CHAPTER 1
Limits and Their Properties

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C H A P T E R 1
Limits and Their Properties

Section 1.1 A Preview of Calculus

Solutions to Even-Numbered Exercises

2. Calculus: velocity is not constant

Distance $\approx$ (20 ft/sec)(15 seconds) = 300 feet

4. Precalculus: rate of change = slope = 0.08

6. Precalculus: Area $= \pi\left(\sqrt{2}\right)^2$

$= 2\pi$

8. Precalculus: Volume $= \pi(3)^2 6 = 54\pi$

10. (a) Area $\approx 5 + \frac{5}{2} + \frac{5}{3} + \frac{5}{4} \approx 10.417$

Area $\approx \frac{1}{2}\left(5 + \frac{5}{1.5} + \frac{5}{2} + \frac{5}{2.5} + \frac{5}{3} + \frac{5}{3.5} + \frac{5}{4} + \frac{5}{4.5}\right) \approx 9.145$

(b) You could improve the approximation by using more rectangles.

Section 1.2 Finding Limits Graphically and Numerically

2.

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$	0.2564	0.2506	0.2501	0.2499	0.2494	0.2439

$\lim_{x\to 2} \frac{x-2}{x^2-4} \approx 0.25$ (Actual limit is $\frac{1}{4}$.)

4.

x	−3.1	−3.01	−3.001	−2.999	−2.99	−2.9
$f(x)$	−0.2485	−0.2498	−0.2500	−0.2500	−0.2502	−0.2516

$\lim_{x\to -3} \frac{\sqrt{1-x}-2}{x+3} \approx -0.25$ (Actual limit is $-\frac{1}{4}$.)

6.

x	3.9	3.99	3.999	4.001	4.01	4.1
$f(x)$	0.0408	0.0401	0.0400	0.0400	0.0399	0.0392

$\lim_{x\to 4} \frac{[x/(x+1)] - (4/5)}{x-4} \approx 0.04$ (Actual limit is $\frac{1}{25}$.)

8.

x	−0.1	−0.01	−0.001	0.001	0.01	0.1
$f(x)$	0.0500	0.0050	0.0005	−0.0005	−0.0050	−0.0500

$\lim_{x\to 0} \frac{\cos x - 1}{x} \approx 0.0000$ (Actual limit is 0.) (Make sure you use radian mode.)

10. $\lim_{x \to 1} (x^2 + 2) = 3$

12. $\lim_{x \to 1} f(x) = \lim_{x \to 1} (x^2 + 2) = 3$

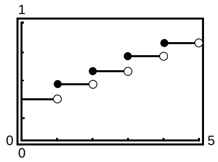
14. $\lim_{x \to 3} \frac{1}{x - 3}$ does not exist since the function increases and decreases without bound as x approaches 3.

16. $\lim_{x \to 0} \sec x = 1$

18. $\lim_{x \to 1} \sin(\pi x) = 0$

20. $C(t) = 0.35 - 0.12[\![-(t - 1)]\!]$

(a)



(b)

t	3	3.3	3.4	3.5	3.6	3.7	4
$C(t)$	0.59	0.71	0.71	0.71	0.71	0.71	0.71

$\lim_{t \to 3.5} C(t) = 0.71$

(c)

t	3	2.5	2.9	3	3.1	3.5	4
$C(t)$	0.47	0.59	0.59	0.59	0.71	0.71	0.71

$\lim_{t \to 3.5} C(t)$ does not exist. The values of C jump from 0.59 to 0.71 at $t = 3$.

22. You need to find δ such that $0 < |x - 2| < \delta$ implies $|f(x) - 3| = |x^2 - 1 - 3| = |x^2 - 4| < 0.2$. That is,

$$-0.2 < x^2 - 4 < 0.2$$
$$4 - 0.2 < x^2 < 4 + 0.2$$
$$3.8 < x^2 < 4.2$$
$$\sqrt{3.8} < x < \sqrt{4.2}$$
$$\sqrt{3.8} - 2 < x - 2 < \sqrt{4.2} - 2$$

So take $\delta = \sqrt{4.2} - 2 \approx 0.0494$.

Then $0 < |x - 2| < \delta$ implies

$$-\left(\sqrt{4.2} - 2\right) < x - 2 < \sqrt{4.2} - 2$$
$$\sqrt{3.8} - 2 < x - 2 < \sqrt{4.2} - 2.$$

Using the first series of equivalent inequalities, you obtain

$$|f(x) - 3| = |x^2 - 4| < \epsilon = 0.2.$$

24. $\lim_{x \to 4} \left(4 - \frac{x}{2}\right) = 2$

$$\left|\left(4 - \frac{x}{2}\right) - 2\right| < 0.01$$
$$\left|2 - \frac{x}{2}\right| < 0.01$$
$$\left|-\frac{1}{2}(x - 4)\right| < 0.01$$
$$0 < |x - 4| < 0.02 = \delta$$

Hence, if $0 < |x - 4| < \delta = 0.02$, you have

$$\left|-\frac{1}{2}(x - 4)\right| < 0.01$$
$$\left|2 - \frac{x}{2}\right| < 0.01$$
$$\left|\left(4 - \frac{x}{2}\right) - 2\right| < 0.01$$
$$|f(x) - L| < 0.01$$

26. $\lim_{x\to 5}(x^2+4) = 29$

$$|(x^2+4)-29| < 0.01$$
$$|x^2-25| < 0.01$$
$$|(x+5)(x-5)| < 0.01$$
$$|x-5| < \frac{0.01}{|x+5|}$$

If we assume $4 < x < 6$, then $\delta = 0.01/11 \approx 0.0009$.

Hence, if $0 < |x-5| < \delta = \frac{0.01}{11}$, you have

$$|x-5| < \frac{0.01}{11} < \frac{1}{|x+5|}(0.01)$$
$$|x-5||x+5| < 0.01$$
$$|x^2-25| < 0.01$$
$$|(x^2+4)-29| < 0.01$$
$$|f(x)-L| < 0.01$$

28. $\lim_{x\to -3}(2x+5) = -1$

Given $\epsilon > 0$:

$$|(2x+5)-(-1)| < \epsilon$$
$$|2x+6| < \epsilon$$
$$2|x+3| < \epsilon$$
$$|x+3| < \frac{\epsilon}{2} = \delta$$

Hence, let $\delta = \epsilon/2$.

Hence, if $0 < |x+3| < \delta = \frac{\epsilon}{2}$, you have

$$|x+3| < \frac{\epsilon}{2}$$
$$|2x+6| < \epsilon$$
$$|(2x+5)-(-1)| < \epsilon$$
$$|f(x)-L| < \epsilon$$

30. $\lim_{x\to 1}\left(\frac{2}{3}x+9\right) = \frac{2}{3}(1)+9 = \frac{29}{3}$

Given $\epsilon > 0$:

$$\left|\left(\tfrac{2}{3}x+9\right)-\tfrac{29}{3}\right| < \epsilon$$
$$\left|\tfrac{2}{3}x-\tfrac{2}{3}\right| < \epsilon$$
$$\tfrac{2}{3}|x-1| < \epsilon$$
$$|x-1| < \tfrac{3}{2}\epsilon$$

Hence, let $\delta = (3/2)\epsilon$.

Hence, if $0 < |x-1| < \delta = \frac{3}{2}\epsilon$, you have

$$|x-1| < \tfrac{3}{2}\epsilon$$
$$\left|\tfrac{2}{3}x-\tfrac{2}{3}\right| < \epsilon$$
$$\left|\left(\tfrac{2}{3}x+9\right)-\tfrac{29}{3}\right| < \epsilon$$
$$|f(x)-L| < \epsilon$$

32. $\lim_{x\to 2}(-1) = -1$

Given $\epsilon > 0$: $|-1-(-1)| < \epsilon$

$$0 < \epsilon$$

Hence, any $\delta > 0$ will work.

