

CHAPTER 1

Limits and Their Properties

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CHAPTER 1

Limits and Their Properties

Section 1.1 A Preview of Calculus

Solutions to Odd-Numbered Exercises

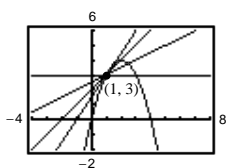
1. Precalculus: $(20 \text{ ft/sec})(15 \text{ seconds}) = 300 \text{ feet}$

3. Calculus required: slope of tangent line at $x = 2$ is rate of change, and equals about 0.16.

5. Precalculus: Area $= \frac{1}{2}bh = \frac{1}{2}(5)(3) = \frac{15}{2}$ sq. units

7. Precalculus: Volume $= (2)(4)(3) = 24$ cubic units

9. (a)



(b) The graphs of y_2 are approximations to the tangent line to y_1 at $x = 1$.

(c) The slope is approximately 2. For a better approximation make the list numbers smaller:
 $\{0.2, 0.1, 0.01, 0.001\}$

11. (a) $D_1 = \sqrt{(5-1)^2 + (1-5)^2} = \sqrt{16+16} \approx 5.66$

(b) $D_2 = \sqrt{1 + (\frac{5}{2})^2} + \sqrt{1 + (\frac{5}{2} - \frac{5}{3})^2} + \sqrt{1 + (\frac{5}{3} - \frac{5}{4})^2} + \sqrt{1 + (\frac{5}{4} - 1)^2}$
 $\approx 2.693 + 1.302 + 1.083 + 1.031 \approx 6.11$

(c) Increase the number of line segments.

Section 1.2 Finding Limits Graphically and Numerically

1.

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$	0.3448	0.3344	0.3334	0.3332	0.3322	0.3226

$$\lim_{x \rightarrow 2} \frac{x-2}{x^2-x-2} \approx 0.3333 \quad (\text{Actual limit is } \frac{1}{3}.)$$

3.

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	0.2911	0.2889	0.2887	0.2887	0.2884	0.2863

$$\lim_{x \rightarrow 0} \frac{\sqrt{x+3} - \sqrt{3}}{x} \approx 0.2887 \quad (\text{Actual limit is } 1/(2\sqrt{3}).)$$

5.

x	2.9	2.99	2.999	3.001	3.01	3.1
$f(x)$	-0.0641	-0.0627	-0.0625	-0.0625	-0.0623	-0.0610

$$\lim_{x \rightarrow 3} \frac{[1/(x+1)] - (1/4)}{x-3} \approx -0.0625 \quad (\text{Actual limit is } -\frac{1}{16}.)$$

7.

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	0.9983	0.99998	1.0000	1.0000	0.99998	0.9983

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} \approx 1.0000 \quad (\text{Actual limit is } 1.) \quad (\text{Make sure you use radian mode.})$$

9. $\lim_{x \rightarrow 3} (4 - x) = 1$

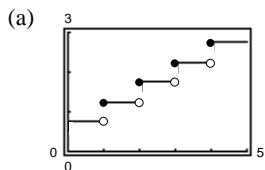
11. $\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} (4 - x) = 2$

13. $\lim_{x \rightarrow 5} \frac{|x-5|}{x-5}$ does not exist. For values of x to the left of 5, $|x-5|/(x-5)$ equals -1 , whereas for values of x to the right of 5, $|x-5|/(x-5)$ equals 1.

15. $\lim_{x \rightarrow \pi/2} \tan x$ does not exist since the function increases and decreases without bound as x approaches $\pi/2$.

17. $\lim_{x \rightarrow 0} \cos(1/x)$ does not exist since the function oscillates between -1 and 1 as x approaches 0.

19. $C(t) = 0.75 - 0.50[-(t-1)]$



(b)

t	3	3.3	3.4	3.5	3.6	3.7	4
C	1.75	2.25	2.25	2.25	2.25	2.25	2.25

$$\lim_{t \rightarrow 3.5} C(t) = 2.25$$

(c)

t	2	2.5	2.9	3	3.1	3.5	4
C	1.25	1.75	1.75	1.75	2.25	2.25	2.25

$\lim_{t \rightarrow 3} C(t)$ does not exist. The values of C jump from 1.75 to 2.25 at $t = 3$.

21. You need to find δ such that $0 < |x-1| < \delta$ implies

$$|f(x) - 1| = \left| \frac{1}{x} - 1 \right| < 0.1. \quad \text{That is,}$$

$$-0.1 < \frac{1}{x} - 1 < 0.1$$

$$1 - 0.1 < \frac{1}{x} < 1 + 0.1$$

$$\frac{9}{10} < \frac{1}{x} < \frac{11}{10}$$

$$\frac{10}{9} > x > \frac{10}{11}$$

$$\frac{10}{9} - 1 > x - 1 > \frac{10}{11} - 1$$

$$\frac{1}{9} > x - 1 > -\frac{1}{11}.$$

So take $\delta = \frac{1}{11}$. Then $0 < |x-1| < \delta$ implies

$$-\frac{1}{11} < x - 1 < \frac{1}{11}$$

$$-\frac{1}{11} < x - 1 < \frac{1}{9}.$$

Using the first series of equivalent inequalities, you obtain

$$|f(x) - 1| = \left| \frac{1}{x} - 1 \right| < \epsilon < 0.1.$$

$$23. \lim_{x \rightarrow 2} (3x + 2) = 8 = L$$

$$|(3x + 2) - 8| < 0.01$$

$$|3x - 6| < 0.01$$

$$3|x - 2| < 0.01$$

$$0 < |x - 2| < \frac{0.01}{3} \approx 0.0033 = \delta$$

Hence, if $0 < |x - 2| < \delta = \frac{0.01}{3}$, you have

$$3|x - 2| < 0.01$$

$$|3x - 6| < 0.01$$

$$|(3x + 2) - 8| < 0.01$$

$$|f(x) - L| < 0.01$$

$$27. \lim_{x \rightarrow 2} (x + 3) = 5$$

Given $\epsilon > 0$:

$$|(x + 3) - 5| < \epsilon$$

$$|x - 2| < \epsilon = \delta$$

Hence, let $\delta = \epsilon$.

Hence, if $0 < |x - 2| < \delta = \epsilon$, you have

$$|x - 2| < \epsilon$$

$$|(x + 3) - 5| < \epsilon$$

$$|f(x) - L| < \epsilon$$

$$31. \lim_{x \rightarrow 6} 3 = 3$$

Given $\epsilon > 0$:

$$|3 - 3| < \epsilon$$

$$0 < \epsilon$$

Hence, any $\delta > 0$ will work.

Hence, for any $\delta > 0$, you have

$$|3 - 3| < \epsilon$$

$$|f(x) - L| < \epsilon$$

$$25. \lim_{x \rightarrow 2} (x^2 - 3) = 1 = L$$

$$|(x^2 - 3) - 1| < 0.01$$

$$|x^2 - 4| < 0.01$$

$$|(x + 2)(x - 2)| < 0.01$$

$$|x + 2| |x - 2| < 0.01$$

$$|x - 2| < \frac{0.01}{|x + 2|}$$

If we assume $1 < x < 3$, then $\delta = 0.01/5 = 0.002$.

Hence, if $0 < |x - 2| < \delta = 0.002$, you have

$$|x - 2| < 0.002 = \frac{1}{5}(0.01) < \frac{1}{|x + 2|}(0.01)$$

$$|x + 2| |x - 2| < 0.01$$

$$|x^2 - 4| < 0.01$$

$$|(x^2 - 3) - 1| < 0.01$$

$$|f(x) - L| < 0.01$$

$$29. \lim_{x \rightarrow -4} \left(\frac{1}{2}x - 1\right) = \frac{1}{2}(-4) - 1 = -3$$

Given $\epsilon > 0$:

$$\left|\left(\frac{1}{2}x - 1\right) - (-3)\right| < \epsilon$$

$$\left|\frac{1}{2}x + 2\right| < \epsilon$$

$$\frac{1}{2}|x - (-4)| < \epsilon$$

$$|x - (-4)| < 2\epsilon$$

Hence, let $\delta = 2\epsilon$.

Hence, if $0 < |x - (-4)| < \delta = 2\epsilon$, you have

$$|x - (-4)| < 2\epsilon$$

$$\left|\frac{1}{2}x + 2\right| < \epsilon$$

$$\left|\left(\frac{1}{2}x - 1\right) + 3\right| < \epsilon$$

$$|f(x) - L| < \epsilon$$

$$33. \lim_{x \rightarrow 0} \sqrt[3]{x} = 0$$

Given $\epsilon > 0$: $|\sqrt[3]{x} - 0| < \epsilon$

$$|\sqrt[3]{x}| < \epsilon$$

$$|x| < \epsilon^3 = \delta$$

Hence, let $\delta = \epsilon^3$.

