CHAPTER 2
Differentiation

C H A P T E R 2
Differentiation

Section 2.1 The Derivative and the Tangent Line Problem

Solutions to Even-Numbered Exercises

2. (a) $m = \frac{1}{4}$

(b) $m = 1$

4. (a) $\frac{f(4) - f(1)}{4 - 1} = \frac{5 - 2}{3} = 1$

$\frac{f(4) - f(3)}{4 - 3} \approx \frac{5 - 4.75}{1} = 0.25$

Thus, $\frac{f(4) - f(1)}{4 - 1} > \frac{f(4) - f(3)}{4 - 3}$

(b) The slope of the tangent line at $(1, 2)$ equals $f'(1)$. This slope is steeper than the slope of the line through $(1, 2)$ and $(4, 5)$. Thus,

$$\frac{f(4) - f(1)}{4 - 1} < f'(1).$$

6. $g(x) = \frac{3}{2}x + 1$ is a line. Slope $= \frac{3}{2}$

8. Slope at $(2, 1)$
$$\begin{aligned} &= \lim_{\Delta x \to 0} \frac{g(2 + \Delta x) - g(2)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{5 - (2 + \Delta x)^2 - 1}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{5 - 4 - 4(\Delta x) - (\Delta x)^2 - 1}{\Delta x} \\ &= \lim_{\Delta x \to 0} (-4 - \Delta x) = -4 \end{aligned}$$

10. Slope at $(-2, 7)$
$$\begin{aligned} &= \lim_{\Delta t \to 0} \frac{h(-2 + \Delta t) - h(-2)}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{(-2 + \Delta t)^2 + 3 - 7}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{4 - 4(\Delta t) + (\Delta t)^2 - 4}{\Delta t} \\ &= \lim_{\Delta t \to 0} (-4 + \Delta t) = -4 \end{aligned}$$

12. $g(x) = -5$
$$\begin{aligned} g'(x) &= \lim_{\Delta x \to 0} \frac{g(x + \Delta x) - g(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{-5 - (-5)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{0}{\Delta x} = 0 \end{aligned}$$

14. $f(x) = 3x + 2$
$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{[3(x + \Delta x) + 2] - [3x + 2]}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{3\Delta x}{\Delta x} \\ &= \lim_{\Delta x \to 0} 3 = 3 \end{aligned}$$

16. $f(x) = 9 - \frac{1}{2}x$
$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{[9 - (1/2)(x + \Delta x)] - [9 - (1/2)x]}{\Delta x} \\ &= \lim_{\Delta x \to 0} \left(-\frac{1}{2}\right) = -\frac{1}{2} \end{aligned}$$

18. $f(x) = 1 - x^2$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{[1 - (x + \Delta x)^2] - [1 - x^2]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{1 - x^2 - 2x\Delta x - (\Delta x)^2 - 1 + x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-2x\Delta x - (\Delta x)^2}{\Delta x} = \lim_{\Delta x \to 0} (-2x - \Delta x) = -2x$$

20. $f(x) = x^3 + x^2$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{[(x + \Delta x)^3 + (x + \Delta x)^2] - [x^3 + x^2]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 + x^2 + 2x\Delta x + (\Delta x)^2 - x^3 - x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 + 2x\Delta x + (\Delta x)^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (3x^2 + 3x\Delta x + (\Delta x)^2 + 2x + (\Delta x)) = 3x^2 + 2x$$

22. $f(x) = \dfrac{1}{x^2}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\dfrac{1}{(x + \Delta x)^2} - \dfrac{1}{x^2}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 - (x + \Delta x)^2}{\Delta x(x + \Delta x)^2 x^2}$$

$$= \lim_{\Delta x \to 0} \frac{-2x\Delta x - (\Delta x)^2}{\Delta x(x + \Delta x)^2 x^2}$$

$$= \lim_{\Delta x \to 0} \frac{-2x - \Delta x}{(x + \Delta x)^2 x^2}$$

$$= \frac{-2x}{x^4}$$

$$= -\frac{2}{x^3}$$

24. $f(x) = \dfrac{4}{\sqrt{x}}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\dfrac{4}{\sqrt{x + \Delta x}} - \dfrac{4}{\sqrt{x}}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{4\sqrt{x} - 4\sqrt{x + \Delta x}}{\Delta x\sqrt{x}\sqrt{x + \Delta x}} \cdot \left(\frac{\sqrt{x} + \sqrt{x + \Delta x}}{\sqrt{x} + \sqrt{x + \Delta x}}\right)$$

$$= \lim_{\Delta x \to 0} \frac{4x - 4(x + \Delta x)}{\Delta x\sqrt{x}\sqrt{x + \Delta x}\left(\sqrt{x} + \sqrt{x + \Delta x}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{-4}{\sqrt{x}\sqrt{x + \Delta x}\left(\sqrt{x} + \sqrt{x + \Delta x}\right)}$$

$$= \frac{-4}{\sqrt{x}\sqrt{x}\left(\sqrt{x} + \sqrt{x}\right)} = \frac{-2}{x\sqrt{x}}$$

26. (a) $f(x) = x^2 + 2x + 1$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{[(x + \Delta x)^2 + 2(x + \Delta x) + 1] - [x^2 + 2x + 1]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2 + 2\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (2x + \Delta x + 2) = 2x + 2$$

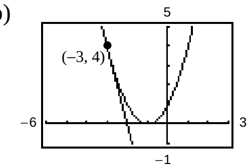
At $(-3, 4)$, the slope of the tangent line is $m = 2(-3) + 2 = -4$.