Hence, for any $\delta > 0$, you have

$$|(-1)-(-1)| < \epsilon$$
$$|f(x)-L| < \epsilon$$

34. $\lim_{x\to 4}\sqrt{x} = \sqrt{4} = 2$

Given $\epsilon > 0$: $\left|\sqrt{x}-2\right| < \epsilon$

$$\left|\sqrt{x}-2\right|\left|\sqrt{x}+2\right| < \epsilon\left|\sqrt{x}+2\right|$$
$$|x-4| < \epsilon\left|\sqrt{x}+2\right|$$

Assuming $1 < x < 9$, you can choose $\delta = 3\epsilon$. Then,

$$0 < |x-4| < \delta = 3\epsilon \Rightarrow |x-4| < \epsilon\left|\sqrt{x}+2\right|$$
$$\Rightarrow \left|\sqrt{x}-2\right| < \epsilon.$$

36. $\lim_{x\to 3}|x-3| = 0$

Given $\epsilon > 0$:

$$|(x-3)-0| < \epsilon$$
$$|x-3| < \epsilon = \delta$$

Hence, let $\delta = \epsilon$.

Hence for $0 < |x-3| < \delta = \epsilon$, you have

$$|x-3| < \epsilon$$
$$||x-3|-0| < \epsilon$$
$$|f(x)-L| < \epsilon$$

38. $\lim_{x \to -3} (x^2 + 3x) = 0$

Given $\epsilon > 0$:

$$|(x^2 + 3x) - 0| < \epsilon$$

$$|x(x + 3)| < \epsilon$$

$$|x + 3| < \frac{\epsilon}{|x|}$$

If we assume $-4 < x < -2$, then $\delta = \epsilon/4$.

Hence for $0 < |x - (-3)| < \delta = \frac{\epsilon}{4}$, you have

$$|x + 3| < \frac{1}{4}\epsilon < \frac{1}{|x|}\epsilon$$

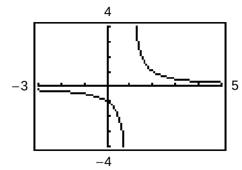
$$|x(x + 3)| < \epsilon$$

$$|x^2 + 3x - 0| < \epsilon$$

$$|f(x) - L| < \epsilon$$

40. $f(x) = \dfrac{x - 3}{x^2 - 4x + 3}$

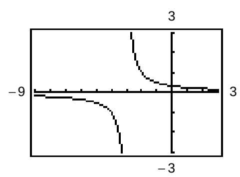
$\lim_{x \to 3} f(x) = \dfrac{1}{2}$



The domain is all $x \neq 1, 3$. The graphing utility does not show the hole at $\left(3, \frac{1}{2}\right)$.

42. $f(x) = \dfrac{x - 3}{x^2 - 9}$

$\lim_{x \to 3} f(x) = \dfrac{1}{6}$



The domain is all $x \neq \pm 3$. The graphing utility does not show the hole at $\left(3, \frac{1}{6}\right)$.

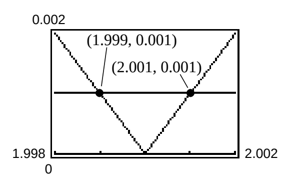
44. (a) No. The fact that $f(2) = 4$ has no bearing on the existence of the limit of $f(x)$ as x approaches 2.

(b) No. The fact that $\lim_{x \to 2} f(x) = 4$ has no bearing on the value of f at 2.

46. Let $p(x)$ be the atmospheric pressure in a plane at altitude x (in feet).

$$\lim_{x \to 0^+} p(x) = 14.7 \text{ lb/in}^2$$

48.



Using the zoom and trace feature, $\delta = 0.001$. That is, for

$$0 < |x - 2| < 0.001, \left|\frac{x^2 - 4}{x - 2} - 4\right| < 0.001.$$

50. True

52. False; let

$$f(x) = \begin{cases} x^2 - 4x, & x \neq 4 \\ 10, & x = 4 \end{cases}.$$

$\lim_{x \to 4} f(x) = \lim_{x \to 4} (x^2 - 4x) = 0$ and $f(4) = 10 \neq 0$

54. $\lim_{x \to 4} \dfrac{x^2 - x - 12}{x - 4} = 7$

n	$4 + [0.1]^n$	$f(4 + [0.1]^n)$
1	4.1	7.1
2	4.01	7.01
3	4.001	7.001
4	4.0001	7.0001

n	$4 - [0.1]^n$	$f(4 - [0.1]^n)$
1	3.9	6.9
2	3.99	6.99
3	3.999	6.999
4	3.9999	6.9999

56. $f(x) = mx + b, m \neq 0$. Let $\epsilon > 0$ be given. Take $\delta = \dfrac{\epsilon}{|m|}$.

If $0 < |x - c| < \delta = \dfrac{\epsilon}{|m|}$, then

$$|m||x - c| < \epsilon$$
$$|mx - mc| < \epsilon$$
$$|(mx + b) - (mc + b)| < \epsilon$$

which shows that $\lim\limits_{x \to c} (mx + b) = mc + b$.

58. $\lim\limits_{xc} g(x) = L, L > 0$. Let $\epsilon = \frac{1}{2}L$. There exists $\delta > 0$ such that $0 < |x - 0| < \delta$ implies $|g(x) - L| < \epsilon = \frac{1}{2}L$.

That is,

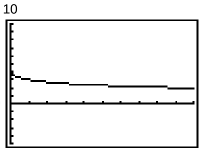
$$-\tfrac{1}{2}L < g(x) - L < \tfrac{1}{2}L$$
$$\tfrac{1}{2}L < \quad g(x) \quad < \tfrac{3}{2}L$$

Hence for x in the interval $(c - \delta, c + \delta), x \neq c$,

$$g(x) > \tfrac{1}{2}L > 0.$$

Section 1.3 Evaluating Limits Analytically

2.

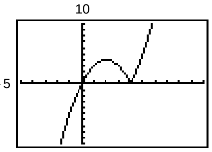


$g(x) = \dfrac{12(\sqrt{x} - 3)}{x - 9}$

(a) $\lim\limits_{x \to 4} g(x) = 2.4$

(b) $\lim\limits_{x \to 0} g(x) = 4$

4.



$f(t) = t|t - 4|$

(a) $\lim\limits_{t \to 4} f(t) = 0$

(b) $\lim\limits_{t \to -1} f(t) = -5$

6. $\lim\limits_{x \to -2} x^3 = (-2)^3 = -8$

8. $\lim\limits_{x \to -3} (3x + 2) = 3(-3) + 2 = -7$

10. $\lim\limits_{x \to 1} (-x^2 + 1) = -(1)^2 + 1 = 0$

12. $\lim\limits_{x \to 1} (3x^3 - 2x^2 + 4) = 3(1)^3 - 2(1)^2 + 4 = 5$

14. $\lim\limits_{x \to -3} \dfrac{2}{x + 2} = \dfrac{2}{-3 + 2} = -2$

16. $\lim\limits_{x \to 3} \dfrac{2x - 3}{x + 5} = \dfrac{2(3) - 3}{3 + 5} = \dfrac{3}{8}$

18. $\lim\limits_{x \to 3} \dfrac{\sqrt{x + 1}}{x - 4} = \dfrac{\sqrt{3 + 1}}{3 - 4} = -2$

