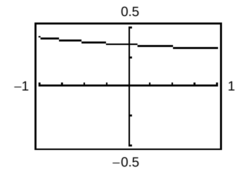
Review Exercises for Chapter 1

2. Precalculus. $L = \sqrt{(9-1)^2 + (3-1)^2} \approx 8.25$

4.

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	0.358	0.354	0.354	0.354	0.353	0.349

$\lim_{x\to 0} f(x) \approx 0.2$



6. $g(x) = \dfrac{3x}{x-2}$ (a) $\lim_{x\to 2} g(x)$ does not exist. (b) $\lim_{x\to 0} g(x) = 0$

8. $\lim_{x\to 9} \sqrt{x} = \sqrt{9} = 3.$

Let $\epsilon > 0$ be given. We need

$$\left|\sqrt{x} - 3\right| < \epsilon \implies \left|\sqrt{x}+3\right|\left|\sqrt{x}-3\right| < \epsilon\left|\sqrt{x}+3\right|$$
$$|x-9| < \epsilon\left|\sqrt{x}+3\right|$$

Assuming $4 < x < 16$, you can choose $\delta = 5\epsilon$.

Hence, for $0 < |x-9| < \delta = 5\epsilon$, you have

$$|x-9| < 5\epsilon < \left|\sqrt{x}+3\right|\epsilon$$
$$\left|\sqrt{x}-3\right| < \epsilon$$
$$|f(x) - L| < \epsilon$$

10. $\lim_{x\to 5} 9 = 9$. Let $\epsilon > 0$ be given. δ can be any positive number. Hence, for $0 < |x-5| < \delta$, you have

$$|9-9| < \epsilon$$
$$|f(x) - L| < \epsilon$$

12. $\lim_{y\to 4} 3|y-1| = 3|4-1| = 9$

14. $\lim_{t\to 3} \dfrac{t^2-9}{t-3} = \lim_{t\to 3} (t+3) = 6$

16. $\lim_{x\to 0} \dfrac{\sqrt{4+x}-2}{x} = \lim_{x\to 0} \dfrac{\sqrt{4+x}-2}{x} \cdot \dfrac{\sqrt{4+x}+2}{\sqrt{4+x}+2}$

$$= \lim_{x\to 0} \frac{1}{\sqrt{4+x}+2} = \frac{1}{4}$$

18. $\lim_{s\to 0} \dfrac{\left(1/\sqrt{1+s}\right)-1}{s} = \lim_{s\to 0} \left[\dfrac{\left(1/\sqrt{1+s}\right)-1}{s} \cdot \dfrac{\left(1/\sqrt{1+s}\right)+1}{\left(1/\sqrt{1+s}\right)+1}\right]$

$$= \lim_{s\to 0} \frac{[1/(1+s)]-1}{s\left[\left(1/\sqrt{1+s}\right)+1\right]} = \lim_{s\to 0} \frac{-1}{(1+s)\left[\left(1/\sqrt{1+s}\right)+1\right]} = -\frac{1}{2}$$

20. $\lim_{x\to -2} \dfrac{x^2-4}{x^3+8} = \lim_{x\to -2} \dfrac{(x+2)(x-2)}{(x+2)(x^2-2x+4)}$

$$= \lim_{x\to -2} \frac{x-2}{x^2-2x+4}$$

$$= -\frac{4}{12} = -\frac{1}{3}$$

22. $\lim_{x\to(\pi/4)} \dfrac{4x}{\tan x} = \dfrac{4(\pi/4)}{1} = \pi$

24. $$\lim_{\Delta x \to 0} \frac{\cos(\pi + \Delta x) + 1}{\Delta x} = \lim_{\Delta x \to 0} \frac{\cos \pi \cos \Delta x - \sin \pi \sin \Delta x + 1}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \left[-\frac{(\cos \Delta x - 1)}{\Delta x} \right] - \lim_{\Delta x \to 0} \left[\sin \pi \frac{\sin \Delta x}{\Delta x} \right]$$
$$= -0 - (0)(1) = 0$$

26. $\lim_{x \to c} [f(x) + 2g(x)] = -\frac{3}{4} + 2\left(\frac{2}{3}\right) = \frac{7}{12}$

