AP Taylor Series practice

AP Calculus BC–3 **2000**

The Taylor series about $x = 5$ for a certain function f converges to $f(x)$ for all x in the interval of convergence. The nth derivative of f at $x = 5$ is given by $f^{(n)}(5) = \dfrac{(-1)^n n!}{2^n (n+2)}$, and $f(5) = \dfrac{1}{2}$.

(a) Write the third-degree Taylor polynomial for f about $x = 5$.

(b) Find the radius of convergence of the Taylor series for f about $x = 5$.

(c) Show that the sixth-degree Taylor polynomial for f about $x = 5$ approximates $f(6)$ with error less than $\dfrac{1}{1000}$.

Question 6

The Maclaurin series for $\ln\left(\dfrac{1}{1-x}\right)$ is $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$ with interval of convergence $-1 \le x < 1$.

(a) Find the Maclaurin series for $\ln\left(\dfrac{1}{1+3x}\right)$ and determine the interval of convergence.

(b) Find the value of $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.

(c) Give a value of p such that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converges, but $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ diverges. Give reasons why your value of p is correct.

(d) Give a value of p such that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges, but $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ converges. Give reasons why your value of p is correct.

Question 6

A function f is defined by

$$f(x) = \frac{1}{3} + \frac{2}{3^2}x + \frac{3}{3^3}x^2 + \cdots + \frac{n+1}{3^{n+1}}x^n + \cdots$$

for all x in the interval of convergence of the given power series.

(a) Find the interval of convergence for this power series. Show the work that leads to your answer.

(b) Find $\displaystyle\lim_{x \to 0} \frac{f(x) - \frac{1}{3}}{x}$.

(c) Write the first three nonzero terms and the general term for an infinite series that represents $\displaystyle\int_0^1 f(x)\,dx$.

(d) Find the sum of the series determined in part (c).

4. The function f has derivatives of all orders for all real numbers x. Assume $f(2) = -3$, $f'(2) = 5$, $f''(2) = 3$, and $f'''(2) = -8$.
   (a) Write the third–degree Taylor polynomial for f about $x = 2$ and use it to approximate $f(1.5)$.
   (b) The fourth derivative of f satisfies the inequality $|f^{(4)}(x)| \leq 3$ for all x in the closed interval $[1.5, 2]$. Use the Lagrange error bound on the approximation to $f(1.5)$ found in part (a) to explain why $f(1.5) \neq -5$.
   (c) Write the fourth–degree Taylor polynomial, $P(x)$, for $g(x) = f(x^2 + 2)$ about $x = 0$. Use P to explain why g must have a relative minimum at $x = 0$.

3. Let f be a function that has derivatives of all orders for all real numbers. Assume $f(0) = 5$, $f'(0) = -3$, $f''(0) = 1$, and $f'''(0) = 4$.
   (a) Write the third-degree Taylor polynomial for f about $x = 0$ and use it to approximate $f(0.2)$.
   (b) Write the fourth-degree Taylor polynomial for g, where $g(x) = f(x^2)$, about $x = 0$.
   (c) Write the third-degree Taylor polynomial for h, where $h(x) = \int_0^x f(t)\,dt$, about $x = 0$.
   (d) Let h be defined as in part (c). Given that $f(1) = 3$, either find the exact value of $h(1)$ or explain why it cannot be determined.

$$f(0) = 0$$
$$f'(0) = 1$$
$$f^{(n+1)}(0) = -n \cdot f^{(n)}(0) \text{ for all } n \geq 1$$

6. A function f has derivatives of all orders for $-1 < x < 1$. The derivatives of f satisfy the conditions above. The Maclaurin series for f converges to $f(x)$ for $|x| < 1$.

   (a) Show that the first four nonzero terms of the Maclaurin series for f are $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$, and write the general term of the Maclaurin series for f.

   (b) Determine whether the Maclaurin series described in part (a) converges absolutely, converges conditionally, or diverges at $x = 1$. Explain your reasoning.

   (c) Write the first four nonzero terms and the general term of the Maclaurin series for $g(x) = \int_0^x f(t)\,dt$.

   (d) Let $P_n\left(\frac{1}{2}\right)$ represent the nth-degree Taylor polynomial for g about $x = 0$ evaluated at $x = \frac{1}{2}$, where g is the function defined in part (c). Use the alternating series error bound to show that

   $$\left| P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right) \right| < \frac{1}{500}.$$