Hence for $0 < |x - 0| < \delta = \epsilon^3$, you have

$$|x| < \epsilon^3$$

$$|\sqrt[3]{x}| < \epsilon$$

$$|\sqrt[3]{x} - 0| < \epsilon$$

$$|f(x) - L| < \epsilon$$

35. $\lim_{x \rightarrow -2} |x - 2| = |(-2) - 2| = 4$

Given $\epsilon > 0$:

$$||x - 2| - 4| < \epsilon$$

$$|-(x - 2) - 4| < \epsilon \quad (x - 2 < 0)$$

$$|-x - 2| = |x + 2| = |x - (-2)| < \epsilon$$

Hence, $\delta = \epsilon$.

Hence for $0 < |x - (-2)| < \delta = \epsilon$, you have

$$|x + 2| < \epsilon$$

$$|-(x + 2)| < \epsilon$$

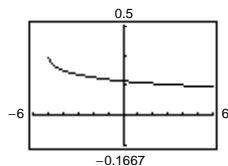
$$|-(x - 2) - 4| < \epsilon$$

$$||x - 2| - 4| < \epsilon \quad (\text{because } x - 2 < 0)$$

$$|f(x) - L| < \epsilon$$

39. $f(x) = \frac{\sqrt{x+5} - 3}{x - 4}$

$$\lim_{x \rightarrow 4} f(x) = \frac{1}{6}$$



The domain is $[-5, 4) \cup (4, \infty)$.

The graphing utility does not show the hole at $(4, \frac{1}{6})$.

37. $\lim_{x \rightarrow 1} (x^2 + 1) = 2$

Given $\epsilon > 0$:

$$|(x^2 + 1) - 2| < \epsilon$$

$$|x^2 - 1| < \epsilon$$

$$|(x + 1)(x - 1)| < \epsilon$$

$$|x - 1| < \frac{\epsilon}{|x + 1|}$$

If we assume $0 < x < 2$, then $\delta = \epsilon/3$.

Hence for $0 < |x - 1| < \delta = \frac{\epsilon}{3}$, you have

$$|x - 1| < \frac{1}{3}\epsilon < \frac{1}{|x + 1|}\epsilon$$

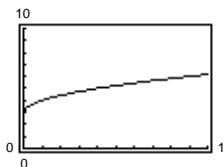
$$|x^2 - 1| < \epsilon$$

$$|(x^2 + 1) - 2| < \epsilon$$

$$|f(x) - 2| < \epsilon$$

41. $f(x) = \frac{x - 9}{\sqrt{x} - 3}$

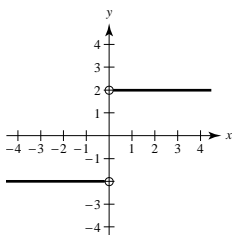
$$\lim_{x \rightarrow 9} f(x) = 6$$



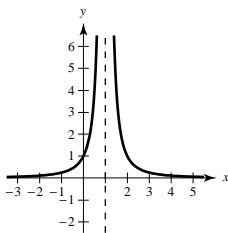
The domain is all $x \geq 0$ except $x = 9$. The graphing utility does not show the hole at $(9, 6)$.

43. $\lim_{x \rightarrow 8} f(x) = 25$ means that the values of f approach 25 as x gets closer and closer to 8.

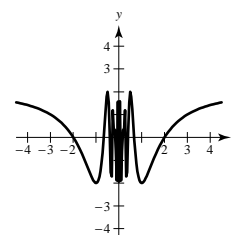
45. (i) The values of f approach different numbers as x approaches c from different sides of c :



(ii) The values of f increase without bound as x approaches c :

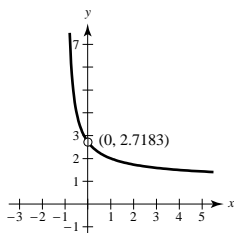


(iii) The values of f oscillate between two fixed numbers as x approaches c :



47. $f(x) = (1 + x)^{1/x}$

$$\lim_{x \rightarrow 0} (1 + x)^{1/x} = e \approx 2.71828$$



x	$f(x)$	x	$f(x)$
-0.1	2.867972	0.1	2.593742
-0.01	2.731999	0.01	2.704814
-0.001	2.719642	0.001	2.716942
-0.0001	2.718418	0.0001	2.718146
-0.00001	2.718295	0.00001	2.718268
-0.000001	2.718283	0.000001	2.718280

49. False; $f(x) = (\sin x)/x$ is undefined when $x = 0$.
From Exercise 7, we have

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1.$$

51. False; let

$$f(x) = \begin{cases} x^2 - 4x, & x \neq 4 \\ 10, & x = 4 \end{cases}$$

$$f(4) = 10$$

$$\lim_{x \rightarrow 4} f(x) = \lim_{x \rightarrow 4} (x^2 - 4x) = 0 \neq 10$$

53. Answers will vary.

55. If $\lim_{x \rightarrow c} f(x) = L_1$ and $\lim_{x \rightarrow c} f(x) = L_2$, then for every $\epsilon > 0$, there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that $|x - c| < \delta_1 \implies |f(x) - L_1| < \epsilon$ and $|x - c| < \delta_2 \implies |f(x) - L_2| < \epsilon$. Let δ equal the smaller of δ_1 and δ_2 . Then for $|x - c| < \delta$, we have
- $$|L_1 - L_2| = |L_1 - f(x) + f(x) - L_2| \leq |L_1 - f(x)| + |f(x) - L_2| < \epsilon + \epsilon.$$

Therefore, $|L_1 - L_2| < 2\epsilon$. Since $\epsilon > 0$ is arbitrary, it follows that $L_1 = L_2$.

57. $\lim_{x \rightarrow c} [f(x) - L] = 0$ means that for every $\epsilon > 0$ there exists $\delta > 0$ such that if

$$0 < |x - c| < \delta,$$

then

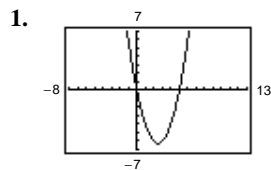
$$|(f(x) - L) - 0| < \epsilon.$$

This means the same as $|f(x) - L| < \epsilon$ when

$$0 < |x - c| < \delta.$$

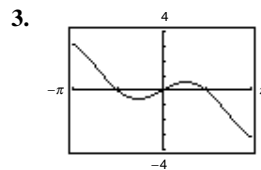
Thus, $\lim_{x \rightarrow c} f(x) = L$.

Section 1.3 Evaluating Limits Analytically



$$h(x) = x^2 - 5x$$

- (a) $\lim_{x \rightarrow 5} h(x) = 0$
(b) $\lim_{x \rightarrow -1} h(x) = 6$



$$f(x) = x \cos x$$

- (a) $\lim_{x \rightarrow 0} f(x) = 0$
(b) $\lim_{x \rightarrow \pi/3} f(x) \approx 0.524$
 $\left(= \frac{\pi}{6} \right)$

5. $\lim_{x \rightarrow 2} x^4 = 2^4 = 16$

7. $\lim_{x \rightarrow 0} (2x - 1) = 2(0) - 1 = -1$

9. $\lim_{x \rightarrow -3} (x^2 + 3x) = (-3)^2 + 3(-3) = 9 - 9 = 0$

11. $\lim_{x \rightarrow -3} (2x^2 + 4x + 1) = 2(-3)^2 + 4(-3) + 1 = 18 - 12 + 1 = 7$

13. $\lim_{x \rightarrow 2} \frac{1}{x} = \frac{1}{2}$

15. $\lim_{x \rightarrow 1} \frac{x - 3}{x^2 + 4} = \frac{1 - 3}{1^2 + 4} = \frac{-2}{5} = -\frac{2}{5}$

17. $\lim_{x \rightarrow 7} \frac{5x}{\sqrt{x} + 2} = \frac{5(7)}{\sqrt{7} + 2} = \frac{35}{\sqrt{7} + 2} = \frac{35}{3}$