20. $\lim\limits_{x \to 4} \sqrt[3]{x + 4} = \sqrt[3]{4 + 4} = 2$

22. $\lim\limits_{x \to 0} (2x - 1)^3 = [2(0) - 1]^3 = -1$

24. (a) $\lim\limits_{x \to -3} f(x) = (-3) + 7 = 4$

(b) $\lim\limits_{x \to 4} g(x) = 4^2 = 16$

(c) $\lim\limits_{x \to -3} g(f(x)) = g(4) = 16$

26. (a) $\lim\limits_{x \to 4} f(x) = 2(4^2) - 3(4) + 1 = 21$

(b) $\lim\limits_{x \to 21} g(x) = \sqrt[3]{21 + 6} = 3$

(c) $\lim\limits_{x \to 4} g(f(x)) = g(21) = 3$

28. $\lim\limits_{x \to \pi} \tan x = \tan \pi = 0$

30. $\lim\limits_{x \to 1} \sin \dfrac{\pi x}{2} = \sin \dfrac{\pi}{2} = 1$

32. $\lim\limits_{x \to \pi} \cos 3x = \cos 3\pi = -1$

34. $\lim\limits_{x \to 5\pi/3} \cos x = \cos \dfrac{5\pi}{3} = \dfrac{1}{2}$

36. $\lim\limits_{x \to 7} \sec\left(\dfrac{\pi x}{6}\right) = \sec \dfrac{7\pi}{6} = \dfrac{-2\sqrt{3}}{3}$

38. (a) $\lim_{x\to c}[4f(x)] = 4\lim_{x\to c} f(x) = 4\left(\frac{3}{2}\right) = 6$

(b) $\lim_{x\to c}[f(x) + g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = \frac{3}{2} + \frac{1}{2} = 2$

(c) $\lim_{x\to c}[f(x)g(x)] = \left[\lim_{x\to c} f(x)\right]\left[\lim_{x\to c} g(x)\right] = \left(\frac{3}{2}\right)\left(\frac{1}{2}\right) = \frac{3}{4}$

(d) $\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{3/2}{1/2} = 3$

40. (a) $\lim_{x\to c}\sqrt[3]{f(x)} = \sqrt[3]{\lim_{x\to c} f(x)} = \sqrt[3]{27} = 3$

(b) $\lim_{x\to c}\frac{f(x)}{18} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} 18} = \frac{27}{18} = \frac{3}{2}$

(c) $\lim_{x\to c}[f(x)]^2 = \left[\lim_{x\to c} f(x)\right]^2 = (27)^2 = 729$

(d) $\lim_{x\to c}[f(x)]^{2/3} = \left[\lim_{x\to c} f(x)\right]^{2/3} = (27)^{2/3} = 9$

42. $f(x) = x - 3$ and $h(x) = \frac{x^2 - 3x}{x}$ agree except at $x = 0$.

(a) $\lim_{x\to -2} h(x) = \lim_{x\to -2} f(x) = -5$

(b) $\lim_{x\to 0} h(x) = \lim_{x\to 0} f(x) = -3$

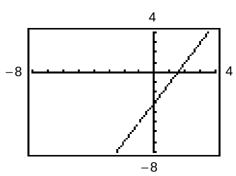
44. $g(x) = \frac{1}{x - 1}$ and $f(x) = \frac{x}{x^2 - x}$ agree except at $x = 0$.

(a) $\lim_{x\to 1} f(x)$ does not exist.

(b) $\lim_{x\to 0} f(x) = -1$

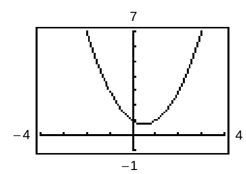
46. $f(x) = \frac{2x^2 - x - 3}{x + 1}$ and $g(x) = 2x - 3$ agree except at $x = -1$.

$\lim_{x\to -1} f(x) = \lim_{x\to -1} g(x) = -5$



48. $f(x) = \frac{x^3 + 1}{x + 1}$ and $g(x) = x^2 - x + 1$ agree except at $x = -1$.

$\lim_{x\to -1} f(x) = \lim_{x\to -1} g(x) = 3$



50. $\lim_{x\to 2}\frac{2 - x}{x^2 - 4} = \lim_{x\to 2}\frac{-(x - 2)}{(x - 2)(x + 2)}$

$= \lim_{x\to 2}\frac{-1}{x + 2} = -\frac{1}{4}$

52. $\lim_{x\to 4}\frac{x^2 - 5x + 4}{x^2 - 2x - 8} = \lim_{x\to 4}\frac{(x - 4)(x - 1)}{(x - 4)(x + 2)}$

$= \lim_{x\to 4}\frac{(x - 1)}{(x + 2)} = \frac{3}{6} = \frac{1}{2}$

54. $\lim_{x\to 0}\frac{\sqrt{2 + x} - \sqrt{2}}{x} = \lim_{x\to 0}\frac{\sqrt{2 + x} - \sqrt{2}}{x}\cdot\frac{\sqrt{2 + x} + \sqrt{2}}{\sqrt{2 + x} + \sqrt{2}}$

$= \lim_{x\to 0}\frac{2 + x - 2}{\left(\sqrt{2 + x} + \sqrt{2}\right)x} = \lim_{x\to 0}\frac{1}{\sqrt{2 + x} + \sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$

56. $\lim_{x\to 3}\frac{\sqrt{x + 1} - 2}{x - 3} = \lim_{x\to 3}\frac{\sqrt{x + 1} - 2}{x - 3}\cdot\frac{\sqrt{x + 1} + 2}{\sqrt{x + 1} + 2} = \lim_{x\to 3}\frac{x - 3}{(x - 3)\left[\sqrt{x + 1} + 2\right]} = \lim_{x\to 3}\frac{1}{\sqrt{x + 1} + 2} = \frac{1}{4}$

58. $\lim_{x\to 0}\frac{\frac{1}{x + 4} - \frac{1}{4}}{x} = \lim_{x\to 0}\frac{\frac{4 - (x + 4)}{4(x + 4)}}{x} = \lim_{x\to 0}\frac{-1}{4(x + 4)} = -\frac{1}{16}$

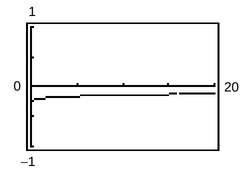
60. $\lim_{\Delta x\to 0}\frac{(x + \Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x\to 0}\frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x\to 0}\frac{\Delta x(2x + \Delta x)}{\Delta x} = \lim_{\Delta x\to 0}(2x + \Delta x) = 2x$

62. $\lim_{\Delta x \to 0} \frac{(x + \Delta x)^3 - x^3}{\Delta x} = \lim_{\Delta x \to 0} \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - x^3}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{\Delta x(3x^2 + 3x\Delta x + (\Delta x)^2)}{\Delta x} = \lim_{\Delta x \to 0} (3x^2 + 3x\Delta x + (\Delta x)^2) = 3x^2$$

64. $f(x) = \frac{4 - \sqrt{x}}{x - 16}$

x	15.9	15.99	15.999	16	16.001	16.01	16.1
$f(x)$	$-.1252$	$-.125$	$-.125$	?	$-.125$	$-.125$	$-.1248$



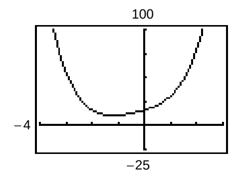
It appears that the limit is -0.125.

Analytically, $\lim_{x \to 16} \frac{4 - \sqrt{x}}{x - 16} = \lim_{x \to 16} \frac{\left(4 - \sqrt{x}\right)}{\left(\sqrt{x} + 4\right)\left(\sqrt{x} - 4\right)}$

$$= \lim_{x \to 16} \frac{-1}{\sqrt{x} + 4} = -\frac{1}{8}.$$

66. $\lim_{x \to 2} \frac{x^5 - 32}{x - 2} = 80$

x	1.9	1.99	1.999	1.9999	2.0	2.0001	2.001	2.01	2.1
$f(x)$	72.39	79.20	79.92	79.99	?	80.01	80.08	80.80	88.41



Analytically, $\lim_{x \to 2} \frac{x^5 - 32}{x - 2} = \lim_{x \to 2} \frac{(x - 2)(x^4 + 2x^3 + 4x^2 + 8x + 16)}{x - 2}$

$$= \lim_{x \to 2} (x^4 + 2x^3 + 4x^2 + 8x + 16) = 80.$$

(*Hint:* Use long division to factor $x^5 - 32$.)