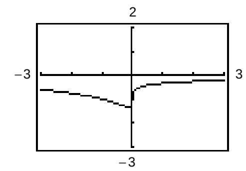
28. $f(x) = \dfrac{1 - \sqrt[3]{x}}{x - 1}$

(a)

x	1.1	1.01	1.001	1.0001
$f(x)$	-0.3228	-0.3322	-0.3332	-0.3333

$\lim_{x \to 1^+} \dfrac{1 - \sqrt[3]{x}}{x - 1} \approx -0.333$ $\left(\text{Actual limit is } -\frac{1}{3}.\right)$

(b)



(c) $$\lim_{x \to 1^+} \frac{1 - \sqrt[3]{x}}{x - 1} = \lim_{x \to 1^+} \frac{1 - \sqrt[3]{x}}{x - 1} \cdot \frac{1 + \sqrt[3]{x} + (\sqrt[3]{x})^2}{1 + \sqrt[3]{x} + (\sqrt[3]{x})^2}$$
$$= \lim_{x \to 1^+} \frac{1 - x}{(x - 1)\left[1 + \sqrt[3]{x} + (\sqrt[3]{x})^2\right]}$$
$$= \lim_{x \to 1^+} \frac{-1}{1 + \sqrt[3]{x} + (\sqrt[3]{x})^2}$$
$$= -\frac{1}{3}$$

30. $s(t) = 0 \implies -4.9t^2 + 200 = 0 \implies t^2 \approx 40.816 \implies t \approx 6.39$ sec

When $t = 6.39$, the velocity is approximately

$$\lim_{t \to a} \frac{s(a) - s(t)}{a - t} = \lim_{t \to a} -4.9(a + t)$$
$$= \lim_{t \to 6.39} -4.9(6.39 + 6.39) = -62.6 \text{ m/sec.}$$

32. $\lim_{x \to 4} [\![x - 1]\!]$ does not exist. The graph jumps from 2 to 3 at $x = 4$.

34. $\lim_{x \to 1^+} g(x) = 1 + 1 = 2.$

36. $\lim_{s \to -2} f(s) = 2$

38. $f(x) = \begin{cases} \dfrac{3x^2 - x - 2}{x - 1}, & x \neq 1 \\ 0, & x = 1 \end{cases}$

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{3x^2 - x - 2}{x - 1}$$
$$= \lim_{x \to 1} (3x + 2) = 5 \neq 0$$

Removable discontinuity at $x = 1$
Continuous on $(-\infty, 1) \cup (1, \infty)$

40. $f(x) = \begin{cases} 5 - x, & x \le 2 \\ 2x - 3, & x > 2 \end{cases}$

$\lim_{x\to 2^-} (5 - x) = 3$

$\lim_{x\to 2^+} (2x - 3) = 1$

Nonremovable discontinuity at $x = 2$
Continuous on $(-\infty, 2) \cup (2, \infty)$

42. $f(x) = \sqrt{\dfrac{x+1}{x}} = \sqrt{1 + \dfrac{1}{x}}$

$\lim_{x\to 0^+} \sqrt{1 + \dfrac{1}{x}} = \infty$

Domain: $(-\infty, -1], (0, \infty)$

Nonremovable discontinuity at $x = 0$
Continuous on $(-\infty, -1] \cup (0, \infty)$

44. $f(x) = \dfrac{x+1}{2x+2}$

$\lim_{x\to -1} \dfrac{x+1}{2(x+1)} = \dfrac{1}{2}$

Removable discontinuity at $x = -1$
Continuous on $(-\infty, -1) \cup (-1, \infty)$

46. $f(x) = \tan 2x$

Nonremovable discontinuities when

$$x = \frac{(2n+1)\pi}{4}$$

Continuous on

$$\left(\frac{(2n-1)\pi}{4}, \frac{(2n+1)\pi}{4}\right)$$

for all integers n.

48. $\lim_{x\to 1^+} (x + 1) = 2$

$\lim_{x\to 3^-} (x + 1) = 4$

Find b and c so that $\lim_{x\to 1^-} (x^2 + bx + c) = 2$ and $\lim_{x\to 3^+} (x^2 + bx + c) = 4$.

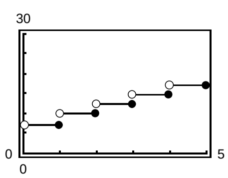
Consequently we get $1 + b + c = 2$ and $9 + 3b + c = 4$.