19. $\lim_{x \rightarrow 3} \sqrt{x + 1} = \sqrt{3 + 1} = 2$

21. $\lim_{x \rightarrow -4} (x + 3)^2 = (-4 + 3)^2 = 1$

25. (a) $\lim_{x \rightarrow 1} f(x) = 4 - 1 = 3$

(b) $\lim_{x \rightarrow 3} g(x) = \sqrt{3 + 1} = 2$

(c) $\lim_{x \rightarrow 1} g(f(x)) = g(3) = 2$

29. $\lim_{x \rightarrow 2} \cos \frac{\pi x}{3} = \cos \frac{\pi 2}{3} = -\frac{1}{2}$

33. $\lim_{x \rightarrow 5\pi/6} \sin x = \sin \frac{5\pi}{6} = \frac{1}{2}$

37. (a) $\lim_{x \rightarrow c} [5g(x)] = 5 \lim_{x \rightarrow c} g(x) = 5(3) = 15$

(b) $\lim_{x \rightarrow c} [f(x) + g(x)] = \lim_{x \rightarrow c} f(x) + \lim_{x \rightarrow c} g(x) = 2 + 3 = 5$

(c) $\lim_{x \rightarrow c} [f(x)g(x)] = [\lim_{x \rightarrow c} f(x)][\lim_{x \rightarrow c} g(x)] = (2)(3) = 6$

(d) $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow c} f(x)}{\lim_{x \rightarrow c} g(x)} = \frac{2}{3}$

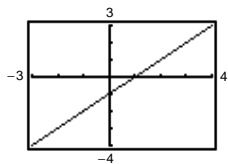
41. $f(x) = -2x + 1$ and $g(x) = \frac{-2x^2 + x}{x}$ agree except at $x = 0$.

(a) $\lim_{x \rightarrow 0} g(x) = \lim_{x \rightarrow 0} f(x) = 1$

(b) $\lim_{x \rightarrow -1} g(x) = \lim_{x \rightarrow -1} f(x) = 3$

45. $f(x) = \frac{x^2 - 1}{x + 1}$ and $g(x) = x - 1$ agree except at $x = -1$.

$$\lim_{x \rightarrow -1} f(x) = \lim_{x \rightarrow -1} g(x) = -2$$



49.
$$\begin{aligned} \lim_{x \rightarrow 5} \frac{x - 5}{x^2 - 25} &= \lim_{x \rightarrow 5} \frac{x - 5}{(x + 5)(x - 5)} \\ &= \lim_{x \rightarrow 5} \frac{1}{x + 5} = \frac{1}{10} \end{aligned}$$

23. (a) $\lim_{x \rightarrow 1} f(x) = 5 - 1 = 4$

(b) $\lim_{x \rightarrow 4} g(x) = 4^3 = 64$

(c) $\lim_{x \rightarrow 1} g(f(x)) = g(f(1)) = g(4) = 64$

27. $\lim_{x \rightarrow \pi/2} \sin x = \sin \frac{\pi}{2} = 1$

31. $\lim_{x \rightarrow 0} \sec 2x = \sec 0 = 1$

35. $\lim_{x \rightarrow 3} \tan\left(\frac{\pi x}{4}\right) = \tan \frac{3\pi}{4} = -1$

39. (a) $\lim_{x \rightarrow c} [f(x)]^3 = [\lim_{x \rightarrow c} f(x)]^3 = (4)^3 = 64$

(b) $\lim_{x \rightarrow c} \sqrt{f(x)} = \sqrt{\lim_{x \rightarrow c} f(x)} = \sqrt{4} = 2$

(c) $\lim_{x \rightarrow c} [3f(x)] = 3 \lim_{x \rightarrow c} f(x) = 3(4) = 12$

(d) $\lim_{x \rightarrow c} [f(x)]^{3/2} = [\lim_{x \rightarrow c} f(x)]^{3/2} = (4)^{3/2} = 8$

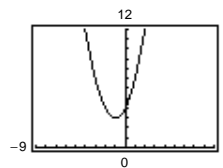
43. $f(x) = x(x + 1)$ and $g(x) = \frac{x^3 - x}{x - 1}$ agree except at $x = 1$.

(a) $\lim_{x \rightarrow 1} g(x) = \lim_{x \rightarrow 1} f(x) = 2$

(b) $\lim_{x \rightarrow -1} g(x) = \lim_{x \rightarrow -1} f(x) = 0$

47. $f(x) = \frac{x^3 - 8}{x - 2}$ and $g(x) = x^2 + 2x + 4$ agree except at $x = 2$.

$$\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} g(x) = 12$$



51.
$$\begin{aligned} \lim_{x \rightarrow -3} \frac{x^2 + x - 6}{x^2 - 9} &= \lim_{x \rightarrow -3} \frac{(x + 3)(x - 2)}{(x + 3)(x - 3)} \\ &= \lim_{x \rightarrow -3} \frac{x - 2}{x - 3} = \frac{-5}{-6} = \frac{5}{6} \end{aligned}$$

$$\begin{aligned}
 53. \lim_{x \rightarrow 0} \frac{\sqrt{x+5} - \sqrt{5}}{x} &= \lim_{x \rightarrow 0} \frac{\sqrt{x+5} - \sqrt{5}}{x} \cdot \frac{\sqrt{x+5} + \sqrt{5}}{\sqrt{x+5} + \sqrt{5}} \\
 &= \lim_{x \rightarrow 0} \frac{(x+5) - 5}{x(\sqrt{x+5} + \sqrt{5})} = \lim_{x \rightarrow 0} \frac{1}{\sqrt{x+5} + \sqrt{5}} = \frac{1}{2\sqrt{5}} = \frac{\sqrt{5}}{10}
 \end{aligned}$$

$$\begin{aligned}
 55. \lim_{x \rightarrow 4} \frac{\sqrt{x+5} - 3}{x-4} &= \lim_{x \rightarrow 4} \frac{\sqrt{x+5} - 3}{x-4} \cdot \frac{\sqrt{x+5} + 3}{\sqrt{x+5} + 3} \\
 &= \lim_{x \rightarrow 4} \frac{(x+5) - 9}{(x-4)(\sqrt{x+5} + 3)} = \lim_{x \rightarrow 4} \frac{1}{\sqrt{x+5} + 3} = \frac{1}{\sqrt{9} + 3} = \frac{1}{6}
 \end{aligned}$$

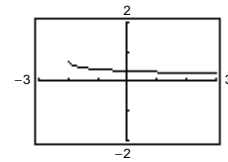
$$57. \lim_{x \rightarrow 0} \frac{\frac{1}{2+x} - \frac{1}{2}}{x} = \lim_{x \rightarrow 0} \frac{\frac{2 - (2+x)}{2(2+x)}}{x} = \lim_{x \rightarrow 0} \frac{-1}{2(2+x)} = -\frac{1}{4}$$

$$59. \lim_{\Delta x \rightarrow 0} \frac{2(x + \Delta x) - 2x}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{2x + 2\Delta x - 2x}{\Delta x} = \lim_{\Delta x \rightarrow 0} 2 = 2$$

$$\begin{aligned}
 61. \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^2 - 2(x + \Delta x) + 1 - (x^2 - 2x + 1)}{\Delta x} &= \lim_{\Delta x \rightarrow 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - 2x - 2\Delta x + 1 - x^2 + 2x - 1}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} (2x + \Delta x - 2) = 2x - 2
 \end{aligned}$$

$$63. \lim_{x \rightarrow 0} \frac{\sqrt{x+2} - \sqrt{2}}{x} \approx 0.354$$

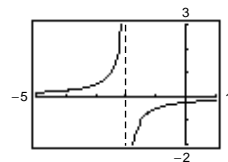
x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	0.358	0.354	0.345	?	0.354	0.353	0.349



$$\begin{aligned}
 \text{Analytically, } \lim_{x \rightarrow 0} \frac{\sqrt{x+2} - \sqrt{2}}{x} &= \lim_{x \rightarrow 0} \frac{\sqrt{x+2} - \sqrt{2}}{x} \cdot \frac{\sqrt{x+2} + \sqrt{2}}{\sqrt{x+2} + \sqrt{2}} \\
 &= \lim_{x \rightarrow 0} \frac{x + 2 - 2}{x(\sqrt{x+2} + \sqrt{2})} = \lim_{x \rightarrow 0} \frac{1}{\sqrt{x+2} + \sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4} \approx 0.354
 \end{aligned}$$

$$65. \lim_{x \rightarrow 0} \frac{\frac{1}{2+x} - \frac{1}{2}}{x} = -\frac{1}{4}$$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	-0.263	-0.251	-0.250	?	-0.250	-0.249	-0.238