68. $\lim_{x \to 0} \frac{3(1 - \cos x)}{x} = \lim_{x \to 0} \left[3\left(\frac{1 - \cos x}{x}\right)\right] = (3)(0) = 0$

70. $\lim_{\theta \to 0} \frac{\cos \theta \tan \theta}{\theta} = \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$

72. $\lim_{x \to 0} \frac{\tan^2 x}{x} = \lim_{x \to 0} \frac{\sin^2 x}{x \cos^2 x} = \lim_{x \to 0} \left[\frac{\sin x}{x} \cdot \frac{\sin x}{\cos^2 x}\right]$

$$= (1)(0) = 0$$

74. $\lim_{\phi \to \pi} \phi \sec \phi = \pi(-1) = -\pi$

76. $\lim_{x \to \pi/4} \frac{1 - \tan x}{\sin x - \cos x} = \lim_{x \to \pi/4} \frac{\cos x - \sin x}{\sin x \cos x - \cos^2 x}$

$$= \lim_{x \to \pi/4} \frac{-(\sin x - \cos x)}{\cos x(\sin x - \cos x)}$$

$$= \lim_{x \to \pi/4} \frac{-1}{\cos x}$$

$$= \lim_{x \to \pi/4} (-\sec x)$$

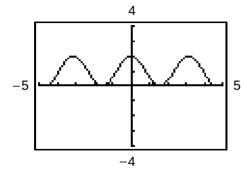
$$= -\sqrt{2}$$

78. $\lim_{x \to 0} \frac{\sin 2x}{\sin 3x} = \lim_{x \to 0} \left[2\left(\frac{\sin 2x}{2x}\right)\left(\frac{1}{3}\right)\left(\frac{3x}{\sin 3x}\right)\right] = 2(1)\left(\frac{1}{3}\right)(1) = \frac{2}{3}$

80. $f(h) = (1 + \cos 2h)$

h	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(h)$	1.98	1.9998	2	?	2	1.9998	1.98

Analytically, $\lim_{h\to 0} (1 + \cos 2h) = 1 + \cos(0) = 1 + 1 = 2.$

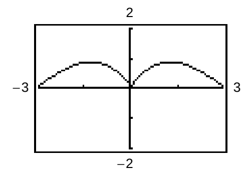


The limit appear to equal 2.

82. $f(x) = \dfrac{\sin x}{\sqrt[3]{x}}$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	0.215	0.0464	0.01	?	0.01	0.0464	0.215

Analytically, $\lim_{x\to 0} \dfrac{\sin x}{\sqrt[3]{x}} = \lim_{x\to 0} \sqrt[3]{x^2}\left(\dfrac{\sin x}{x}\right) = (0)(1) = 0.$



The limit appear to equal 0.

84. $$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h\to 0} \frac{x + h - x}{h\left(\sqrt{x+h} + \sqrt{x}\right)} = \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

86. $$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - 4(x+h) - (x^2 - 4x)}{h} = \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - 4x - 4h - x^2 + 4x}{h}$$

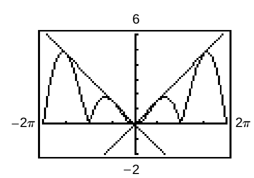
$$= \lim_{h\to 0} \frac{h(2x + h - 4)}{h} = \lim_{h\to 0} (2x + h - 4) = 2x - 4$$

88. $$\lim_{x\to a} [b - |x - a|] \le \lim_{x\to a} f(x) \le \lim_{x\to a} [b + |x - a|]$$

$$b \le \lim_{x\to a} f(x) \le b$$

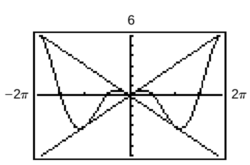
Therefore, $\lim_{x\to a} f(x) = b.$

90. $f(x) = |x \sin x|$



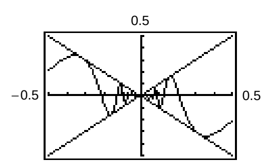
$\lim_{x\to 0} |x \sin x| = 0$

92. $f(x) = |x| \cos x$



$\lim_{x\to 0} |x| \cos x = 0$

94. $h(x) = x \cos \dfrac{1}{x}$

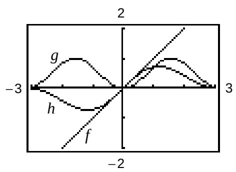


$\lim_{x\to 0} \left(x \cos \dfrac{1}{x}\right) = 0$

96. $f(x) = \dfrac{x^2 - 1}{x - 1}$ and $g(x) = x + 1$ agree at all points except $x = 1$.

98. If a function f is squeezed between two functions h and g, $h(x) \le f(x) \le g(x)$, and h and g have the same limit L as $x \to c$, then $\lim_{x\to c} f(x)$ exists and equals L.

100. $f(x) = x,\ g(x) = \sin^2 x,\ h(x) = \dfrac{\sin^2 x}{x}$



When you are "close to" 0 the magnitude of g is "smaller" than the magnitude of f and the magnitude of g is approaching zero "faster" than the magnitude of f.
Thus, $|g|/|f| \approx 0$ when x is "close to" 0

102. $s(t) = -16t^2 + 1000 = 0$ when $t = \sqrt{\dfrac{1000}{16}} = \dfrac{5\sqrt{10}}{2}$ seconds

$$\lim_{t\to 5\sqrt{10}/2} \frac{s\left(\frac{5\sqrt{10}}{2}\right) - s(t)}{\frac{5\sqrt{10}}{2} - t} = \lim_{t\to 5\sqrt{10}/2} \frac{0 - (-16t^2 + 1000)}{\frac{5\sqrt{10}}{2} - t}$$

$$= \lim_{t\to 5\sqrt{10}/2} \frac{16\left(t^2 - \frac{125}{2}\right)}{\frac{5\sqrt{10}}{2} - t} = \lim_{t\to 5\sqrt{10}/2} \frac{16\left(t + \frac{5\sqrt{10}}{2}\right)\left(t - \frac{5\sqrt{10}}{2}\right)}{-\left(t - \frac{5\sqrt{10}}{2}\right)}$$

$$= \lim_{t\to 5\sqrt{10}/2} -16\left(t + \frac{5\sqrt{10}}{2}\right) = -80\sqrt{10} \text{ ft/sec} \approx -253 \text{ ft/sec}$$

104. $-4.9t^2 + 150 = 0$ when $t = \sqrt{\dfrac{150}{4.9}} = \sqrt{\dfrac{1500}{49}} \approx 5.53$ seconds.

The velocity at time $t = a$ is

$$\lim_{t\to a} \frac{s(a) - s(t)}{a - t} = \lim_{t\to a} \frac{(-4.9a^2 + 150) - (-4.9t^2 + 150)}{a - t} = \lim_{t\to a} \frac{-4.9(a - t)(a + t)}{a - t}$$

$$= \lim_{t\to a} -4.9(a + t) = -2a(4.9) = -9.8a \text{ m/sec.}$$

Hence, if $a = \sqrt{1500/49}$, the velocity is $-9.8\sqrt{1500/49} \approx -54.2$ m/sec.

106. Suppose, on the contrary, that $\lim_{x\to c} g(x)$ exists. Then, since $\lim_{x\to c} f(x)$ exists, so would $\lim_{x\to c} [f(x) + g(x)]$, which is a contradiction. Hence, $\lim_{x\to c} g(x)$ does not exist.

108. Given $f(x) = x^n$, n is a positive integer, then

$$\lim_{x\to c} x^n = \lim_{x\to c} (xx^{n-1}) = \left[\lim_{x\to c} x\right]\left[\lim_{x\to c} x^{n-1}\right]$$

$$= c\left[\lim_{x\to c} (xx^{n-2})\right] = c\left[\lim_{x\to c} x\right]\left[\lim_{x\to c} x^{n-2}\right]$$

$$= c(c)\lim_{x\to c} (xx^{n-3}) = \cdots = c^n.$$

110. Given $\lim_{x\to c} f(x) = 0$:

For every $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - 0| < \epsilon$ whenever $0 < |x - c| < \delta$.

