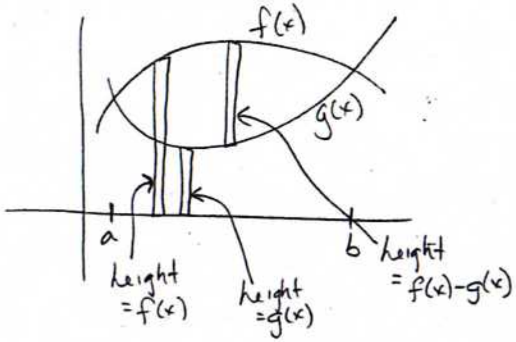
Area Between Curves



So, given $f(x)$ and $g(x) \geq 0$

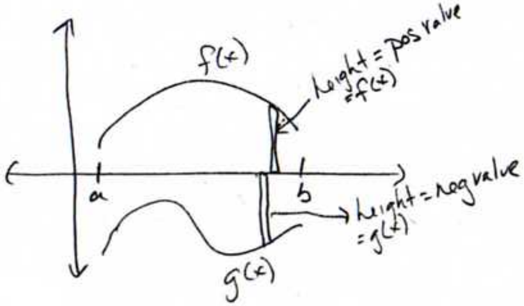
$\int_a^b f(x)\,dx$ gives area between $f(x)$ + x-axis

and $\int_a^b g(x)\,dx$ gives area between $g(x)$ + x-axis, <u>then</u>

If $f(x) \geq g(x)$

$\int_a^b [f(x)-g(x)]\,dx$ gives area between curves

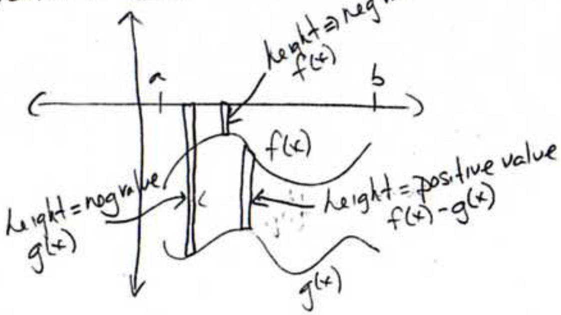
What if not above x-axis? Will $\int_a^b [f(x)-g(x)]\,dx$ give <u>Area</u> or <u>Signed Area</u>??



$\int_a^b [f(x)-g(x)]\,dx$ will take the pos. heights for $f(x)$ <u>minus</u> the <u>neg.</u> heights for $g(x)$, so get <u>positive</u> values

$\Rightarrow$ <u>AREA</u>!

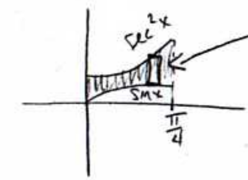
What if <u>both</u> below x-axis?



<u>AREA</u> $\Rightarrow$ regardless of placement with the x-axis, if $f(x) \geq g(x)$ {Above} then Area between them $= \int_a^b [f(x)-g(x)]\,dx$

Ex. Find area between $y = \sec^2 x$ and $y = \sin x$ on $[0, \frac{\pi}{4}]$

① Graph!! Determine which function is <u>above</u> the other on interval. Draw a "sample rectangle"



top = $\sec^2 x$

bottom = $\sin x$

$\int_a^b (\text{top} - \text{bottom})\,dx$

② Set up integral:

$$\int_0^{\pi/4} (\sec^2 x - \sin x)\,dx$$

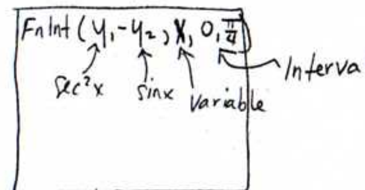
③ Evaluate either by hand <u>OR</u> with calculator

$$= \tan x + \cos x \Big|_0^{\pi/4} = \tan\tfrac{\pi}{4} + \cos\tfrac{\pi}{4} - (\tan 0 + \cos 0)$$

$$= 1 + \tfrac{\sqrt{2}}{2} - (0+1)$$

$$= \boxed{\tfrac{\sqrt{2}}{2} \text{ units}^2}$$

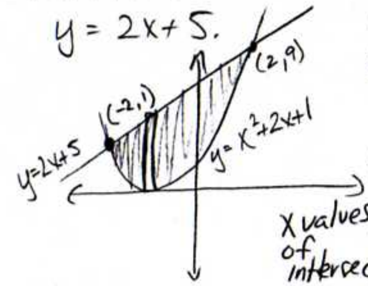
In calculator, do MATH 9 FnInt



* See Ex 1 p. 413

Sometimes the interval isn't given but they ask you to find the <u>bounded</u> region. This means you need to graph the given functions and find where they <u>intersect</u> first.

Ex. Find the area of the region bounded by $y = x^2 + 2x + 1$ and $y = 2x + 5$.



① Graph + find intersections (use calculator)

② Determine "top" + "bottom" functions

top $\Rightarrow 2x+5$

bottom $\Rightarrow x^2 + 2x + 1$

③ Set up integral + solve

X values of intersection

$$\int_{-2}^{2} [\underbrace{(2x+5)}_{\text{top}} - \underbrace{(x^2+2x+1)}_{\text{bottom}}]\,dx = \boxed{\tfrac{32}{3} \text{ units}^2}$$

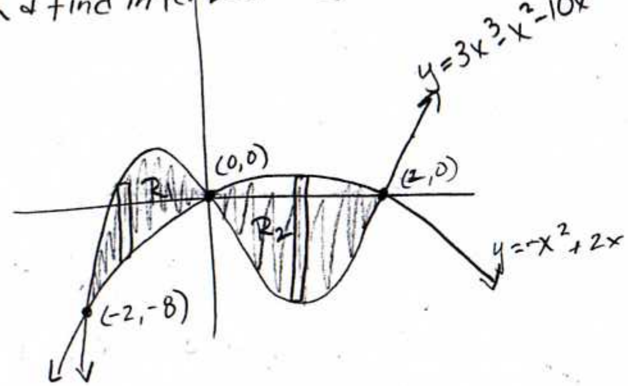
(3)

Try: Find the area of the region bounded by $y=2-x^2$ and $y=-x$. (Ans. $\frac{9}{2}$ units)

Sometimes the "top" + "bottom" change when either the curves intersect at more than 2 points or if have more than 2 curves. When finding bounded area need ALL area that is bounded by the curves.