$$\text{Analytically, } \lim_{x \rightarrow 0} \frac{\frac{1}{2+x} - \frac{1}{2}}{x} = \lim_{x \rightarrow 0} \frac{2 - (2+x)}{2(2+x)} \cdot \frac{1}{x} = \lim_{x \rightarrow 0} \frac{-x}{2(2+x)} \cdot \frac{1}{x} = \lim_{x \rightarrow 0} \frac{-1}{2(2+x)} = -\frac{1}{4}$$

$$67. \lim_{x \rightarrow 0} \frac{\sin x}{5x} = \lim_{x \rightarrow 0} \left[\left(\frac{\sin x}{x} \right) \left(\frac{1}{5} \right) \right] = (1) \left(\frac{1}{5} \right) = \frac{1}{5}$$

$$69. \lim_{x \rightarrow 0} \frac{\sin x(1 - \cos x)}{2x^2} = \lim_{x \rightarrow 0} \left[\frac{1}{2} \cdot \frac{\sin x}{x} \cdot \frac{1 - \cos x}{x} \right]$$

$$= \frac{1}{2}(1)(0) = 0$$

$$71. \lim_{x \rightarrow 0} \frac{\sin^2 x}{x} = \lim_{x \rightarrow 0} \left[\frac{\sin x}{x} \sin x \right] = (1) \sin 0 = 0$$

$$73. \lim_{h \rightarrow 0} \frac{(1 - \cos h)^2}{h} = \lim_{h \rightarrow 0} \left[\frac{1 - \cos h}{h} (1 - \cos h) \right]$$

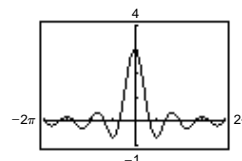
$$= (0)(0) = 0$$

$$75. \lim_{x \rightarrow \pi/2} \frac{\cos x}{\cot x} = \lim_{x \rightarrow \pi/2} \sin x = 1$$

$$77. \lim_{t \rightarrow 0} \frac{\sin 3t}{2t} = \lim_{t \rightarrow 0} \left(\frac{\sin 3t}{3t} \right) \left(\frac{3}{2} \right) = (1) \left(\frac{3}{2} \right) = \frac{3}{2}$$

$$79. f(t) = \frac{\sin 3t}{t}$$

t	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(t)$	2.96	2.9996	3	?	3	2.9996	2.96

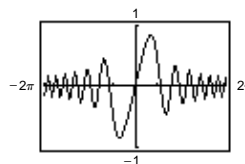


The limit appear to equal 3.

$$\text{Analytically, } \lim_{t \rightarrow 0} \frac{\sin 3t}{t} = \lim_{t \rightarrow 0} 3 \left(\frac{\sin 3t}{3t} \right) = 3(1) = 3.$$

$$81. f(x) = \frac{\sin x^2}{x}$$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	-0.099998	-0.01	-0.001	?	0.001	0.01	0.099998



$$\text{Analytically, } \lim_{x \rightarrow 0} \frac{\sin x^2}{x} = \lim_{x \rightarrow 0} x \left(\frac{\sin x^2}{x^2} \right) = 0(1) = 0.$$

$$83. \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{2(x+h) + 3 - (2x+3)}{h} = \lim_{h \rightarrow 0} \frac{2x+2h+3-2x-3}{h} = \lim_{h \rightarrow 0} \frac{2h}{h} = 2$$

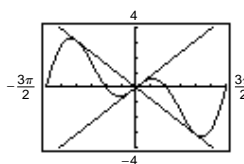
$$85. \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\frac{4}{x+h} - \frac{4}{x}}{h} = \lim_{h \rightarrow 0} \frac{4x - 4(x+h)}{(x+h)hx} = \lim_{h \rightarrow 0} \frac{-4}{(x+h)x} = \frac{-4}{x^2}$$

$$87. \lim_{x \rightarrow 0} (4 - x^2) \leq \lim_{x \rightarrow 0} f(x) \leq \lim_{x \rightarrow 0} (4 + x^2)$$

$$4 \leq \lim_{x \rightarrow 0} f(x) \leq 4$$

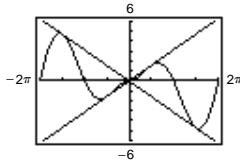
$$\text{Therefore, } \lim_{x \rightarrow 0} f(x) = 4.$$

$$89. f(x) = x \cos x$$



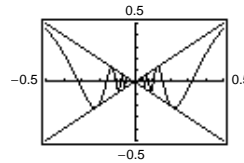
$$\lim_{x \rightarrow 0} (x \cos x) = 0$$

91. $f(x) = |x| \sin x$



$$\lim_{x \rightarrow 0} |x| \sin x = 0$$

93. $f(x) = x \sin \frac{1}{x}$



$$\lim_{x \rightarrow 0} \left(x \sin \frac{1}{x} \right) = 0$$

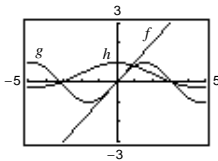
95. We say that two functions f and g agree at all but one point (on an open interval) if $f(x) = g(x)$ for all x in the interval except for $x = c$, where c is in the interval.

97. An indeterminate form is obtained when evaluating a limit using direct substitution produces a meaningless fractional expression such as $0/0$. That is,

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)}$$

$$\text{for which } \lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} g(x) = 0$$

99. $f(x) = x$, $g(x) = \sin x$, $h(x) = \frac{\sin x}{x}$



When you are “close to” 0 the magnitude of f is approximately equal to the magnitude of g . Thus, $|g|/|f| \approx 1$ when x is “close to” 0.

101. $s(t) = -16t^2 + 1000$

$$\lim_{t \rightarrow 5} \frac{s(5) - s(t)}{5 - t} = \lim_{t \rightarrow 5} \frac{600 - (-16t^2 + 1000)}{5 - t} = \lim_{t \rightarrow 5} \frac{16(t + 5)(t - 5)}{-(t - 5)} = \lim_{t \rightarrow 5} -16(t + 5) = -160 \text{ ft/sec.}$$

$$\text{Speed} = 160 \text{ ft/sec}$$

103. $s(t) = -4.9t^2 + 150$

$$\begin{aligned} \lim_{t \rightarrow 3} \frac{s(3) - s(t)}{3 - t} &= \lim_{t \rightarrow 3} \frac{-4.9(3^2) + 150 - (-4.9t^2 + 150)}{3 - t} = \lim_{t \rightarrow 3} \frac{-4.9(9 - t^2)}{3 - t} \\ &= \lim_{x \rightarrow 3} \frac{-4.9(3 - t)(3 + t)}{3 - t} = \lim_{x \rightarrow 3} -4.9(3 + t) = -29.4 \text{ m/sec} \end{aligned}$$

105. Let $f(x) = 1/x$ and $g(x) = -1/x$. $\lim_{x \rightarrow 0^+} f(x)$ and $\lim_{x \rightarrow 0^-} g(x)$ do not exist.

$$\lim_{x \rightarrow 0} [f(x) + g(x)] = \lim_{x \rightarrow 0} \left[\frac{1}{x} + \left(-\frac{1}{x} \right) \right] = \lim_{x \rightarrow 0} [0] = 0$$

107. Given $f(x) = b$, show that for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - b| < \epsilon$ whenever $|x - c| < \delta$. Since $|f(x) - b| = |b - b| = 0 < \epsilon$ for any $\epsilon > 0$, then any value of $\delta > 0$ will work.

109. If $b = 0$, then the property is true because both sides are equal to 0. If $b \neq 0$, let $\epsilon > 0$ be given. Since $\lim_{x \rightarrow c} f(x) = L$, there exists $\delta > 0$ such that $|f(x) - L| < \epsilon/|b|$ whenever $0 < |x - c| < \delta$. Hence, whenever $0 < |x - c| < \delta$, we have

$$|b||f(x) - L| < \epsilon \quad \text{or} \quad |bf(x) - bL| < \epsilon$$

which implies that $\lim_{x \rightarrow c} [bf(x)] = bL$.

$$111. \quad -M|f(x)| \leq f(x)g(x) \leq M|f(x)|$$

$$\lim_{x \rightarrow c} (-M|f(x)|) \leq \lim_{x \rightarrow c} f(x)g(x) \leq \lim_{x \rightarrow c} (M|f(x)|)$$

$$-M(0) \leq \lim_{x \rightarrow c} f(x)g(x) \leq M(0)$$

$$0 \leq \lim_{x \rightarrow c} f(x)g(x) \leq 0$$

Therefore, $\lim_{x \rightarrow c} f(x)g(x) = 0$.