Now $|f(x) - 0| = |f(x)| = ||f(x)| - 0| < \epsilon$ for $|x - c| < \delta$. Therefore, $\lim_{x\to c} |f(x)| = 0$.

112. (a) If $\lim_{x\to c} |f(x)| = 0$, then $\lim_{x\to c} [-|f(x)|] = 0$.

$$-|f(x)| \le f(x) \le |f(x)|$$

$$\lim_{x\to c} [-|f(x)|] \le \lim_{x\to c} f(x) \le \lim_{x\to c} |f(x)|$$

$$0 \le \lim_{x\to c} f(x) \le 0$$

Therefore, $\lim_{x\to c} f(x) = 0$.

(b) Given $\lim_{x\to c} f(x) = L$:

For every $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - c| < \delta$. Since $||f(x)| - |L|| \le |f(x) - L| < \epsilon$ for $|x - c| < \delta$, then $\lim_{x\to c} |f(x)| = |L|$.

114. True. $\lim_{x\to 0} x^3 = 0^3 = 0$

116. False. Let $f(x) = \begin{cases} x & x \ne 1 \\ 3 & x = 1 \end{cases}$, $c = 1$

Then $\lim_{x\to 1} f(x) = 1$ but $f(1) \ne 1$.

118. False. Let $f(x) = \frac{1}{2}x^2$ and $g(x) = x^2$. Then $f(x) < g(x)$ for all $x \ne 0$. But $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0$.

120. $$\lim_{x\to 0} \frac{1 - \cos x}{x} = \lim_{x\to 0} \frac{1 - \cos x}{x} \cdot \frac{1 + \cos x}{1 + \cos x}$$

$$= \lim_{x\to 0} \frac{1 - \cos^2 x}{x(1 + \cos x)} = \lim_{x\to 0} \frac{\sin^2 x}{x(1 + \cos x)}$$

$$= \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{1 + \cos x}$$

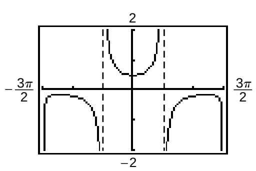
$$= \left[\lim_{x\to 0} \frac{\sin x}{x}\right]\left[\lim_{x\to 0} \frac{\sin x}{1 + \cos x}\right]$$

$$= (1)(0) = 0$$

122. $f(x) = \dfrac{\sec x - 1}{x^2}$

(a) The domain of f is all $x \ne 0$, $\pi/2 + n\pi$.

(b)



The domain is not obvious. The hole at $x = 0$ is not apparent.

(c) $\lim_{x\to 0} f(x) = \dfrac{1}{2}$

(d) $$\frac{\sec x - 1}{x^2} = \frac{\sec x - 1}{x^2} \cdot \frac{\sec x + 1}{\sec x + 1} = \frac{\sec^2 x - 1}{x^2(\sec x + 1)}$$

$$= \frac{\tan^2 x}{x^2(\sec x + 1)} = \frac{1}{\cos^2 x}\left(\frac{\sin^2 x}{x^2}\right)\frac{1}{\sec x + 1}$$

Hence, $$\lim_{x\to 0} \frac{\sec x - 1}{x^2} = \lim_{x\to 0} \frac{1}{\cos^2 x}\left(\frac{\sin^2 x}{x^2}\right)\frac{1}{\sec x + 1}$$

$$= 1(1)\left(\frac{1}{2}\right) = \frac{1}{2}.$$

124. The calculator was set in degree mode, instead of radian mode.

Section 1.4 Continuity and One-Sided Limits

2. (a) $\lim_{x\to -2^+} f(x) = -2$

(b) $\lim_{x\to -2^-} f(x) = -2$

(c) $\lim_{x\to -2} f(x) = -2$

The function is continuous at $x = -2$.

4. (a) $\lim_{x\to -2^+} f(x) = 2$

(b) $\lim_{x\to -2^-} f(x) = 2$

(c) $\lim_{x\to -2} f(x) = 2$

The function is NOT continuous at $x = -2$.

6. (a) $\lim_{x\to -1^+} f(x) = 0$

(b) $\lim_{x\to -1^-} f(x) = 2$

(c) $\lim_{x\to -1} f(x)$ does not exist.

The function is NOT continuous at $x = -1$.

8. $\lim_{x\to 2^+} \frac{2-x}{x^2-4} = \lim_{x\to 2^+} -\frac{1}{x+2} = -\frac{1}{4}$

10. $$\lim_{x\to 4^-} \frac{\sqrt{x}-2}{x-4} = \lim_{x\to 4^-} \frac{\sqrt{x}-2}{x-4}\cdot\frac{\sqrt{x}+2}{\sqrt{x}+2}$$
$$= \lim_{x\to 4^-} \frac{x-4}{(x-4)(\sqrt{x}+2)}$$
$$= \lim_{x\to 4^-} \frac{1}{\sqrt{x}+2} = \frac{1}{4}$$

12. $\lim_{x\to 2^+} \frac{|x-2|}{x-2} = \lim_{x\to 2^+} \frac{x-2}{x-2} = 1$

14. $$\lim_{\Delta x\to 0^+} \frac{(x+\Delta x)^2 + (x+\Delta x) - (x^2+x)}{\Delta x} = \lim_{\Delta x\to 0^+} \frac{x^2 + 2x(\Delta x) + (\Delta x)^2 + x + \Delta x - x^2 - x}{\Delta x}$$
$$= \lim_{\Delta x\to 0^+} \frac{2x(\Delta x) + (\Delta x)^2 + \Delta x}{\Delta x}$$
$$= \lim_{\Delta x\to 0^+} (2x + \Delta x + 1)$$
$$= 2x + 0 + 1 = 2x + 1$$

16. $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (-x^2 + 4x - 2) = 2$

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x^2 - 4x + 6) = 2$

$\lim_{x\to 2} f(x) = 2$

18. $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (1 - x) = 0$

20. $\lim_{x\to \pi/2} \sec x$ does not exist since $\lim_{x\to (\pi/2)^+} \sec x$ and $\lim_{x\to (\pi/2)^-} \sec x$ do not exist.

22. $\lim_{x\to 2^+} (2x - [\![x]\!]) = 2(2) - 2 = 2$

24. $\lim_{x\to 1} \left(1 - \left[\!\left[-\frac{x}{2}\right]\!\right]\right) = 1 - (-1) = 2$

26. $f(x) = \frac{x^2-1}{x+1}$

has a discontinuity at $x = -1$ since $f(-1)$ is not defined.

28. $f(x) = \begin{cases} x, & x < 1 \\ 2, & x = 1 \\ 2x - 1, & x > 1 \end{cases}$ has discontinuity at $x = 1$ since $f(1) = 2 \neq \lim_{x\to 1} f(x) = 1$.

30. $f(t) = 3 - \sqrt{9 - t^2}$ is continuous on $[-3, 3]$.

32. $g(2)$ is not defined. g is continuous on $[-1, 2)$.

34. $f(x) = \dfrac{1}{x^2 + 1}$ is continuous for all real x.

36. $f(x) = \cos \dfrac{\pi x}{2}$ is continuous for all real x.

38. $f(x) = \dfrac{x}{x^2 - 1}$ has nonremovable discontinuities at $x = 1$ and $x = -1$ since $\lim_{x\to 1} f(x)$ and $\lim_{x\to -1} f(x)$ do not exist.

40. $f(x) = \dfrac{x - 3}{x^2 - 9}$ has a nonremovable discontinuity at $x = -3$ since $\lim_{x\to -3} f(x)$ does not exist, and has a removable discontinuity at $x = 3$ since

$$\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{1}{x + 3} = \frac{1}{6}.$$

42. $f(x) = \dfrac{x - 1}{(x + 2)(x - 1)}$

has a nonremovable discontinuity at $x = -2$ since $\lim_{x\to -2} f(x)$ does not exist, and has a removable discontinuity at $x = 1$ since

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{1}{x + 2} = \frac{1}{3}.$$

44. $f(x) = \dfrac{|x - 3|}{x - 3}$

has a nonremovable discontinuity at $x = 3$ since $\lim_{x\to 3} f(x)$ does not exist.