The equation of the tangent line is

$$y - 4 = -4(x + 3)$$

$$y = -4x - 8.$$

(b)



28. (a) $f(x) = x^3 + 1$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{[(x + \Delta x)^3 + 1] - (x^3 + 1)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^3 + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3 + 1 - x^3 - 1}{\Delta x}$$

$$= \lim_{\Delta x \to 0} [3x^2 + 3x(\Delta x) + (\Delta x)^2] = 3x^2$$

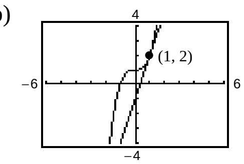
At $(1, 2)$, the slope of the tangent line is $m = 3(1)^2 = 3$.

The equation of the tangent line is

$$y - 2 = 3(x - 1)$$

$$y = 3x - 1.$$

(b)



30. (a) $f(x) = \sqrt{x - 1}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{x + \Delta x - 1} - \sqrt{x - 1}}{\Delta x} \cdot \left(\frac{\sqrt{x + \Delta x - 1} + \sqrt{x - 1}}{\sqrt{x + \Delta x - 1} + \sqrt{x - 1}}\right)$$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x - 1) - (x - 1)}{\Delta x\left(\sqrt{x + \Delta x - 1} + \sqrt{x - 1}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x + \Delta x - 1} + \sqrt{x - 1}} = \frac{1}{2\sqrt{x - 1}}$$

At $(5, 2)$, the slope of the tangent line is

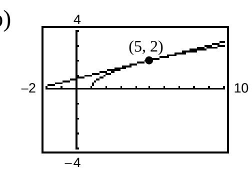
$$m = \frac{1}{2\sqrt{5 - 1}} = \frac{1}{4}$$

The equation of the tangent line is

$$y - 2 = \frac{1}{4}(x - 5)$$

$$y = \frac{1}{4}x + \frac{3}{4}$$

(b)



32. (a) $f(x) = \dfrac{1}{x+1}$

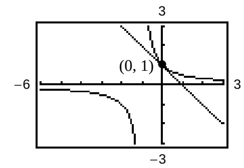
$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{\dfrac{1}{x+\Delta x+1} - \dfrac{1}{x+1}}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{(x+1) - (x+\Delta x+1)}{\Delta x(x+\Delta x+1)(x+1)}$$
$$= \lim_{\Delta x \to 0} -\frac{1}{(x+\Delta x+1)(x+1)}$$
$$= -\frac{1}{(x+1)^2}$$

At $(0, 1)$, the slope of the tangent line is

$$m = \frac{-1}{(0+1)^2} = -1.$$

The equation of the tangent line is $y = -x + 1$.

(b)



34. Using the limit definition of derivative, $f'(x) = 3x^2$. Since the slope of the given line is 3, we have

$$3x^2 = 3$$
$$x^2 = 1 \Rightarrow x = \pm 1.$$

Therefore, at the points $(1, 3)$ and $(-1, 1)$ the tangent lines are parallel to $3x - y - 4 = 0$. These lines have equations

$y - 3 = 3(x - 1)$ and $y - 1 = 3(x + 1)$

$y = 3x$ $\qquad$ $y = 3x + 4$

36. Using the limit definition of derivative, $f'(x) = \dfrac{-1}{2(x-1)^{3/2}}$.

Since the slope of the given line is $-\dfrac{1}{2}$, we have

$$\frac{-1}{2(x-1)^{3/2}} = -\frac{1}{2}$$
$$1 = (x-1)^{3/2}$$
$$1 = x - 1 \Rightarrow x = 2$$

At the point $(2, 1)$, the tangent line is parallel to $x + 2y + 7 = 0$. The equation of the tangent line is

$$y - 1 = -\frac{1}{2}(x - 2)$$
$$y = -\frac{1}{2}x + 2$$

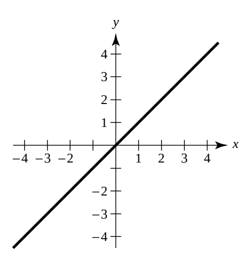
38. $h(-1) = 4$ because the tangent line passes through $(-1, 4)$

$$h'(-1) = \frac{6-4}{3-(-1)} = \frac{2}{4} = \frac{1}{2}$$

40. $f(x) = x^2 \implies f'(x) = 2x$ (d)

42. f' does not exist at $x = 0$. Matches (c)

44.



Answers will vary.

Sample answer: $y = x$

46. (a) Yes. $\lim_{\Delta x\to 0}\dfrac{f(x+2\Delta x)-f(x)}{2Dx}=\lim_{\Delta x\to 0}\dfrac{f(x+\Delta x)-f(x)}{\Delta x}=f'(x)$

(b) No. The numerator does not approach zero.

(c) Yes. $\lim_{\Delta x\to 0}\dfrac{f(x+\Delta x)-f(x-\Delta x)}{2\Delta x}=\lim_{\Delta x\to 0}\dfrac{f(x+\Delta x)-f(x)-f(x-\Delta x)+f(x)}{2\Delta x}$

$$=\lim_{\Delta x\to 0}\left[\frac{f(x+\Delta x)-f(x)}{2\Delta x}+\frac{f(x-\Delta x)-f(x)}{2(-\Delta x)}\right]$$

$$=\frac{1}{2}f'(x)+\frac{1}{2}f'(x)=f'(x)$$

(d) Yes. $\lim_{\Delta x\to 0}\dfrac{f(x+\Delta x)-f(x)}{\Delta x}=f'(x)$

48. Let (x_0, y_0) be a point of tangency on the graph of f. By the limit definition for the derivative, $f'(x)=2x$. The slope of the line through $(1, -3)$ and (x_0, y_0) equals the derivative of f at x_0:

