sylvie's Old World Cheeses has found that the cost, in dollars per kilogram, of the cheese it produces is

$$c(x) = -0.012x + 6.50,$$

where x is the number of kilograms of cheese produced and $0 \le x \le 300$.

a) Draw a sketch of the cost function. Label each axes with the correct units.

b) If you wanted to find the total cost of producing 200 kg of cheese how is this represented on your sketch?

c) Find the total cost of producing 200 kg of cheese.

5. A particle is moving along the x-axis with velocity given by $v(t) = 2t + 1$, where velocity is measured in feet/sec.

a) Draw a sketch of the velocity function for $0 \le t \le 5$.

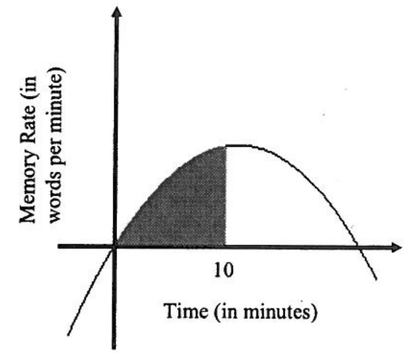
b) What does the area under the graph of velocity represent?

c) If the object originally began at $x = 3$, where is the object located at $t = 2$? … What about $t = 5$?

6. Suppose that in a memory experiment, the rate of memorizing is given by $M\ t\ = -0.009t^2 + 0.2t$, where $M\ t$ is the memory rate, in words per minute. The graph is shown below.

a) Explain what the shaded area represents in the context of this problem.

b) Use MRAM with 5 rectangles of equal width to approximate the area of the shaded region.



7. Complete each sentence with ALWAYS, SOMETIMES, or NEVER.

a) If a function is concave up, then LRAM will ___________ overestimate the actual area under the curve.

b) If a function is decreasing, then RRAM will ___________ overestimate the actual area under the curve.

8. Which rule is simply the average of LRAM and RRAM?

9. If f is a positive, continuous function on an interval $[a, b]$, which of the following rectangular approximation methods has a limit equal to the actual area under the curve from a to b as the number of rectangles approaches infinity?

I. LRAM
II. RRAM
III. MRAM

A I and II only
B III only
C I and III only
D I, II, and III
E None of these

10. A truck moves with positive velocity $v(t)$ from time $t = 3$ to time $t = 15$. The area under the graph of $v(t)$ between $t = 3$ and $t = 15$ gives

A the velocity of the truck at $t = 15$
B the acceleration of the truck at $t = 15$
C the position of the truck at $t = 15$
D the distance traveled by the truck from $t = 3$ to $t = 15$
E The average position of ht e truck in the interval $t = 3$ and $t = 15$.