115. True.

119. Let

$$f(x) = \begin{cases} 4, & \text{if } x \geq 0 \\ -4, & \text{if } x < 0 \end{cases}$$

$$\lim_{x \rightarrow 0} |f(x)| = \lim_{x \rightarrow 0} 4 = 4.$$

$\lim_{x \rightarrow 0} f(x)$ does not exist since for $x < 0$, $f(x) = -4$ and for $x \geq 0$, $f(x) = 4$.

$$121. \quad f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$$

$$g(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ x, & \text{if } x \text{ is irrational} \end{cases}$$

$\lim_{x \rightarrow 0} f(x)$ does not exist.

No matter how “close to” 0 x is, there are still an infinite number of rational and irrational numbers so that $\lim_{x \rightarrow 0} f(x)$ does not exist.

$$\lim_{x \rightarrow 0} g(x) = 0.$$

When x is “close to” 0, both parts of the function are “close to” 0.

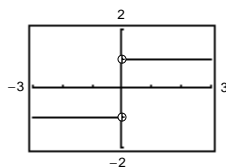
$$123. \quad (a) \quad \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} = \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} \cdot \frac{1 + \cos x}{1 + \cos x}$$

$$= \lim_{x \rightarrow 0} \frac{1 - \cos^2 x}{x^2(1 + \cos x)}$$

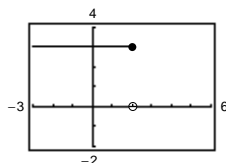
$$= \lim_{x \rightarrow 0} \frac{\sin^2 x}{x^2} \cdot \frac{1}{1 + \cos x}$$

$$= (1) \left(\frac{1}{2} \right) = \frac{1}{2}$$

113. False. As x approaches 0 from the left, $\frac{|x|}{x} = -1$.



117. False. The limit does not exist.



$$(b) \quad \text{Thus, } \frac{1 - \cos x}{x^2} \approx \frac{1}{2} \Rightarrow 1 - \cos x \approx \frac{1}{2}x^2$$

$$\Rightarrow \cos x \approx 1 - \frac{1}{2}x^2 \text{ for } x \approx 0.$$

$$(c) \quad \cos(0.1) \approx 1 - \frac{1}{2}(0.1)^2 = 0.995$$

$$(d) \quad \cos(0.1) \approx 0.9950, \text{ which agrees with part (c).}$$

Section 1.4 Continuity and One-Sided Limits

1. (a) $\lim_{x \rightarrow 3^+} f(x) = 1$

(b) $\lim_{x \rightarrow 3^-} f(x) = 1$

(c) $\lim_{x \rightarrow 3} f(x) = 1$

The function is continuous at $x = 3$.

3. (a) $\lim_{x \rightarrow 3^+} f(x) = 0$

(b) $\lim_{x \rightarrow 3^-} f(x) = 0$

(c) $\lim_{x \rightarrow 3} f(x) = 0$

The function is NOT continuous at $x = 3$.

5. (a) $\lim_{x \rightarrow 4^+} f(x) = 2$

(b) $\lim_{x \rightarrow 4^-} f(x) = -2$

(c) $\lim_{x \rightarrow 4} f(x)$ does not exist

The function is NOT continuous at $x = 4$.

7. $\lim_{x \rightarrow 5^+} \frac{x-5}{x^2-25} = \lim_{x \rightarrow 5^+} \frac{1}{x+5} = \frac{1}{10}$

9. $\lim_{x \rightarrow -3^-} \frac{x}{\sqrt{x^2-9}}$ does not exist because $\frac{x}{\sqrt{x^2-9}}$ grows without bound as $x \rightarrow -3^-$.

11. $\lim_{x \rightarrow 0^-} \frac{|x|}{x} = \lim_{x \rightarrow 0^-} \frac{-x}{x} = -1$.

$$\begin{aligned} 13. \lim_{\Delta x \rightarrow 0^-} \frac{\frac{1}{x+\Delta x} - \frac{1}{x}}{\Delta x} &= \lim_{\Delta x \rightarrow 0^-} \frac{x - (x + \Delta x)}{x(x + \Delta x)} \cdot \frac{1}{\Delta x} = \lim_{\Delta x \rightarrow 0^-} \frac{-\Delta x}{x(x + \Delta x)} \cdot \frac{1}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0^-} \frac{-1}{x(x + \Delta x)} \\ &= \frac{-1}{x(x + 0)} = -\frac{1}{x^2} \end{aligned}$$

15. $\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} \frac{x+2}{2} = \frac{5}{2}$

17. $\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (x+1) = 2$

$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (x^3 + 1) = 2$

$\lim_{x \rightarrow 1} f(x) = 2$

19. $\lim_{x \rightarrow \pi} \cot x$ does not exist since

$\lim_{x \rightarrow \pi^+} \cot x$ and $\lim_{x \rightarrow \pi^-} \cot x$ do not exist.

21. $\lim_{x \rightarrow 4^-} (3\llbracket x \rrbracket - 5) = 3(3) - 5 = 4$

$(\llbracket x \rrbracket = 3 \text{ for } 3 < x < 4)$

23. $\lim_{x \rightarrow 3} (2 - \llbracket -x \rrbracket)$ does not exist

because

$\lim_{x \rightarrow 3^-} (2 - \llbracket -x \rrbracket) = 2 - (-3) = 5$

and

$\lim_{x \rightarrow 3^+} (2 - \llbracket -x \rrbracket) = 2 - (-4) = 6$.

25. $f(x) = \frac{1}{x^2 - 4}$

has discontinuities at $x = -2$ and $x = 2$ since $f(-2)$ and $f(2)$ are not defined.

27. $f(x) = \frac{\llbracket x \rrbracket}{2} + x$

has discontinuities at each integer k since $\lim_{x \rightarrow k^-} f(x) \neq \lim_{x \rightarrow k^+} f(x)$.

29. $g(x) = \sqrt{25 - x^2}$ is continuous on $[-5, 5]$.

31. $\lim_{x \rightarrow 0^-} f(x) = 3 = \lim_{x \rightarrow 0^+} f(x)$.
 f is continuous on $[-1, 4]$.

33. $f(x) = x^2 - 2x + 1$ is continuous for all real x .

35. $f(x) = 3x - \cos x$ is continuous for all real x .

37. $f(x) = \frac{x}{x^2 - x}$ is not continuous at $x = 0, 1$. Since

$\frac{x}{x^2 - x} = \frac{1}{x - 1}$ for $x \neq 0, x = 0$ is a removable discontinuity, whereas $x = 1$ is a nonremovable discontinuity.

39. $f(x) = \frac{x}{x^2 + 1}$ is continuous for all real x .

41. $f(x) = \frac{x + 2}{(x + 2)(x - 5)}$

has a nonremovable discontinuity at $x = 5$ since $\lim_{x \rightarrow 5} f(x)$ does not exist, and has a removable discontinuity at $x = -2$ since

$$\lim_{x \rightarrow -2} f(x) = \lim_{x \rightarrow -2} \frac{1}{x - 5} = -\frac{1}{7}.$$

43. $f(x) = \frac{|x + 2|}{x + 2}$ has a nonremovable discontinuity at $x = -2$ since $\lim_{x \rightarrow -2} f(x)$ does not exist.

45. $f(x) = \begin{cases} x, & x \leq 1 \\ x^2, & x > 1 \end{cases}$

has a **possible** discontinuity at $x = 1$.

1. $f(1) = 1$

2. $\left. \begin{array}{l} \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} x = 1 \\ \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} x^2 = 1 \end{array} \right\} \lim_{x \rightarrow 1} f(x) = 1$

3. $f(1) = \lim_{x \rightarrow 1} f(x)$

f is continuous at $x = 1$, therefore, f is continuous for all real x .

47. $f(x) = \begin{cases} \frac{x}{2} + 1, & x \leq 2 \\ 3 - x, & x > 2 \end{cases}$ has a **possible** discontinuity at $x = 2$.

1. $f(2) = \frac{2}{2} + 1 = 2$

2. $\left. \begin{array}{l} \lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} \left(\frac{x}{2} + 1 \right) = 2 \\ \lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (3 - x) = 1 \end{array} \right\} \lim_{x \rightarrow 2} f(x) \text{ does not exist.}$

Therefore, f has a nonremovable discontinuity at $x = 2$.