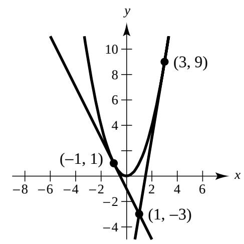
$$\frac{-3-y_0}{1-x_0}=2x_0$$

$$-3-y_0=(1-x_0)2x_0$$

$$-3-x_0^2=2x_0-2x_0^2$$

$$x_0^2-2x_0-3=0$$

$$(x_0-3)(x_0+1)=0 \Longrightarrow x_0=3, -1$$



Therefore, the points of tangency are $(3, 9)$ and $(-1, 1)$, and the corresponding slopes are 6 and -2. The equations of the tangent lines are

$$y+3=6(x-1) \qquad y+3=-2(x-1)$$

$$y=6x-9 \qquad y=-2x-1$$

50. (a) $f(x)=x^2$

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{(x+\Delta x)^2-x^2}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{x^2+2x(\Delta x)+(\Delta x)^2-x^2}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\Delta x(2x+\Delta x)}{\Delta x}$$

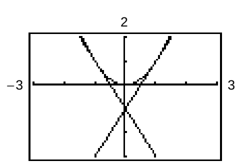
$$=\lim_{\Delta x\to 0}(2x+\Delta x)=2x$$

At $x=-1$, $f'(-1)=-2$ and the tangent line is

$y-1=-2(x+1)$ or $y=-2x-1$.

At $x=0$, $f'(0)=0$ and the tangent line is $y=0$.

At $x=1$, $f'(1)=2$ and the tangent line is $y=2x-1$.



For this function, the slopes of the tangent lines are always distinct for different values of x.

(b) $g'(x)=\lim_{\Delta x\to 0}\dfrac{g(x+\Delta x)-g(x)}{\Delta x}$

$$=\lim_{\Delta x\to 0}\frac{(x+\Delta x)^3-x^3}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{x^3+3x^2(\Delta x)+3x(\Delta x)^2+(\Delta x)^3-x^3}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\Delta x(3x^2+3x(\Delta x)+(\Delta x)^2)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}(3x^2+3x(\Delta x)+(\Delta x)^2)=3x^2$$

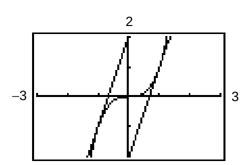
At $x=-1$, $g'(-1)=3$ and the tangent line is

$y+1=3(x+1)$ or $y=3x+2$.

At $x=0$, $g'(0)=0$ and the tangent line is $y=0$.

At $x=1$, $g'(1)=3$ and the tangent line is

$y-1=3(x-1)$ or $y=3x-2$.

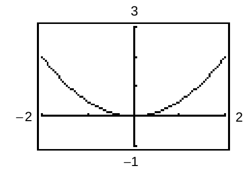


For this function, the slopes of the tangent lines are sometimes the same.

52. $f(x) = \frac{1}{2}x^2$

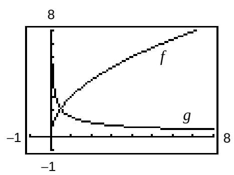
By the limit definition of the derivative we have $f'(x) = x$.

x	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
$f(x)$	2	1.125	0.5	0.125	0	0.125	0.5	1.125	2
$f'(x)$	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2



54. $g(x) = \dfrac{f(x + 0.01) - f(x)}{0.01}$

$$= \left(3\sqrt{x + 0.01} - 3\sqrt{3}\right)100$$

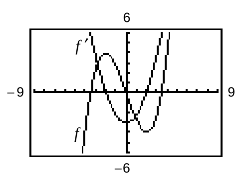


The graph of $g(x)$ is approximately the graph of $f'(x)$.

56. $f(2) = \dfrac{1}{4}(2^3) = 2,\ f(2.1) = 2.31525$

$$f'(2) \approx \frac{2.31525 - 2}{2.1 - 2} = 3.1525\ [\text{Exact: } f'(2) = 3]$$

58. $f(x) = \dfrac{x^3}{4} - 3x$ and $f'(x) = \dfrac{3}{4}x^2 - 3$



60. $f(x) = x + \dfrac{1}{x}$

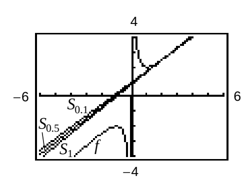
$$S_{\Delta x}(x) = \frac{f(2 + \Delta x) - f(2)}{\Delta x}(x - 2) + f(2) = \frac{(2 + \Delta x) + \dfrac{1}{2 + \Delta x} - \dfrac{5}{2}}{\Delta x}(x - 2) + \frac{5}{2}$$

$$= \frac{2(2 + \Delta x)^2 + 2 - 5(2 + \Delta x)}{2(2 + \Delta x)\,\Delta x}(x - 2) + \frac{5}{2} = \frac{(2\Delta x + 3)}{2(2 + \Delta x)}(x - 2) + \frac{5}{2}$$

(a) $\Delta x = 1$: $S_{\Delta x} = \dfrac{5}{6}(x - 2) + \dfrac{5}{2} = \dfrac{5}{6}x + \dfrac{5}{6}$

$\Delta x = 0.5$: $S_{\Delta x} = \dfrac{4}{5}(x - 2) + \dfrac{5}{2} = \dfrac{4}{5}x + \dfrac{9}{10}$

$\Delta x = 0.1$: $S_{\Delta x} = \dfrac{16}{21}(x - 2) + \dfrac{5}{2} = \dfrac{16}{21}x + \dfrac{41}{42}$



(b) As $\Delta x \to 0$, the line approaches the tangent line to f at $\left(2, \frac{5}{2}\right)$.