46. $f(x) = \begin{cases} -2x + 3, & x < 1 \\ x^2, & x \ge 1 \end{cases}$

has a **possible** discontinuity at $x = 1$.

1. $f(1) = 1^2 = 1$

2. $\left.\begin{array}{l} \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (-2x + 3) = 1 \\ \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x^2 = 1 \end{array}\right\} \lim_{x\to 1} f(x) = 1$

3. $f(1) = \lim_{x\to 1} f(x)$

f is continuous at $x = 1$, therefore, f is continuous for all real x.

48. $f(x) = \begin{cases} -2x, & x \le 2 \\ x^2 - 4x + 1, & x > 2 \end{cases}$ has a **possible** discontinuity at $x = 2$.

1. $f(2) = -2(2) = -4$

2. $\left.\begin{array}{l} \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (-2x) = -4 \\ \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (x^2 - 4x + 1) = -3 \end{array}\right\} \lim_{x\to 2} f(x)$ does not exist.

Therefore, f has a nonremovable discontinuity at $x = 2$.

50. $f(x) = \begin{cases} \csc \dfrac{\pi x}{6}, & |x - 3| \le 2 \\ 2, & |x - 3| > 2 \end{cases} = \begin{cases} \csc \dfrac{\pi x}{6}, & 1 \le x \le 5 \\ 2, & x < 1 \text{ or } x > 5 \end{cases}$ has **possible** discontinuities at $x = 1$, $x = 5$.

1. $f(1) = \csc \dfrac{\pi}{6} = 2$ $\qquad f(5) = \csc \dfrac{5\pi}{6} = 2$

2. $\lim_{x\to 1} f(x) = 2$ $\qquad \lim_{x\to 5} f(x) = 2$

3. $f(1) = \lim_{x\to 1} f(x)$ $\qquad f(5) = \lim_{x\to 5} f(x)$

f is continuous at $x = 1$ and $x = 5$, therefore, f is continuous for all real x.

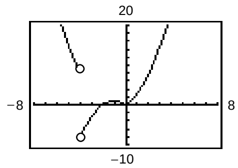
52. $f(x) = \tan \dfrac{\pi x}{2}$ has nonremovable discontinuities at each $2k + 1$, k is an integer.

54. $f(x) = 3 - [\![x]\!]$ has nonremovable discontinuities at each integer k.

56. $\displaystyle\lim_{x\to 0^+} f(x) = 0$

$\displaystyle\lim_{x\to 0^-} f(x) = 0$

f is not continuous at $x = -4$



58. $\displaystyle\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} \frac{4 \sin x}{x} = 4$

$\displaystyle\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} (a - 2x) = a$

Let $a = 4$.

60. $\displaystyle\lim_{x\to a} g(x) = \lim_{x\to a} \frac{x^2 - a^2}{x - a}$

$\displaystyle= \lim_{x\to a} (x + a) = 2a$

Find a such that $2a = 8 \implies a = 4$.

62. $f(g(x)) = \dfrac{1}{\sqrt{x - 1}}$

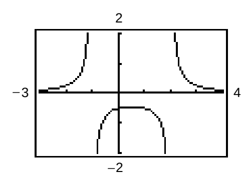
Nonremovable discontinuity at $x = 1$. Continuous for all $x > 1$.
Because $f \circ g$ is not defined for $x < 1$, it is better to say that $f \circ g$ is discontinuous from the right at $x = 1$.

64. $f(g(x)) = \sin x^2$

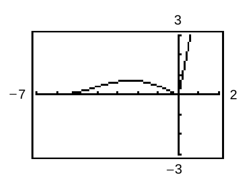
Continuous for all real x

66. $h(x) = \dfrac{1}{(x + 1)(x - 2)}$

Nonremovable discontinuity at $x = -1$ and $x = 2$.



68. $f(x) = \begin{cases} \dfrac{\cos x - 1}{x}, & x < 0 \\ 5x, & x \geq 0 \end{cases}$



$f(0) = 5(0) = 0$

$\displaystyle\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{(\cos x - 1)}{x} = 0$

$\displaystyle\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (5x) = 0$

Therefore, $\displaystyle\lim_{x\to 0} f(x) = 0 = f(0)$ and f is continuous on the entire real line. ($x = 0$ was the only possible discontinuity.)

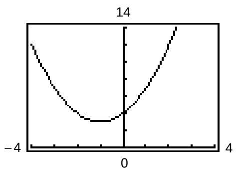
70. $f(x) = x\sqrt{x + 3}$

Continuous on $[-3, \infty]$

72. $f(x) = \dfrac{x + 1}{\sqrt{x}}$

Continuous on $(0, \infty)$

74. $f(x) = \dfrac{x^3 - 8}{x - 2}$



The graph **appears** to be continuous on the interval $[-4, 4]$. Since $f(2)$ is not defined, we know that f has a discontinuity at $x = 2$. This discontinuity is removable so it does not show up on the graph.

76. $f(x) = x^3 + 3x - 2$ is continuous on $[0, 1]$.

$f(0) = -2$ and $f(1) = 2$

By the Intermediate Value Theorem, $f(x) = 0$ for at least one value of c between 0 and 1.

78. $f(x) = \dfrac{-4}{x} + \tan\dfrac{\pi x}{8}$ is continuous on $[1, 3]$.

$f(1) = -4 + \tan\dfrac{\pi}{8} < 0$ and $f(3) = -\dfrac{4}{3} + \tan\dfrac{3\pi}{8} > 0.$

By the Intermediate Value Theorem, $f(1) = 0$ for at least one value of c between 1 and 3.

80. $f(x) = x^3 + 3x - 2$

$f(x)$ is continuous on $[0, 1]$.

$f(0) = -2$ and $f(1) = 2$

By the Intermediate Value Theorem, $f(x) = 0$ for at least one value of c between 0 and 1. Using a graphing utility, we find that $x \approx 0.5961$.

82. $h(\theta) = 1 + \theta - 3\tan\theta$

h is continuous on $[0, 1]$.

$h(0) = 1 > 0$ and $h(1) \approx -2.67 < 0.$

By the Intermediate Value Theorem, $h(\theta) = 0$ for at least one value θ between 0 and 1. Using a graphing utility, we find that $\theta \approx 0.4503$.

84. $f(x) = x^2 - 6x + 8$

f is continuous on $[0, 3]$.

$f(0) = 8$ and $f(3) = -1$

$-1 < 0 < 8$

The Intermediate Value Theorem applies.

$$x^2 - 6x + 8 = 0$$
$$(x - 2)(x - 4) = 0$$
$$x = 2 \text{ or } x = 4$$
$$c = 2 \ (x = 4 \text{ is not in the interval.})$$

Thus, $f(2) = 0$.

86. $f(x) = \dfrac{x^2 + x}{x - 1}$

f is continuous on $\left[\frac{5}{2}, 4\right]$. The nonremovable discontinuity, $x = 1$, lies outside the interval.

$$f\left(\frac{5}{2}\right) = \frac{35}{6} \text{ and } f(4) = \frac{20}{3}$$

$$\frac{35}{6} < 6 < \frac{20}{3}$$

The Intermediate Value Theorem applies.

$$\frac{x^2 + x}{x - 1} = 6$$
$$x^2 + x = 6x - 6$$
$$x^2 - 5x + 6 = 0$$
$$(x - 2)(x - 3) = 0$$
$$x = 2 \text{ or } x = 3$$
$$c = 3 \ (x = 2 \text{ is not in the interval.})$$

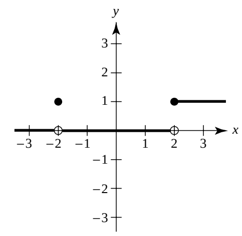
Thus, $f(3) = 6$.