49. $f(x) = \begin{cases} \tan \frac{\pi x}{4}, & |x| < 1 \\ x, & |x| \geq 1 \end{cases} = \begin{cases} \tan \frac{\pi x}{4}, & -1 < x < 1 \\ x, & x \leq -1 \text{ or } x \geq 1 \end{cases}$ has **possible** discontinuities at $x = -1, x = 1$.

1. $f(-1) = -1$ $f(1) = 1$

2. $\lim_{x \rightarrow -1} f(x) = -1$ $\lim_{x \rightarrow 1} f(x) = 1$

3. $f(-1) = \lim_{x \rightarrow -1} f(x)$ $f(1) = \lim_{x \rightarrow 1} f(x)$

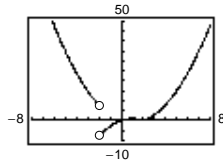
f is continuous at $x = \pm 1$, therefore, f is continuous for all real x .

51. $f(x) = \csc 2x$ has nonremovable discontinuities at integer multiples of $\pi/2$.

55. $\lim_{x \rightarrow 0^+} f(x) = 0$

$\lim_{x \rightarrow 0^-} f(x) = 0$

f is not continuous at $x = -2$.



53. $f(x) = \llbracket x - 1 \rrbracket$ has nonremovable discontinuities at each integer k .

57. $f(2) = 8$

Find a so that $\lim_{x \rightarrow 2^+} ax^2 = 8 \implies a = \frac{8}{2^2} = 2$.

59. Find a and b such that $\lim_{x \rightarrow -1^+} (ax + b) = -a + b = 2$ and $\lim_{x \rightarrow 3^-} (ax + b) = 3a + b = -2$.

$$a - b = -2$$

$$\begin{array}{r} (+) \\ 3a + b = -2 \end{array}$$

$$4a = -4$$

$$a = -1$$

$$b = 2 + (-1) = 1$$

$$f(x) = \begin{cases} 2, & x \leq -1 \\ -x + 1, & -1 < x < 3 \\ -2, & x \geq 3 \end{cases}$$

61. $f(g(x)) = (x - 1)^2$

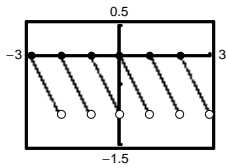
Continuous for all real x .

63. $f(g(x)) = \frac{1}{(x^2 + 5) - 6} = \frac{1}{x^2 - 1}$

Nonremovable discontinuities at $x = \pm 1$

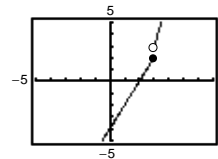
65. $y = \llbracket x \rrbracket - x$

Nonremovable discontinuity at each integer



67. $f(x) = \begin{cases} 2x - 4, & x \leq 3 \\ x^2 - 2x, & x > 3 \end{cases}$

Nonremovable discontinuity at $x = 3$



69. $f(x) = \frac{x}{x^2 + 1}$

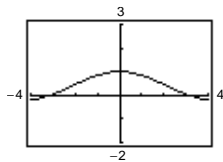
Continuous on $(-\infty, \infty)$

71. $f(x) = \sec \frac{\pi x}{4}$

Continuous on:

$\dots, (-6, -2), (-2, 2), (2, 6), (6, 10), \dots$

73. $f(x) = \frac{\sin x}{x}$



The graph **appears** to be continuous on the interval $[-4, 4]$. Since $f(0)$ is not defined, we know that f has a discontinuity at $x = 0$. This discontinuity is removable so it does not show up on the graph.

75. $f(x) = \frac{1}{16}x^4 - x^3 + 3$ is continuous on $[1, 2]$.

$f(1) = \frac{33}{16}$ and $f(2) = -4$. By the Intermediate Value Theorem, $f(c) = 0$ for at least one value of c between 1 and 2.

77. $f(x) = x^2 - 2 - \cos x$ is continuous on $[0, \pi]$.

$f(0) = -3$ and $f(\pi) = \pi^2 - 1 > 0$. By the Intermediate Value Theorem, $f(c) = 0$ for the least one value of c between 0 and π .

81. $g(t) = 2 \cos t - 3t$

g is continuous on $[0, 1]$.

$g(0) = 2 > 0$ and $g(1) \approx -1.9 < 0$.

By the Intermediate Value Theorem, $g(t) = 0$ for at least one value c between 0 and 1. Using a graphing utility, we find that $t \approx 0.5636$.

85. $f(x) = x^3 - x^2 + x - 2$

f is continuous on $[0, 3]$.

$f(0) = -2$ and $f(3) = 19$

$-2 < 4 < 19$

The Intermediate Value Theorem applies.

$$x^3 - x^2 + x - 2 = 4$$

$$x^3 - x^2 + x - 6 = 0$$

$$(x - 2)(x^2 + x + 3) = 0$$

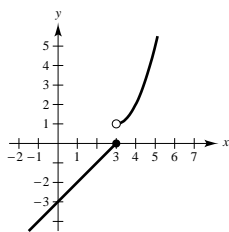
$$x = 2$$

($x^2 + x + 3$ has no real solution.)

$$c = 2$$

Thus, $f(2) = 4$.

89.



The function is not continuous at $x = 3$ because

$$\lim_{x \rightarrow 3^+} f(x) = 1 \neq 0 = \lim_{x \rightarrow 3^-} f(x).$$

79. $f(x) = x^3 + x - 1$

$f(x)$ is continuous on $[0, 1]$.

$f(0) = -1$ and $f(1) = 1$

By the Intermediate Value Theorem, $f(x) = 0$ for at least one value of c between 0 and 1. Using a graphing utility, we find that $x \approx 0.6823$.

83. $f(x) = x^2 + x - 1$

f is continuous on $[0, 5]$.

$f(0) = -1$ and $f(5) = 29$

$-1 < 11 < 29$

The Intermediate Value Theorem applies.

$$x^2 + x - 1 = 11$$

$$x^2 + x - 12 = 0$$

$$(x + 4)(x - 3) = 0$$

$$x = -4 \text{ or } x = 3$$

$c = 3$ ($x = -4$ is not in the interval.)

Thus, $f(3) = 11$.

87. (a) The limit does not exist at $x = c$.

(b) The function is not defined at $x = c$.

(c) The limit exists at $x = c$, but it is not equal to the value of the function at $x = c$.

(d) The limit does not exist at $x = c$.

91. The functions agree for integer values of x :

$$\left. \begin{aligned} g(x) &= 3 - \lfloor -x \rfloor = 3 - (-x) = 3 + x \\ f(x) &= 3 + \lfloor x \rfloor = 3 + x \end{aligned} \right\} \text{ for } x \text{ an integer}$$

However, for non-integer values of x , the functions differ by 1.

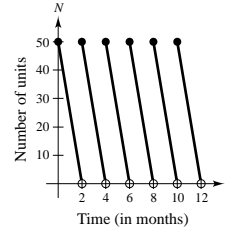
$$f(x) = 3 + \lfloor x \rfloor = g(x) - 1 = 2 - \lfloor -x \rfloor.$$

For example, $f(\frac{1}{2}) = 3 + 0 = 3$, $g(\frac{1}{2}) = 3 - (-1) = 4$.

93. $N(t) = 25\left(2\left\lceil\frac{t+2}{2}\right\rceil - t\right)$

t	0	1	1.8	2	3	3.8
$N(t)$	50	25	5	50	25	5

Discontinuous at every positive even integer. The company replenishes its inventory every two months.


 95. Let $V = \frac{4}{3}\pi r^3$ be the volume of a sphere of radius r .

$$V(1) = \frac{4}{3}\pi \approx 4.19$$

$$V(5) = \frac{4}{3}\pi(5^3) \approx 523.6$$

Since $4.19 < 275 < 523.6$, the Intermediate Value Theorem implies that there is at least one value r between 1 and 5 such that $V(r) = 275$. (In fact, $r \approx 4.0341$.)

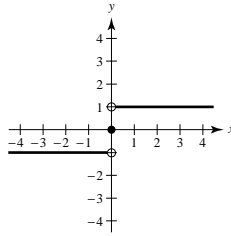
 97. Let c be any real number. Then $\lim_{x \rightarrow c} f(x)$ does not exist since there are both rational and irrational numbers arbitrarily close to c . Therefore, f is not continuous at c .

99. $\operatorname{sgn}(x) = \begin{cases} -1, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ 1, & \text{if } x > 0 \end{cases}$

(a) $\lim_{x \rightarrow 0^-} \operatorname{sgn}(x) = -1$

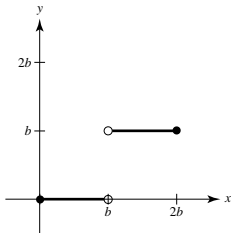
(b) $\lim_{x \rightarrow 0^+} \operatorname{sgn}(x) = 1$

(c) $\lim_{x \rightarrow 0} \operatorname{sgn}(x)$ does not exist.