62. $g(x) = x(x - 1) = x^2 - x,\ c = 1$

$$g'(1) = \lim_{x\to 1}\frac{g(x) - g(1)}{x - 1} = \lim_{x\to 1}\frac{x^2 - x - 0}{x - 1} = \lim_{x\to 1}\frac{x(x - 1)}{x - 1}$$

$$= \lim_{x\to 1} x = 1$$

64. $f(x) = x^3 + 2x, c = 1$

$$f'(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{x^3 + 2x - 3}{x - 1} = \lim_{x \to 1} \frac{(x - 1)(x^2 + x + 3)}{x - 1} = \lim_{x \to 1} (x^2 + x + 3) = 5$$

66. $f(x) = \frac{1}{x}, c = 3$

$$f'(3) = \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3} = \lim_{x \to 3} \frac{(1/x) - (1/3)}{x - 3} = \lim_{x \to 3} \frac{3 - x}{3x} \cdot \frac{1}{x - 3} = \lim_{x \to 3} \left(-\frac{1}{3x}\right) = -\frac{1}{9}$$

68. $g(x) = (x + 3)^{1/3}, c = -3$

$$g'(-3) = \lim_{x \to -3} \frac{g(x) - g(-3)}{x - (-3)} = \lim_{x \to -3} \frac{(x + 3)^{1/3} - 0}{x + 3} = \lim_{x \to -3} \frac{1}{(x + 3)^{2/3}}$$

Does not exist.

70. $f(x) = |x - 4|, c = 4$

$$f'(4) = \lim_{x \to 4} \frac{f(x) - f(4)}{x - 4} = \lim_{x \to 4} \frac{|x - 4| - 0}{x - 4} = \lim_{x \to 4} \frac{|x - 4|}{x - 4}$$

Does not exist.

72. $f(x)$ is differentiable everywhere except at $x = \pm 3$. (Sharp turns in the graph.)

74. $f(x)$ is differentiable everywhere except at $x = 1$. (Discontinuity)

76. $f(x)$ is differentiable everywhere except at $x = 0$. (Sharp turn in the graph)

78. $f(x)$ is differentiable everywhere except at $x = \pm 2$. (Discontinuities)

80. $f(x)$ is differentiable everywhere except at $x = 1$. (Discontinuity)

82. $f(x) = \sqrt{1 - x^2}$

The derivative from the left does not exist because

$$\lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^-} \frac{\sqrt{1 - x^2} - 0}{x - 1} = \lim_{x \to 1^-} \frac{\sqrt{1 - x^2}}{x - 1} \cdot \frac{\sqrt{1 - x^2}}{\sqrt{1 - x^2}} = \lim_{x \to 1^-} -\frac{1 + x}{\sqrt{1 - x^2}} = -\infty. \text{ (Vertical tangent)}$$

The limit from the right does not exist since f is undefined for $x > 1$. Therefore, f is not differentiable at $x = 1$.

84. $f(x) = \begin{cases} x, & x \leq 1 \\ x^2, & x > 1 \end{cases}$

The derivative from the left is

$$\lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^-} \frac{x - 1}{x - 1} = \lim_{x \to 1^-} 1 = 1.$$

The derivative from the right is

$$\lim_{x \to 1^+} = \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^+} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1^+} (x + 1) = 2.$$

These one-sided limits are not equal. Therefore, f is not differentiable at $x = 1$.

86. Note that f is continuous at $x = 2$. $f(x) = \begin{cases} \frac{1}{2}x + 1, & x < 2 \\ \sqrt{2x}, & x \geq 2 \end{cases}$

The derivative from the left is

$$\lim_{x\to 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x\to 2^-} \frac{\left(\frac{1}{2}x + 1\right) - 2}{x - 2} = \lim_{x\to 2^-} \frac{\frac{1}{2}(x - 2)}{x - 2} = \frac{1}{2}.$$

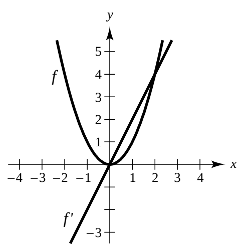
The derivative from the right is

$$\lim_{x\to 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x\to 2^+} \frac{\sqrt{2x} - 2}{x - 2} \cdot \frac{\sqrt{2x} + 2}{\sqrt{2x} + 2}$$

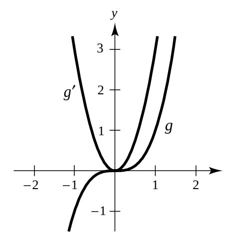
$$= \lim_{x\to 2^+} \frac{2x - 4}{(x - 2)\left(\sqrt{2x} + 2\right)} = \lim_{x\to 2^+} \frac{2(x - 2)}{(x - 2)\left(\sqrt{2x} + 2\right)} = \lim_{x\to 2^+} \frac{2}{\sqrt{2x} + 2} = \frac{1}{2}.$$

The one-sided limits are equal. Therefore, f is differentiable at $x = 2$. $\left(f'(2) = \frac{1}{2}\right)$

88. (a) $f(x) = x^2$ and $f'(x) = 2x$



(b) $g(x) = x^3$ and $g'(x) = 3x^2$



(c) The derivative is a polynomial of degree 1 less than the original function. If $h(x) = x^n$, then $h'(x) = nx^{n-1}$.

(d) If $f(x) = x^4$, then $f'(x) = \lim_{\Delta x\to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$

$$= \lim_{\Delta x\to 0} \frac{(x + \Delta x)^4 - x^4}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \frac{x^4 + 4x^3(\Delta x) + 6x^2(\Delta x)^2 + 4x(\Delta x)^3 + (\Delta x)^4 - x^4}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \frac{\Delta x(4x^3 + 6x^2(\Delta x) + 4x(\Delta x)^2 + (\Delta x)^3)}{\Delta x}$$

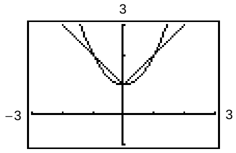
$$= \lim_{\Delta x\to 0} (4x^3 + 6x^2(\Delta x) + 4x(\Delta x)^2 + (\Delta x)^3) = 4x^3$$

Hence, if $f(x) = x^4$, then $f'(x) = 4x^3$ which is consistent with the conjecture. However, this is not a proof, since you must verify the conjecture for all integer values of n, $n \geq 2$.

90. False. $y = |x - 2|$ is continuous at $x = 2$, but is not differentiable at $x = 2$. (Sharp turn in the graph)

92. True—see Theorem 2.1

94.