Ex. Find area between $y=3x^3-x^2-10x$ and $y=-x^2+2x$.

① Graph & find intersections



② Determine "top" + "bottom" ⇒ notice changes

R_1: $[-2,0]$ top $=3x^3-x^2-10x$, bott $=-x^2+2x$

R_2: $[0,2]$ top $=-x^2+2x$, bott $=3x^3-x^2-10x$

③ Do integrals of each region separately + add tog.

$$R_1=\int_{-2}^{0}[(3x^3-x^2-10x)-(-x^2+2x)]dx=12$$

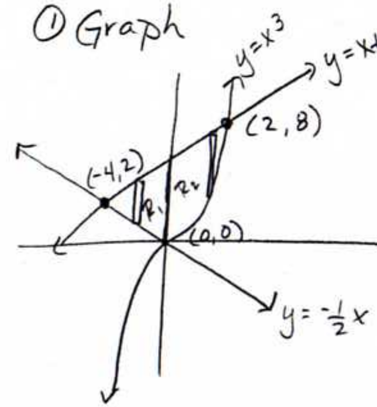
not always the same!

$$+\; R_2=\int_{0}^{2}[(-x^2+2x)-(3x^3-x^2-10x)]dx=12$$

$\boxed{24 \text{ units}^2}$

Ex. Find area of region bounded by $y-x=6$, $y-x^3=0$, and $2y+x=0$.

① Graph



② Determine "top" + "bottom"

R_1: $[-4,0]$ top $=x+6$, bott $=-\frac{1}{2}x$

R_2: $[0,2]$ top $=x+6$, bott $=x^3$

③ Set up integrals + solve

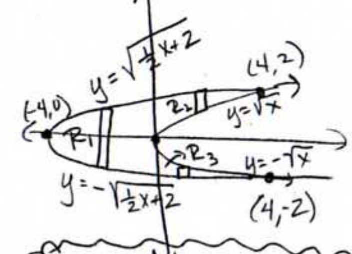
$$R_1=\int_{-4}^{0}[(x+6)-(-\tfrac{1}{2}x)]dx=12$$

$$+\; R_2=\int_{0}^{2}[(x+6)-(x^3)]dx=10$$

$\boxed{22 \text{ units}}$

Ex. Find area of region bounded by $2y^2=x+4$ and $y^2=x$.

Graph:



* Notice when solved for y, there are 4 curves!

3 regions or use symmetry + 2 regions

R_1 $[-4,0]$ top $=\sqrt{\frac{1}{2}x+2}$, bott $=-\sqrt{\frac{1}{2}x+2}$

R_2 $[0,4]$ top : $\sqrt{\frac{1}{2}x+2}$, bott : $\sqrt{x}$

R_3 $[0,4]$ top : $-\sqrt{x}$, bott : $-\sqrt{\frac{1}{2}x+2}$

Area = so could do 2 times one of them

$$\int_{-4}^{0}\left(\sqrt{\tfrac{1}{2}x+2}\right)-\left(-\sqrt{\tfrac{1}{2}x+2}\right)dx$$

$$+\int_{0}^{4}\left(\sqrt{\tfrac{1}{2}x+2}\right)-(\sqrt{x})dx+\int_{0}^{4}(-\sqrt{x})-\left(-\sqrt{\tfrac{1}{2}x+2}\right)dx$$

$=\boxed{\frac{32}{3}\text{ units}^2}$

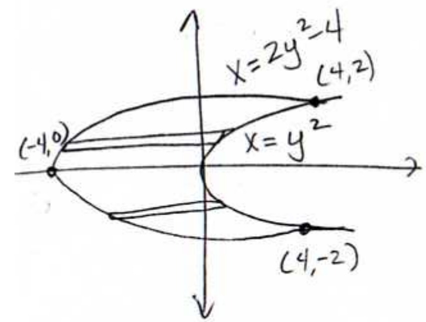
Lets do the same example, but in terms of $\underline{y}$!!

If when have x's & dx's, you do $\int_{x_1}^{x_2}(\text{top} - \text{bottom})\,dx$

then with y's & dy's, you do $\int_{y_1}^{y_2}(\text{right} - \text{left})\,dy$

* Remember definite integral in terms of y gave area between curve & the y-axis.

Ex. $2y^2 = x+4$ and $y^2 = x$



① Graph & find intersections
② Label in terms of y!
③ Draw horizontal rectangles
* Notice here, the right is always $x=y^2$ & the left is always $x=2y^2-4$!!

Only one region!!

④ Set up integral: $\int_{y_1}^{y_2}(\text{right} - \text{left})\,dy$

y values of intersections → $\int_{-2}^{2}\left[(y^2) - (2y^2-4)\right]dy = \boxed{\frac{32}{3}\text{ units}^2}$

Same answer!! as last ex.

Should be - still finding area of region!

* Look @ Ex 5 p. 416

Hwk: p 418 # 3-48 m3, 59, 61, 63-66