Solving simultaneously, $b = -3$ and $c = 4$.

50. $C = 9.80 + 2.50[-[\![-x]\!] - 1], \ x > 0$

$= 9.80 - 2.50[[\![-x]\!] + 1]$

C has a nonremovable discontinuity at each integer.



52. $f(x) = \sqrt{(x-1)x}$

(a) Domain: $(-\infty, 0] \cup [1, \infty)$

(b) $\lim_{x\to 0^-} f(x) = 0$

(c) $\lim_{x\to 1^+} f(x) = 0$

54. $h(x) = \dfrac{4x}{4 - x^2}$

Vertical asymptotes at $x = 2$ and $x = -2$

56. $f(x) = \csc \pi x$

Vertical asymptote at every integer k

58. $\lim_{x\to (1/2)^+} \dfrac{x}{2x - 1} = \infty$

60. $\lim_{x\to -1^-} \dfrac{x+1}{x^4 - 1} = \lim_{x\to -1^-} \dfrac{1}{(x^2+1)(x-1)} = -\dfrac{1}{4}$

62. $\lim_{x\to -1^+} \dfrac{x^2 - 2x + 1}{x + 1} = \infty$

64. $\lim_{x\to 2^-} \dfrac{1}{\sqrt[3]{x^2 - 4}} = -\infty$

66. $\lim_{x\to 0^+} \dfrac{\sec x}{x} = \infty$

68. $\lim_{x\to 0^-} \dfrac{\cos^2 x}{x} = -\infty$

70. $f(x) = \dfrac{\tan 2x}{x}$

(a)

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	2.0271	2.0003	2.0000	2.0000	2.0003	2.0271

$\lim_{x \to 0} \dfrac{\tan 2x}{x} = 2$

(b) Yes, define

$$f(x) = \begin{cases} \dfrac{\tan 2x}{x}, & x \neq 0 \\ 2, & x = 0 \end{cases}.$$

Now $f(x)$ is continuous at $x = 0$.

Problem Solving for Chapter 1

2. (a) Area $\triangle PAO = \frac{1}{2}bh = \frac{1}{2}(1)(x) = \frac{x}{2}$

Area $\triangle PBO = \frac{1}{2}bh = \frac{1}{2}(1)(y) = \frac{y}{2} = \frac{x^2}{2}$

(b) $a(x) = \dfrac{\text{Area } \triangle PBO}{\text{Area } \triangle PAO} = \dfrac{x^2/2}{x/2} = x$

x	4	2	1	0.1	0.01
Area $\triangle PAO$	2	1	1/2	1/20	1/200
Area $\triangle PBO$	8	2	1/2	1/200	1/20,000
$a(x)$	4	2	1	1/10	1/100

(c) $\lim_{x \to 0^+} a(x) = \lim_{x \to 0^+} x = 0$

4. (a) Slope $= \dfrac{4-0}{3-0} = \dfrac{4}{3}$

(b) Slope $= -\dfrac{3}{4}$ Tangent line: $y - 4 = -\dfrac{3}{4}(x - 3)$

$$y = -\frac{3}{4}x + \frac{25}{4}$$

(c) Let $Q = (x, y) = \left(x, \sqrt{25 - x^2}\right)$

$$m_x = \frac{\sqrt{25 - x^2} - 4}{x - 3}$$

(d)
$$\begin{aligned}\lim_{x \to 3} m_x &= \lim_{x \to 3} \frac{\sqrt{25 - x^2} - 4}{x - 3} \cdot \frac{\sqrt{25 - x^2} + 4}{\sqrt{25 - x^2} + 4} \\ &= \lim_{x \to 3} \frac{25 - x^2 - 16}{(x - 3)\left(\sqrt{25 - x^2} + 4\right)} \\ &= \lim_{x \to 3} \frac{(3 - x)(3 + x)}{(x - 3)\left(\sqrt{25 - x^2} + 4\right)} \\ &= \lim_{x \to 3} \frac{-(3 + x)}{\sqrt{25 - x^2} + 4} = \frac{-6}{4 + 4} = -\frac{3}{4}\end{aligned}$$

This is the slope of the tangent line at P.