As you zoom in, the graph of $y_1 = x^2 + 1$ appears to be locally the graph of a horizontal line, whereas the graph of $y_2 = |x| + 1$ always has a sharp corner at $(0, 1)$. y_2 is not differentiable at $(0, 1)$.

Section 2.2 Basic Differentiation Rules and Rates of Change

2. (a) $y = x^{-1/2}$
$y' = -\frac{1}{2}x^{-3/2}$
$y'(1) = -\frac{1}{2}$

(b) $y = x^{-1}$
$y' = -x^{-2}$
$y'(1) = -1$

(c) $y = x^{-3/2}$
$y' = -\frac{3}{2}x^{-5/2}$
$y'(1) = -\frac{3}{2}$

(d) $y = x^{-2}$
$y' = -2x^{-3}$
$y'(1) = -2$

4. $f(x) = -2$
$f'(x) = 0$

6. $y = x^8$
$y' = 8x^7$

8. $y = \dfrac{1}{x^8} = x^{-8}$
$y' = 8x^{-9} = \dfrac{-8}{x^9}$

10. $y = \sqrt[4]{x} = x^{1/4}$
$y' = \dfrac{1}{4}x^{-3/4} = \dfrac{1}{4x^{3/4}}$

12. $g(x) = 3x - 1$
$g'(x) = 3$

14. $y = t^2 + 2t - 3$
$y' = 2t + 2$

16. $y = 8 - x^3$
$y' = -3x^2$

18. $f(x) = 2x^3 - x^2 + 3x$
$f'(x) = 6x^2 - 2x + 3$

20. $g(t) = \pi \cos t$
$g'(t) = -\pi \sin t$

22. $y = 5 + \sin x$
$y' = \cos x$

24. $y = \dfrac{5}{(2x)^3} + 2\cos x = \dfrac{5}{8}x^{-3} + 2\cos x$
$y' = \dfrac{5}{8}(-3)x^{-4} - 2\sin x = \dfrac{-15}{8x^4} - 2\sin x$

	Function	*Rewrite*	*Derivative*	*Simplify*
26.	$y = \dfrac{2}{3x^2}$	$y = \dfrac{2}{3}x^{-2}$	$y' = -\dfrac{4}{3}x^{-3}$	$y' = -\dfrac{4}{3x^3}$
28.	$y = \dfrac{\pi}{(3x)^2}$	$y = \dfrac{\pi}{9}x^{-2}$	$y' = -\dfrac{2\pi}{9}x^{-3}$	$y' = -\dfrac{2\pi}{9x^3}$
30.	$y = \dfrac{4}{x^{-3}}$	$y = 4x^3$	$y' = 12x^2$	$y' = 12x^2$

32. $f(t) = 3 - \dfrac{3}{5t}, \left(\dfrac{3}{5}, 2\right)$
$f'(t) = \dfrac{3}{5t^2}$
$f'\left(\dfrac{3}{5}\right) = \dfrac{5}{3}$

34. $y = 3x^3 - 6, (2, 18)$
$y' = 9x^2$
$y'(2) = 36$

36. $f(x) = 3(5 - x)^2, \ (5, 0)$
$= 3x^2 - 30x + 75$
$f'(x) = 6x - 30$
$f'(5) = 0$

38. $g(t) = 2 + 3\cos t, \ (\pi, -1)$
$g'(t) = -3\sin t$
$g'(\pi) = 0$

40. $f(x) = x^2 - 3x - 3x^{-2}$
$f'(x) = 2x - 3 + 6x^{-3}$
$= 2x - 3 + \dfrac{6}{x^3}$

42. $f(x) = x + x^{-2}$
$f'(x) = 1 - 2x^{-3}$
$= 1 - \dfrac{2}{x^3}$

44. $h(x) = \dfrac{2x^2 - 3x + 1}{x} = 2x - 3 + x^{-1}$
$h'(x) = 2 - \dfrac{1}{x^2} = \dfrac{2x^2 - 1}{x^2}$

46. $y = 3x(6x - 5x^2) = 18x^2 - 15x^3$
$y' = 36x - 45x^2$

48. $f(x) = \sqrt[3]{x} + \sqrt[5]{x} = x^{1/3} + x^{1/5}$
$f'(x) = \dfrac{1}{3}x^{-2/3} + \dfrac{1}{5}x^{-4/5} = \dfrac{1}{3x^{2/3}} + \dfrac{1}{5x^{4/5}}$

50. $f(t) = t^{2/3} - t^{1/3} + 4$
$f'(t) = \dfrac{2}{3}t^{-1/3} - \dfrac{1}{3}t^{-2/3} = \dfrac{2}{3t^{1/3}} - \dfrac{1}{3t^{2/3}}$

52. $f(x) = \dfrac{2}{\sqrt[3]{x}} + 3\cos x = 2x^{-1/3} + 3\cos x$

$f'(x) = \dfrac{-2}{3}x^{-4/3} - 3\sin x = \dfrac{-2}{3x^{4/3}} - 3\sin x$

54. (a) $y = x^3 + x$

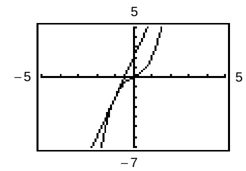
$y' = 3x^2 + 1$

At $(-1, -2)$: $y' = 3(-1)^2 + 1 = 4$.

Tangent line: $y + 2 = 4(x + 1)$

$4x - y + 2 = 0$

(b)



56. (a) $y = (x^2 + 2x)(x + 1)$

$= x^3 + 3x^2 + 2x$

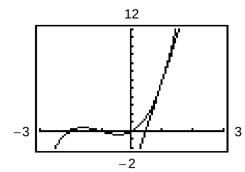
$y' = 3x^2 + 6x + 2$

At $(1, 6)$: $y' = 3(1)^2 + 6(1) + 2 = 11$.