88. A discontinuity at $x = c$ is removable if you can define (or redefine) the function at $x = c$ in such a way that the new function is continuous at $x = c$. Answers will vary.

(a) $f(x) = \dfrac{|x - 2|}{x - 2}$

(b) $f(x) = \dfrac{\sin(x + 2)}{x + 2}$

(c) $f(x) = \begin{cases} 1, & \text{if } x \geq 2 \\ 0, & \text{if } -2 < x < 2 \\ 1, & \text{if } x = -2 \\ 0, & \text{if } x < -2 \end{cases}$



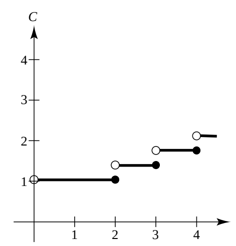
90. If f and g are continuous for all real x, then so is $f + g$ (Theorem 1.11, part 2). However, f/g might not be continuous if $g(x) = 0$. For example, let $f(x) = x$ and $g(x) = x^2 - 1$. Then f and g are continuous for all real x, but f/g is not continuous at $x = \pm 1$.

92. $C = \begin{cases} 1.04, & 0 < t \leq 2 \\ 1.04 + 0.36[\![t - 1]\!], & t > 2, t \text{ is not an integer} \\ 1.04 + 0.36(t - 2), & t > 2, t \text{ is an integer} \end{cases}$

Nonremovable discontinuity at each integer greater than 2.

You can also write C as

$$C = \begin{cases} 1.04, & 0 < t \leq 2 \\ 1.04 - 0.36[\![2 - t]\!], & t > 2 \end{cases}.$$



94. Let $s(t)$ be the position function for the run up to the campsite. $s(0) = 0$ ($t = 0$ corresponds to 8:00 A.M., $s(20) = k$ (distance to campsite)). Let $r(t)$ be the position function for the run back down the mountain: $r(0) = k$, $r(10) = 0$. Let $f(t) = s(t) - r(t)$.

When $t = 0$ (8:00 A.M.), $f(0) = s(0) - r(0) = 0 - k < 0$.

When $t = 10$ (8:10 A.M.), $f(10) = s(10) - r(10) > 0$.

Since $f(0) < 0$ and $f(10) > 0$, then there must be a value t in the interval $[0, 10]$ such that $f(t) = 0$. If $f(t) = 0$, then $s(t) - r(t) = 0$, which gives us $s(t) = r(t)$. Therefore, at some time t, where $0 \leq t \leq 10$, the position functions for the run up and the run down are equal.

96. Suppose there exists x_1 in $[a, b]$ such that $f(x_1) > 0$ and there exists x_2 in $[a, b]$ such that $f(x_2) < 0$. Then by the Intermediate Value Theorem, $f(x)$ must equal zero for some value of x in $[x_1, x_2]$ (or $[x_2, x_1]$ if $x_2 < x_1$). Thus, f would have a zero in $[a, b]$, which is a contradiction. Therefore, $f(x) > 0$ for all x in $[a, b]$ or $f(x) < 0$ for all x in $[a, b]$.

98. If $x = 0$, then $f(0) = 0$ and $\lim_{x \to 0} f(x) = 0$. Hence, f is continuous at $x = 0$.

If $x \neq 0$, then $\lim_{t \to x} f(t) = 0$ for x rational, whereas $\lim_{t \to x} f(t) = \lim_{t \to x} kt = kx \neq 0$ for x irrational. Hence, f is not continuous for all $x \neq 0$.

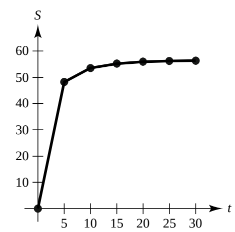
100. True

1. $f(c) = L$ is defined.
2. $\lim_{x \to c} f(x) = L$ exists.
3. $f(c) = \lim_{x \to c} f(x)$

All of the conditions for continuity are met.

102. False; a rational function can be written as $P(x)/Q(x)$ where P and Q are polynomials of degree m and n, respectively. It can have, at most, n discontinuities.

104. (a)



(b) There appears to be a limiting speed and a possible cause is air resistance.

106. Let y be a real number. If $y = 0$, then $x = 0$. If $y > 0$, then let $0 < x_0 < \pi/2$ such that $M = \tan x_0 > y$ (this is possible since the tangent function increases without bound on $[0, \pi/2)$). By the Intermediate Value Theorem, $f(x) = \tan x$ is continuous on $[0, x_0]$ and $0 < y < M$, which implies that there exists x between 0 and x_0 such that $\tan x = y$. The argument is similar if $y < 0$.

108. **1.** $f(c)$ is defined.

2. $\lim_{x \to c} f(x) = \lim_{\Delta x \to 0} f(c + \Delta x) = f(c)$ exists.

$[$Let $x = c + \Delta x$. As $x \to c$, $\Delta x \to 0]$

3. $\lim_{x \to c} f(x) = f(c)$.

Therefore, f is continuous at $x = c$.

110. Define $f(x) = f_2(x) - f_1(x)$. Since f_1 and f_2 are continuous on $[a, b]$, so is f.

$$f(a) = f_2(a) - f_1(a) > 0 \quad \text{and} \quad f(b) = f_2(b) - f_1(b) < 0.$$

By the Intermediate Value Theorem, there exists c in $[a, b]$ such that $f(c) = 0$.

$$f(c) = f_2(c) - f_1(c) = 0 \implies f_1(c) = f_2(c)$$

Section 1.5 Infinite Limits

2. $$\lim_{x \to -2^+} \frac{1}{x + 2} = \infty$$

$$\lim_{x \to -2^-} \frac{1}{x + 2} = -\infty$$

4. $$\lim_{x \to -2^+} \sec \frac{\pi x}{4} = \infty$$

$$\lim_{x \to -2^-} \sec \frac{\pi x}{4} = -\infty$$

6. $f(x) = \dfrac{x}{x^2 - 9}$

x	-3.5	-3.1	-3.01	-3.001	-2.999	-2.99	-2.9	-2.5
$f(x)$	-1.077	-5.082	-50.08	-500.1	499.9	49.92	4.915	0.9091

$$\lim_{x \to -3^-} f(x) = -\infty$$

$$\lim_{x \to -3^+} f(x) = \infty$$

8. $f(x) = \sec \dfrac{\pi x}{6}$

x	-3.5	-3.1	-3.01	-3.001	-2.999	-2.99	-2.9	-2.5
$f(x)$	-3.864	-19.11	-191.0	-1910	1910	191.0	19.11	3.864

$\lim_{x\to -3^-} f(x) = -\infty$

$\lim_{x\to -3^+} f(x) = \infty$

10. $\lim_{x\to 2^+} \dfrac{4}{(x-2)^3} = \infty$

$\lim_{x\to 2^-} \dfrac{4}{(x-2)^3} = -\infty$

Therefore, $x = 2$ is a vertical asymptote.

12. $\lim_{x\to 0^-} \dfrac{2+x}{x^2(1-x)} = \lim_{x\to 0^+} \dfrac{2+x}{x^2(1-x)} = \infty$

Therefore, $x = 0$ is a vertical asymptote.

$\lim_{x\to 1^-} \dfrac{2+x}{x^2(1-x)} = \infty$

$\lim_{x\to 1^+} \dfrac{2+x}{x^2(1-x)} = -\infty$

Therefore, $x = 1$ is a vertical asymptote.

14. No vertical asymptote since the denominator is never zero.

16. $\lim_{s\to -5^-} h(s) = -\infty$ and $\lim_{s\to -5^+} h(s) = \infty$.
Therefore, $s = -5$ is a vertical asymptote.

$\lim_{s\to 5^-} h(s) = -\infty$ and $\lim_{s\to 5^+} h(s) = \infty$.
Therefore, $s = 5$ is a vertical asymptote.