 101. True; if $f(x) = g(x)$, $x \neq c$, then $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} g(x)$ and at least one of these limits (if they exist) does not equal the corresponding function at $x = c$.

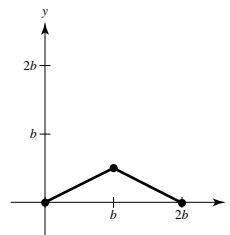
 103. False; $f(1)$ is not defined and $\lim_{x \rightarrow 1} f(x)$ does not exist.

105. (a) $f(x) = \begin{cases} 0 & 0 \leq x < b \\ b & b < x \leq 2b \end{cases}$



NOT continuous at $x = b$.

(b) $g(x) = \begin{cases} \frac{x}{2} & 0 \leq x \leq b \\ b - \frac{x}{2} & b < x \leq 2b \end{cases}$



Continuous on $[0, 2b]$.

$$107. f(x) = \frac{\sqrt{x+c^2}-c}{x}, \quad c > 0$$

Domain: $x + c^2 \geq 0 \Rightarrow x \geq -c^2$ and $x \neq 0$, $[-c^2, 0) \cup (0, \infty)$

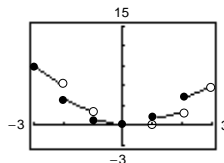
$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sqrt{x+c^2}-c}{x} &= \lim_{x \rightarrow 0} \frac{\sqrt{x+c^2}-c}{x} \cdot \frac{\sqrt{x+c^2}+c}{\sqrt{x+c^2}+c} \\ &= \lim_{x \rightarrow 0} \frac{(x+c^2)-c^2}{x[\sqrt{x+c^2}+c]} = \lim_{x \rightarrow 0} \frac{1}{\sqrt{x+c^2}+c} = \frac{1}{2c} \end{aligned}$$

Define $f(0) = 1/(2c)$ to make f continuous at $x = 0$.

$$109. h(x) = x \llbracket x \rrbracket$$

h has nonremovable discontinuities at

$$x = \pm 1, \pm 2, \pm 3, \dots$$



Section 1.5 Infinite Limits

$$1. \lim_{x \rightarrow -2^+} 2 \left| \frac{x}{x^2 - 4} \right| = \infty$$

$$3. \lim_{x \rightarrow -2^+} \tan \frac{\pi x}{4} = -\infty$$

$$\lim_{x \rightarrow -2^-} 2 \left| \frac{x}{x^2 - 4} \right| = \infty$$

$$\lim_{x \rightarrow -2^-} \tan \frac{\pi x}{4} = \infty$$

$$5. f(x) = \frac{1}{x^2 - 9}$$

x	-3.5	-3.1	-3.01	-3.001	-2.999	-2.99	-2.9	-2.5
$f(x)$	0.308	1.639	16.64	166.6	-166.7	-16.69	-1.695	-0.364

$$\lim_{x \rightarrow -3^-} f(x) = \infty$$

$$\lim_{x \rightarrow -3^+} f(x) = -\infty$$

$$7. f(x) = \frac{x^2}{x^2 - 9}$$

x	-3.5	-3.1	-3.01	-3.001	-2.999	-2.99	-2.9	-2.5
$f(x)$	3.769	15.75	150.8	1501	-1499	-149.3	-14.25	-2.273

$$\lim_{x \rightarrow -3^-} f(x) = \infty$$

$$\lim_{x \rightarrow -3^+} f(x) = -\infty$$

$$9. \lim_{x \rightarrow 0^+} \frac{1}{x^2} = \infty = \lim_{x \rightarrow 0^-} \frac{1}{x^2}$$

Therefore, $x = 0$ is a vertical asymptote.

$$13. \lim_{x \rightarrow -2} \frac{x^2}{x^2 - 4} = \infty \text{ and } \lim_{x \rightarrow -2^+} \frac{x^2}{x^2 - 4} = -\infty$$

Therefore, $x = -2$ is a vertical asymptote.

$$\lim_{x \rightarrow 2^-} \frac{x^2}{x^2 - 4} = -\infty \text{ and } \lim_{x \rightarrow 2^+} \frac{x^2}{x^2 - 4} = \infty$$

Therefore, $x = 2$ is a vertical asymptote.

$$17. f(x) = \tan 2x = \frac{\sin 2x}{\cos 2x} \text{ has vertical asymptotes at}$$

$$x = \frac{(2n + 1)\pi}{4} = \frac{\pi}{4} + \frac{n\pi}{2}, n \text{ any integer.}$$

$$21. \lim_{x \rightarrow -2^+} \frac{x}{(x + 2)(x - 1)} = \infty$$

$$\lim_{x \rightarrow -2^-} \frac{x}{(x + 2)(x - 1)} = -\infty$$

Therefore, $x = -2$ is a vertical asymptote.

$$\lim_{x \rightarrow 1^+} \frac{x}{(x + 2)(x - 1)} = \infty$$

$$\lim_{x \rightarrow 1^-} \frac{x}{(x + 2)(x - 1)} = -\infty$$

Therefore, $x = 1$ is a vertical asymptote.

$$25. f(x) = \frac{(x - 5)(x + 3)}{(x - 5)(x^2 + 1)} = \frac{x + 3}{x^2 + 1}, x \neq 5$$

No vertical asymptotes. The graph has a hole at $x = 5$.

$$11. \lim_{x \rightarrow 2^+} \frac{x^2 - 2}{(x - 2)(x + 1)} = \infty$$

$$\lim_{x \rightarrow 2^-} \frac{x^2 - 2}{(x - 2)(x + 1)} = -\infty$$

Therefore, $x = 2$ is a vertical asymptote.

$$\lim_{x \rightarrow -1^+} \frac{x^2 - 2}{(x - 2)(x + 1)} = \infty$$

$$\lim_{x \rightarrow -1^-} \frac{x^2 - 2}{(x - 2)(x + 1)} = -\infty$$

Therefore, $x = -1$ is a vertical asymptote.

15. No vertical asymptote since the denominator is never zero.

$$19. \lim_{t \rightarrow 0^+} \left(1 - \frac{4}{t^2}\right) = -\infty = \lim_{t \rightarrow 0^-} \left(1 - \frac{4}{t^2}\right)$$

Therefore, $t = 0$ is a vertical asymptote.

$$23. f(x) = \frac{x^3 + 1}{x + 1} = \frac{(x + 1)(x^2 - x + 1)}{x + 1}$$

has no vertical asymptote since

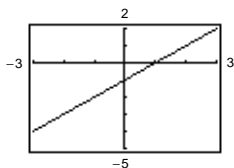
$$\lim_{x \rightarrow -1} f(x) = \lim_{x \rightarrow -1} (x^2 - x + 1) = 3$$

$$27. s(t) = \frac{t}{\sin t} \text{ has vertical asymptotes at } t = n\pi, n$$

a nonzero integer. There is no vertical asymptote at $t = 0$ since

$$\lim_{t \rightarrow 0} \frac{t}{\sin t} = 1.$$

29. $\lim_{x \rightarrow -1} \frac{x^2 - 1}{x + 1} = \lim_{x \rightarrow -1} (x - 1) = -2$



Removable discontinuity at $x = -1$

33. $\lim_{x \rightarrow 2^+} \frac{x - 3}{x - 2} = -\infty$

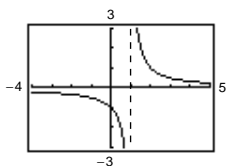
37. $\lim_{x \rightarrow -3^-} \frac{x^2 + 2x - 3}{x^2 + x - 6} = \lim_{x \rightarrow -3^-} \frac{x - 1}{x - 2} = \frac{4}{5}$

41. $\lim_{x \rightarrow 0^+} \left(1 + \frac{1}{x}\right) = -\infty$

45. $\lim_{x \rightarrow \pi} \frac{\sqrt{x}}{\csc x} = \lim_{x \rightarrow \pi} (\sqrt{x} \sin x) = 0$

49. $f(x) = \frac{x^2 + x + 1}{x^3 - 1}$

$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} \frac{1}{x - 1} = \infty$

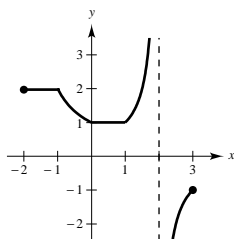


53. A limit in which $f(x)$ increases or decreases without bound as x approaches c is called an infinite limit. ∞ is not a number. Rather, the symbol

$\lim_{x \rightarrow c} f(x) = \infty$

says how the limit fails to exist.