Tangent line: $y - 6 = 11(x - 1)$

$0 = 11x - y - 5$

(b)



58. $y = x^3 + x$

$y' = 3x^2 + 1 > 0$ for all x.

Therefore, there are no horizontal tangents.

60. $y = x^2 + 1$

$y' = 2x = 0 \implies x = 0$

At $x = 0$, $y = 1$.

Horizontal tangent: $(0, 1)$

62. $y = \sqrt{3}x + 2\cos x,\ 0 \le x < 2\pi$

$y' = \sqrt{3} - 2\sin x = 0$

$\sin x = \dfrac{\sqrt{3}}{2} \implies x = \dfrac{\pi}{3}$ or $\dfrac{2\pi}{3}$

At $x = \dfrac{\pi}{3}$, $y = \dfrac{\sqrt{3}\pi + 3}{3}$.

At $x = \dfrac{2\pi}{3}$, $y = \dfrac{2\sqrt{3}\pi - 3}{3}$.

Horizontal tangents: $\left(\dfrac{\pi}{3}, \dfrac{\sqrt{3}\pi + 3}{3}\right), \left(\dfrac{2\pi}{3}, \dfrac{2\sqrt{3}\pi - 3}{3}\right)$

64. $k - x^2 = -4x + 7$ Equate functions

$-2x = -4$ Equate derivatives

Hence, $x = 2$ and $k - 4 = -8 + 7 \implies k = 3$

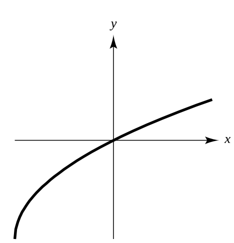
66. $k\sqrt{x} = x + 4$ Equate functions

$\dfrac{k}{2\sqrt{x}} = 1$ Equate derivatives

Hence, $k = 2\sqrt{x}$ and

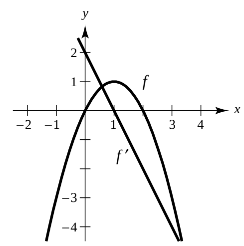
$(2\sqrt{x})\sqrt{x} = x + 4 \implies 2x = x + 4 \implies x = 4 \implies k = 4$

68. The graph of a function f such that $f' > 0$ for all x and the rate of change the function is decreasing (i.e. $f'' < 0$) would, in general, look like the graph at the right.



70. $g(x) = -5f(x) \Longrightarrow g'(x) = -5f'(x)$

72.



If f is quadratic, then its derivative is a linear function.

$$f(x) = ax^2 + bx + c$$

$$f'(x) = 2ax + b$$

74. m_1 is the slope of the line tangent to $y = x$. m_2 is the slope of the line tangent to $y = 1/x$. Since

$$y = x \implies y' = 1 \implies m_1 = 1 \text{ and } y = \frac{1}{x} \implies y' = \frac{-1}{x^2} \implies m_2 = \frac{-1}{x^2}.$$

The points of intersection of $y = x$ and $y = 1/x$ are

$$x = \frac{1}{x} \implies x^2 = 1 \implies x = \pm 1.$$

At $x = \pm 1$, $m_2 = -1$. Since $m_2 = -1/m_1$, these tangent lines are perpendicular at the points intersection.

76. $f(x) = \dfrac{2}{x}$, $(5, 0)$

$$f'(x) = -\frac{2}{x^2}$$

$$-\frac{2}{x^2} = \frac{0 - y}{5 - x}$$

$$-10 + 2x = -x^2 y$$

$$-10 + 2x = -x^2\left(\frac{2}{x}\right)$$

$$-10 + 2x = -2x$$

$$4x = 10$$

$$x = \frac{5}{2}, y = \frac{4}{5}$$

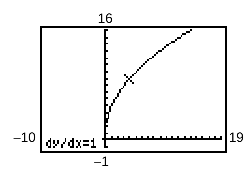
The point $\left(\frac{5}{2}, \frac{4}{5}\right)$ is on the graph of f. The slope of the tangent line is $f'\left(\frac{5}{2}\right) = -\frac{8}{25}$.

Tangent line: $$y - \frac{4}{5} = -\frac{8}{25}\left(x - \frac{5}{2}\right)$$

$$25y - 20 = -8x + 20$$

$$8x + 25y - 40 = 0$$

78. $f'(4) = 1$

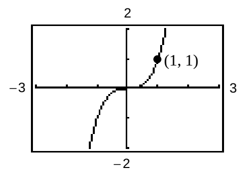


80. (a) Nearby point: $(1.0073138, 1.0221024)$

Secant line: $y - 1 = \frac{1.0221024 - 1}{1.0073138 - 1}(x - 1)$

$y = 3.022(x - 1) + 1$

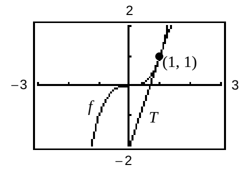
(Answers will vary.)



(b) $f'(x) = 3x^2$

$T(x) = 3(x - 1) + 1 = 3x - 2$

(c) The accuracy worsens at you move away from $(1, 1)$.



(d)

Δx	-3	-2	-1	-0.5	-0.1	0	0.1	0.5	1	2	3
$f(x)$	-8	-1	0	0.125	0.729	1	1.331	3.375	8	27	64
$T(x)$	-8	-5	-2	-0.5	0.7	1	1.3	2.5	4	7	10

The accuracy decreases more rapidly than in Exercise 59 because $y = x^3$ is less "linear" than $y = x^{3/2}$.