18. $f(x) = \sec \pi x = \dfrac{1}{\cos \pi x}$ has vertical asymptotes at

$x = \dfrac{2n+1}{2}$, n any integer.

20. $g(x) = \dfrac{(1/2)x^3 - x^2 - 4x}{3x^2 - 6x - 24} = \dfrac{1}{6}\dfrac{x(x^2 - 2x - 8)}{x^2 - 2x - 8}$

$= \dfrac{1}{6}x,$

$x \neq -2, 4$

No vertical asymptotes. The graph has holes at $x = -2$ and $x = 4$.

22. $f(x) = \dfrac{4(x^2 + x - 6)}{x(x^3 - 2x^2 - 9x + 18)} = \dfrac{4(x+3)(x-2)}{x(x-2)(x^2-9)} = \dfrac{4}{x(x-3)}, x \neq -3, 2$

Vertical asymptotes at $x = 0$ and $x = 3$. The graph has holes at $x = -3$ and $x = 2$.

24. $h(x) = \dfrac{x^2 - 4}{x^3 + 2x^2 + x + 2} = \dfrac{(x+2)(x-2)}{(x+2)(x^2+1)}$

has no vertical asymptote since

$\lim_{x\to -2} h(x) = \lim_{x\to -2} \dfrac{x-2}{x^2+1} = -\dfrac{4}{5}.$

26. $h(t) = \dfrac{t(t-2)}{(t-2)(t+2)(t^2+4)} = \dfrac{t}{(t+2)(t^2+4)}, t \neq 2$

Vertical asymptote at $t = -2$. The graph has a hole at $t = 2$.

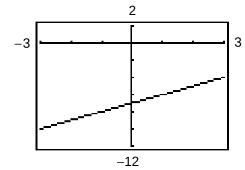
28. $g(\theta) = \dfrac{\tan \theta}{\theta} = \dfrac{\sin \theta}{\theta \cos \theta}$ has vertical asymptotes at

$$\theta = \frac{(2n+1)\pi}{2} = \frac{\pi}{2} + n\pi, \; n \text{ any integer.}$$

There is no vertical asymptote at $\theta = 0$ since

$$\lim_{\theta \to 0} \frac{\tan \theta}{\theta} = 1.$$

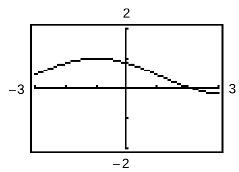
30. $\displaystyle \lim_{x \to -1} \frac{x^2 - 6x - 7}{x + 1} = \lim_{x \to -1} (x - 7) = -8$



Removable discontinuity at $x = -1$

32. $\displaystyle \lim_{x \to -1} \frac{\sin(x+1)}{x+1} = 1$

Removable discontinuity at $x = -1$



34. $\displaystyle \lim_{x \to 1^+} \frac{2 + x}{1 - x} = -\infty$

36. $\displaystyle \lim_{x \to 4^-} \frac{x^2}{x^2 + 16} = \frac{1}{2}$

38. $\displaystyle \lim_{x \to -(1/2)^+} \frac{6x^2 + x - 1}{4x^2 - 4x - 3} = \lim_{x \to -(1/2)^+} \frac{3x - 1}{2x - 3} = \frac{5}{8}$

40. $\displaystyle \lim_{x \to 3} \frac{x - 2}{x^2} = \frac{1}{9}$

42. $\displaystyle \lim_{x \to 0^-} \left(x^2 - \frac{1}{x}\right) = \infty$

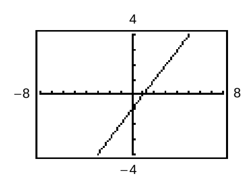
44. $\displaystyle \lim_{x \to (\pi/2)^+} \frac{-2}{\cos x} = \infty$

46. $\displaystyle \lim_{x \to 0} \frac{(x + 2)}{\cot x} = \lim_{x \to 0} [(x + 2)\tan x] = 0$

48. $\displaystyle \lim_{x \to (1/2)^-} x^2 \tan \pi x = \infty$ and $\displaystyle \lim_{x \to (1/2)^+} x^2 \tan \pi x = -\infty$.
Therefore, $\displaystyle \lim_{x \to (1/2)} x^2 \tan \pi x$ does not exist.

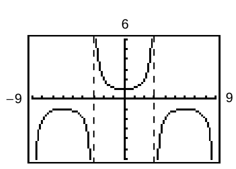
50. $f(x) = \dfrac{x^3 - 1}{x^2 + x + 1}$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x - 1) = 0$$



52. $f(x) = \sec \dfrac{\pi x}{6}$

$$\lim_{x \to 3^+} f(x) = -\infty$$



54. The line $x = c$ is a vertical asymptote if the graph of f approaches $\pm\infty$ as x approaches c.

56. No. For example, $f(x) = \dfrac{1}{x^2 + 1}$ has no vertical asymptote.

58. $P = \dfrac{k}{V}$

$\displaystyle \lim_{V \to 0^+} \frac{k}{V} = k(\infty) = \infty$ (In this case we know that $k > 0$.)

60. (a) $r = 50\pi \sec^2 \frac{\pi}{6} = \frac{200\pi}{3}$ ft/sec

(b) $r = 50\pi \sec^2 \frac{\pi}{3} = 200\pi$ ft/sec

(c) $\lim_{\theta \to (\pi/2)^-} [50\pi \sec^2 \theta] = \infty$

62. $m = \frac{m_0}{\sqrt{1 - (v^2/c^2)}}$

$$\lim_{v \to c^-} m = \lim_{v \to c^-} \frac{m_0}{\sqrt{1 - (v^2/c^2)}} = \infty$$

64. (a) Average speed $= \frac{\text{Total distance}}{\text{Total time}}$

$$50 = \frac{2d}{(d/x) + (d/y)}$$

$$50 = \frac{2xy}{y + x}$$

$$50y + 50x = 2xy$$

$$50x = 2xy - 50y$$

$$50x = 2y(x - 25)$$

$$\frac{25x}{x - 25} = y$$

Domain: $x > 25$

(b)

x	30	40	50	60
y	150	66.667	50	42.857

(c) $\lim_{x \to 25^+} \frac{25x}{x - 25} = \infty$

As x gets close to 25 mph, y becomes larger and larger.

66. (a) $A = \frac{1}{2}bh - \frac{1}{2}r^2 \theta = \frac{1}{2}(10)(10 \tan \theta) - \frac{1}{2}(10)^2\theta$

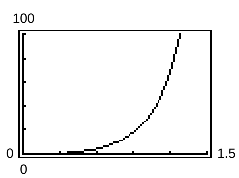
$$= 50 \tan \theta - 50\, \theta$$

Domain: $\left(0, \frac{\pi}{2}\right)$

(b)

ϕ	0.3	0.6	0.9	1.2	1.5
$f(\theta)$	0.47	4.21	18.0	68.6	630.1

(c)



(d) $\lim_{\theta \to \pi/2^-} A = \infty$

68. False; for instance, let

$$f(x) = \frac{x^2 - 1}{x - 1}.$$

The graph of f has a hole at $(1, 2)$, not a vertical asymptote.

70. True

72. Let $f(x) = \frac{1}{x^2}$ and $g(x) = \frac{1}{x^4}$, and $c = 0$.

$\lim_{x \to 0} \frac{1}{x^2} = \infty$ and $\lim_{x \to 0} \frac{1}{x^4} = \infty$, but

$$\lim_{x \to 0} \left(\frac{1}{x^2} - \frac{1}{x^4}\right) = \lim_{x \to 0} \left(\frac{x^2 - 1}{x^4}\right) = -\infty \neq 0.$$

74. Given $\lim_{x \to c} f(x) = \infty$, let $g(x) = 1$. then $\lim_{x \to c} \frac{g(x)}{f(x)} = 0$ by Theorem 1.15.