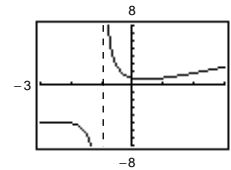
57.



31. $\lim_{x \rightarrow -1^+} \frac{x^2 + 1}{x + 1} = \infty$

$\lim_{x \rightarrow -1^-} \frac{x^2 + 1}{x + 1} = -\infty$

Vertical asymptote at $x = -1$



35. $\lim_{x \rightarrow 3^+} \frac{x^2}{(x - 3)(x + 3)} = \infty$

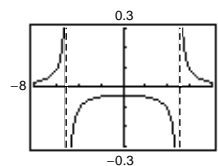
39. $\lim_{x \rightarrow 1} \frac{x^2 - x}{(x^2 + 1)(x - 1)} = \lim_{x \rightarrow 1} \frac{x}{x^2 + 1} = \frac{1}{2}$

43. $\lim_{x \rightarrow 0^+} \frac{2}{\sin x} = \infty$

47. $\lim_{x \rightarrow (1/2)^-} x \sec(\pi x) = \infty$ and $\lim_{x \rightarrow (1/2)^+} x \sec(\pi x) = -\infty$.
Therefore, $\lim_{x \rightarrow (1/2)} x \sec(\pi x)$ does not exist.

51. $f(x) = \frac{1}{x^2 - 25}$

$\lim_{x \rightarrow 5^-} f(x) = -\infty$



55. One answer is $f(x) = \frac{x - 3}{(x - 6)(x + 2)} = \frac{x - 3}{x^2 - 4x - 12}$.

59. $S = \frac{k}{1 - r}$, $0 < |r| < 1$. Assume $k \neq 0$.

$\lim_{r \rightarrow 1} S = \lim_{r \rightarrow 1} \frac{k}{1 - r} = \infty$ (or $-\infty$ if $k < 0$)

$$61. C = \frac{528x}{100 - x}, 0 \leq x < 100$$

(a) $C(25) = \$176$ million

(b) $C(50) = \$528$ million

(c) $C(75) = \$1584$ million

(d) $\lim_{x \rightarrow 100^-} \frac{528}{100 - x} = \infty$ Thus, it is not possible.

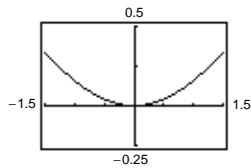
$$63. (a) r = \frac{2(7)}{\sqrt{625 - 49}} = \frac{7}{12} \text{ ft/sec}$$

$$(b) r = \frac{2(15)}{\sqrt{625 - 225}} = \frac{3}{2} \text{ ft/sec}$$

$$(c) \lim_{x \rightarrow 25^-} \frac{2x}{\sqrt{625 - x^2}} = \infty$$

65. (a)

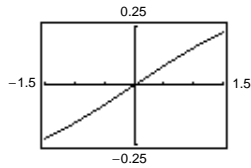
x	1	0.5	0.2	0.1	0.01	0.001	0.0001
$f(x)$	0.1585	0.0411	0.0067	0.0017	≈ 0	≈ 0	≈ 0



$$\lim_{x \rightarrow 0^+} \frac{x - \sin x}{x} = 0$$

(b)

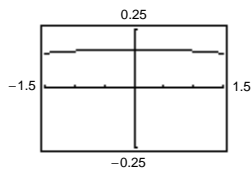
x	1	0.5	0.2	0.1	0.01	0.001	0.0001
$f(x)$	0.1585	0.0823	0.0333	0.0167	0.0017	≈ 0	≈ 0



$$\lim_{x \rightarrow 0^+} \frac{x - \sin x}{x^2} = 0$$

(c)

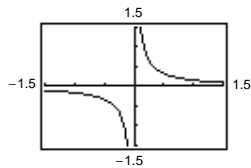
x	1	0.5	0.2	0.1	0.01	0.001	0.0001
$f(x)$	0.1585	0.1646	0.1663	0.1666	0.1667	0.1667	0.1667



$$\lim_{x \rightarrow 0^+} \frac{x - \sin x}{x^3} = 0.1167 \text{ (1/6)}$$

(d)

x	1	0.5	0.2	0.1	0.01	0.001	0.0001
$f(x)$	0.1585	0.3292	0.8317	1.6658	16.67	166.7	1667.0



$$\lim_{x \rightarrow 0^+} \frac{x - \sin x}{x^4} = \infty$$

$$\text{For } n \geq 3, \lim_{x \rightarrow 0^+} \frac{x - \sin x}{x^n} = \infty.$$

67. (a) Because the circumference of the motor is half that of the saw arbor, the saw makes $1700/2 = 850$ revolutions per minute.
- (c) $2(20 \cot \phi) + 2(10 \cot \phi)$: straight sections.
The angle subtended in each circle is

$$2\pi - \left(2\left(\frac{\pi}{2} - \phi\right)\right) = \pi + 2\phi.$$

Thus, the length of the belt around the pulleys is

$$20(\pi + 2\phi) + 10(\pi + 2\phi) = 30(\pi + 2\phi).$$

$$\text{Total length} = 60 \cot \phi + 30(\pi + 2\phi)$$

$$\text{Domain: } \left(0, \frac{\pi}{2}\right)$$

69. False; for instance, let

$$f(x) = \frac{x^2 - 1}{x - 1} \text{ or}$$

$$g(x) = \frac{x}{x^2 + 1}.$$

73. Given $\lim_{x \rightarrow c} f(x) = \infty$ and $\lim_{x \rightarrow c} g(x) = L$:

(2) Product:

If $L > 0$, then for $\epsilon = L/2 > 0$ there exists $\delta_1 > 0$ such that $|g(x) - L| < L/2$ whenever $0 < |x - c| < \delta_1$. Thus, $L/2 < g(x) < 3L/2$. Since $\lim_{x \rightarrow c} f(x) = \infty$ then for $M > 0$, there exists $\delta_2 > 0$ such that $f(x) > M(2/L)$ whenever $|x - c| < \delta_2$. Let δ be the smaller of δ_1 and δ_2 . Then for $0 < |x - c| < \delta$, we have $f(x)g(x) > M(2/L)(L/2) = M$. Therefore $\lim_{x \rightarrow c} f(x)g(x) = \infty$. The proof is similar for $L < 0$.

(3) Quotient: Let $\epsilon > 0$ be given.

There exists $\delta_1 > 0$ such that $f(x) > 3L/2\epsilon$ whenever $0 < |x - c| < \delta_1$ and there exists $\delta_2 > 0$ such that $|g(x) - L| < L/2$ whenever $0 < |x - c| < \delta_2$. This inequality gives us $L/2 < g(x) < 3L/2$. Let δ be the smaller of δ_1 and δ_2 . Then for $0 < |x - c| < \delta$, we have

$$\left| \frac{g(x)}{f(x)} \right| < \frac{3L/2}{3L/2\epsilon} = \epsilon.$$

Therefore, $\lim_{x \rightarrow c} \frac{g(x)}{f(x)} = 0$.

75. Given $\lim_{x \rightarrow c} \frac{1}{f(x)} = 0$.

Suppose $\lim_{x \rightarrow c} f(x)$ exists and equals L . Then,

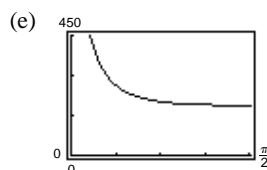
$$\lim_{x \rightarrow c} \frac{1}{f(x)} = \frac{\lim_{x \rightarrow c} 1}{\lim_{x \rightarrow c} f(x)} = \frac{1}{L} = 0.$$

This is not possible. Thus, $\lim_{x \rightarrow c} f(x)$ does not exist.

- (b) The direction of rotation is reversed.

(d)

ϕ	0.3	0.6	0.9	1.2	1.5
L	306.2	217.9	195.9	189.6	188.5



(f) $\lim_{\phi \rightarrow (\pi/2)^-} L = 60\pi \approx 188.5$

(All the belts are around pulleys.)

(g) $\lim_{\phi \rightarrow 0^+} L = \infty$

71. False; let

$$f(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 3, & x = 0. \end{cases}$$

The graph of f has a vertical asymptote at $x = 0$, but $f(0) = 3$.