82. True. If $f(x) = g(x) + c$, then $f'(x) = g'(x) + 0 = g'(x)$.

84. True. If $y = x/\pi = (1/\pi) \cdot x$, then $dy/dx = (1/\pi)(1) = 1/\pi$.

86. False. If $f(x) = \frac{1}{x^n} = x^{-n}$, then $f'(x) = -nx^{-n-1} = \frac{-n}{x^{n+1}}$

88. $f(t) = t^2 - 3, [2, 2.1]$

$f'(t) = 2t$

Instantaneous rate of change:

$(2, 1) \implies f'(2) = 2(2) = 4$

$(2.1, 1.41) \implies f'(2.1) = 4.2$

Average rate of change:

$$\frac{f(2.1) - f(2)}{2.1 - 2} = \frac{1.41 - 1}{0.1} = 4.1$$

90. $f(x) = \sin x, \left[0, \frac{\pi}{6}\right]$

$f'(x) = \cos x$

Instantaneous rate of change:

$(0, 0) \implies f'(0) = 1$

$\left(\frac{\pi}{6}, \frac{1}{2}\right) \implies f'\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \approx 0.866$

Average rate of change:

$$\frac{f(\pi/6) - f(0)}{(\pi/6) - 0} = \frac{(1/2) - 0}{(\pi/6) - 0} = \frac{3}{\pi} \approx 0.955$$

92.

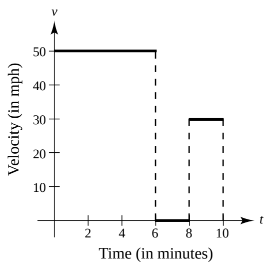
$$s(t) = -16t^2 - 22t + 220$$
$$v(t) = -32t - 22$$
$$v(3) = -118 \text{ ft/sec}$$
$$s(t) = -16t^2 - 22t + 220$$
$$= 112 \text{ (height after falling 108 ft)}$$
$$-16t^2 - 22t + 108 = 0$$
$$-2(t - 2)(8t + 27) = 0$$
$$t = 2$$
$$v(2) = -32(2) - 22$$
$$= -86 \text{ ft/sec}$$

94. $s(t) = -4.9t^2 + v_0 t + s_0$

$= -4.9t^2 + s_0 = 0$ when $t = 6.8$.

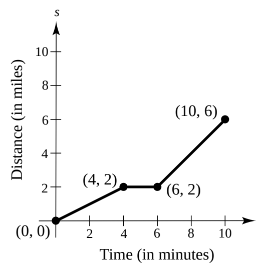
$s_0 = 4.9t^2 = 4.9(6.8)^2 = 226.6$ m

96.



(The velocity has been converted to miles per hour)

98. This graph corresponds with Exercise 75.



100. $s(t) = -\frac{1}{2}at^2 + c$ and $s'(t) = -at$.

Average velocity: $\frac{s(t_0 + \Delta t) - s(t_0 - \Delta t)}{(t_0 + \Delta t) - (t_0 - \Delta t)} = \frac{[-(1/2)a(t_0 + \Delta t)^2 + c] - [-(1/2)a(t_0 - \Delta t)^2 + c)]}{2\Delta t}$

$$= \frac{-(1/2)a(t_0^2 + 2t_0\Delta t + (\Delta t)^2) + (1/2)a(t_0^2 - 2t_0\Delta t + (\Delta t)^2)}{2\Delta t}$$

$$= \frac{-2at_0\Delta t}{2\Delta t}$$

$$= -at_0$$

$= s'(t_0)$ Instantaneous velocity at $t = t_0$

102. $V = s^3, \frac{dV}{ds} = 3s^2$

When $s = 4$ cm, $\frac{dV}{ds} = 48$ cm^2.

104. $C =$ (gallons of fuel used)(cost per gallon)

$$= \left(\frac{15{,}000}{x}\right)(1.25) = \frac{18{,}750}{x}$$

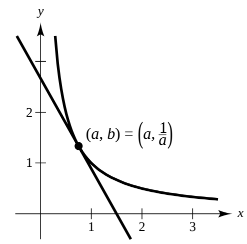
$$\frac{dC}{dx} = -\frac{18{,}750}{x^2}$$

x	10	15	20	25	30	35	40
C	1875	1250	537.5	750	625	535.71	468.75
$\frac{dC}{dx}$	-187.5	-83.333	-46.875	-30	-20.833	-15.306	-11.719

The driver who gets 15 miles per gallon would benefit more from a 1 mile per gallon increase in fuel efficiency. The rate of change is larger when $x = 15$.

106. $\frac{dT}{dt} = K(T - T_a)$

108. $y = \dfrac{1}{x},\ x > 0$



$y' = -\dfrac{1}{x^2}$

At (a, b), the equation of the tangent line is

$$y - \frac{1}{a} = -\frac{1}{a^2}(x - a) \quad \text{or} \quad y = -\frac{x}{a^2} + \frac{2}{a}.$$

The x-intercept is $(2a, 0)$.

The y-intercept is $\left(0, \dfrac{2}{a}\right)$.

The area of the triangle is $A = \dfrac{1}{2}bh = \dfrac{1}{2}(2a)\left(\dfrac{2}{a}\right) = 2.$

110. $y = x^2$

$y' = 2x$

(a) Tangent lines through $(0, a)$:

$$y - a = 2x(x - 0)$$
$$x^2 - a = 2x^2$$
$$-a = x^2$$
$$\pm\sqrt{-a} = x$$

The points of tangency are $\left(\pm\sqrt{-a}, -a\right)$. At $\left(\sqrt{-a}, -a\right)$ the slope is $y'\left(\sqrt{-a}\right) = 2\sqrt{-a}$. At $\left(-\sqrt{-a}, -a\right)$ the slope is $y'\left(-\sqrt{-a}\right) = -2\sqrt{-a}$.

Tangent lines: $y + a = 2\sqrt{-a}\left(x - \sqrt{-a}\right)$ and $y + a = -2\sqrt{-a}\left(x + \sqrt{-a}\right)$

$y = 2\sqrt{-a}\,x + a \qquad\qquad y = -2\sqrt{-a}\,x + a$

Restriction: a must be negative.