6.
$$\begin{aligned}\frac{\sqrt{a + bx} - \sqrt{3}}{x} &= \frac{\sqrt{a + bx} - \sqrt{3}}{x} \cdot \frac{\sqrt{a + bx} + \sqrt{3}}{\sqrt{a + bx} + \sqrt{3}} \\ &= \frac{(a + bx) - 3}{x\left(\sqrt{a + bx} + \sqrt{3}\right)}\end{aligned}$$

Letting $a = 3$ simplifies the numerator.

Thus,

$$\begin{aligned}\lim_{x \to 0} \frac{\sqrt{3 + bx} - \sqrt{3}}{x} &= \lim_{x \to 0} \frac{bx}{x\left(\sqrt{3 + bx} + \sqrt{3}\right)} \\ &= \lim_{x \to 0} \frac{b}{\sqrt{3 + bx} + \sqrt{3}}.\end{aligned}$$

Setting $\dfrac{b}{\sqrt{3} + \sqrt{3}} = \sqrt{3}$, you obtain $b = 6$.

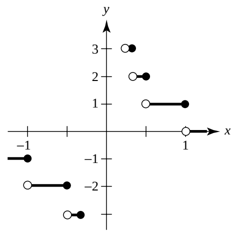
Thus, $a = 3$ and $b = 6$.

8. $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (a^2 - 2) = a^2 - 2$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \dfrac{ax}{\tan x} = a \left(\text{because } \lim_{x \to 0} \dfrac{\tan x}{x} = 1\right)$

Thus,

$$\begin{aligned}a^2 - 2 &= a \\ a^2 - a - 2 &= 0 \\ (a - 2)(a + 1) &= 0 \\ a &= -1, 2\end{aligned}$$

10. 

(a) $f\left(\frac{1}{4}\right) = [\![4]\!] = 4$

$f(3) = \left[\!\left[\frac{1}{3}\right]\!\right] = 0$

$f(1) = [\![1]\!] = 1$

(b) $\lim_{x \to 1^-} f(x) = 1$

$\lim_{x \to 1^+} f(x) = 0$

$\lim_{x \to 0^-} f(x) = -\infty$

$\lim_{x \to 0^+} f(x) = \infty$

(c) f is continuous for all real numbers except

$x = 0, \pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \ldots$

12. (a) $v^2 = \dfrac{192{,}000}{r} + v_0^{\,2} - 48$

$\dfrac{192{,}000}{r} = v^2 - v_0^{\,2} + 48$

$r = \dfrac{192{,}000}{v - v_0^{\,2} + 48}$

$\lim_{v \to 0} r = \dfrac{192{,}000}{48 - v_0^{\,2}}$

Let $v_0 = \sqrt{48} = 4\sqrt{3}$ feet/sec.

(b) $v^2 = \dfrac{1920}{r} + v_0^{\,2} - 2.17$

$\dfrac{1920}{r} = v^2 - v_0^{\,2} + 2.17$

$r = \dfrac{1920}{v^2 - v_0^{\,2} + 2.17}$

$\lim_{v \to 0} r = \dfrac{1920}{2.17 - v_0^{\,2}}$

Let $v_0 = \sqrt{2.17}$ mi/sec (≈ 1.47 mi/sec).

(c) $r = \dfrac{10{,}600}{v^2 - v_0^{\,2} + 6.99}$

$\lim_{v \to 0} r = \dfrac{10{,}600}{6.99 - v_0^{\,2}}$

Let $v_0 = \sqrt{6.99} \approx 2.64$ mi/sec.

Since this is smaller than the escape velocity for earth, the mass is less.

14. Let $a \neq 0$ and let $\epsilon > 0$ be given. There exists $\delta_1 > 0$ such that if $0 < |x - 0| < \delta$, then $|f(x) - L| < \epsilon$. Let $\delta = \delta_1/|a|$. Then for $0 < |x - 0| < \delta = \delta_1/|a|$, you have

$$|x| < \frac{\delta_1}{|a|}$$

$$|ax| < \delta_1$$

$$|f(ax) - L| < \epsilon.$$

As a counterexample, let $f(x) = \begin{cases} 1 & x \neq 0 \\ 2 & x = 0 \end{cases}$.

Then $\lim_{x \to 0} f(x) = 1 = L$,

but $\lim_{x \to 0} f(ax) = \lim_{x \to 0} f(0) = 2$.