(b) Tangent lines through $(a, 0)$:

$$y - 0 = 2x(x - a)$$
$$x^2 = 2x^2 - 2ax$$
$$0 = x^2 - 2ax = x(x - 2a)$$

The points of tangency are $(0, 0)$ and $(2a, 4a^2)$. At $(0, 0)$ the slope is $y'(0) = 0$. At $(2a, 4a^2)$ the slope is $y'(2a) = 4a$.

Tangent lines: $y - 0 = 0(x - 0)$ and $y - 4a^2 = 4a(x - 2a)$

$y = 0 \qquad\qquad y = 4ax - 4a^2$

Restriction: None, a can be any real number.

112. $f_1(x) = |\sin x|$ is differentiable for all $x \neq n\pi$, n an integer.

$f_2(x) = \sin|x|$ is differentiable for all $x \neq 0$.

You can verify this by graphing f_1 and f_2 and observing the locations of the sharp turns.

Section 2.3 The Product and Quotient Rules and Higher-Order Derivatives

2. $f(x) = (6x + 5)(x^3 - 2)$

$$f'(x) = (6x + 5)(3x^2) + (x^3 - 2)(6) = 18x^3 + 15x^2 + 6x^3 - 12 = 24x^3 + 15x^2 - 12$$

4. $g(s) = \sqrt{s}(4 - s^2) = s^{1/2}(4 - s^2)$

$$g'(s) = s^{1/2}(-2s) + (4 - s^2)\frac{1}{2}s^{-1/2} = -2s^{3/2} + \frac{4 - s^2}{2s^{1/2}} = \frac{4 - 5s^2}{2s^{1/2}}$$

6. $g(x) = \sqrt{x}\sin x$

$$g'(x) = \sqrt{x}\cos x + \sin x\left(\frac{1}{2\sqrt{x}}\right) = \sqrt{x}\cos x + \frac{1}{2\sqrt{x}}\sin x$$

8. $g(t) = \dfrac{t^2 + 2}{2t - 7}$

$$g'(t) = \frac{(2t - 7)(2t) - (t^2 + 2)(2)}{(2t - 7)^2} = \frac{2t^2 - 14t - 4}{(2t - 7)^2}$$

10. $h(s) = \dfrac{s}{\sqrt{s} - 1}$

$$h'(s) = \frac{(\sqrt{s} - 1)(1) - s\left(\frac{1}{2}s^{-1/2}\right)}{\left(\sqrt{s} - 1\right)^2} = \frac{\sqrt{s} - 1 - \frac{1}{2}\sqrt{s}}{\left(\sqrt{s} - 1\right)^2} = \frac{\sqrt{s} - 2}{2\left(\sqrt{s} - 1\right)^2}$$

12. $f(t) = \dfrac{\cos t}{t^3}$

$$f'(t) = \frac{t^3(-\sin t) - \cos t(3t^2)}{(t^3)^2} = -\frac{t\sin t + 3\cos t}{t^4}$$

14. $f(x) = (x^2 - 2x + 1)(x^3 - 1)$

$$f'(x) = (x^2 - 2x + 1)(3x^2) + (x^3 - 1)(2x - 2) = 3x^2(x - 1)^2 + 2(x - 1)^2(x^2 + x + 1) = (x - 1)^2(5x^2 + 2x + 2)$$

$f'(1) = 0$

16. $f(x) = \dfrac{x + 1}{x - 1}$

$$f'(x) = \frac{(x - 1)(1) - (x + 1)(1)}{(x - 1)^2} = \frac{x - 1 - x - 1}{(x - 1)^2} = -\frac{2}{(x - 1)^2}$$

$$f'(2) = -\frac{2}{(2 - 1)^2} = -2$$

18. $f(x) = \dfrac{\sin x}{x}$

$$f'(x) = \frac{(x)(\cos x) - (\sin x)(1)}{x^2} = \frac{x\cos x - \sin x}{x^2}$$

$$f'\left(\frac{\pi}{6}\right) = \frac{(\pi/6)\left(\sqrt{3}/2\right) - (1/2)}{\pi^2/36} = \frac{3\sqrt{3}\pi - 18}{\pi^2} = \frac{3\left(\sqrt{3}\pi - 6\right)}{\pi^2}$$

	Function	Rewrite	Derivative	Simplify
20.	$y = \dfrac{5x^2 - 3}{4}$	$y = \dfrac{5}{4}x^2 - \dfrac{3}{4}$	$y' = \dfrac{10}{4}x$	$y' = \dfrac{5x}{2}$

Function	*Rewrite*	*Derivative*	*Simplify*
22. $y = \frac{4}{5x^2}$	$y = \frac{4}{5}x^{-2}$	$y' = -\frac{8}{5}x^{-3}$	$y' = -\frac{8}{5x^3}$
24. $y = \frac{3x^2 - 5}{7}$	$y = \frac{3}{7}x^2 - \frac{5}{7}$	$y' = \frac{6x}{7}$	$y' = \frac{6}{7}x$

26. $f(x) = \frac{x^3 + 3x + 2}{x^2 - 1}$

$$f'(x) = \frac{(x^2 - 1)(3x^2 + 3) - (x^3 + 3x + 2)(2x)}{(x^2 - 1)^2}$$

$$= \frac{x^4 - 6x^2 - 4x - 3}{(x^2 - 1)^2}$$

28. $f(x) = x^4\left[1 - \frac{2}{x + 1}\right] = x^4\left[\frac{x - 1}{x + 1}\right]$

$$f'(x) = x^4\left[\frac{(x + 1) - (x - 1)}{(x + 1)^2}\right] + \left[\frac{x - 1}{x + 1}\right](4x^3)$$

$$= 2x^3\left[\frac{2x^2 + x - 2}{(x + 1)^2}\right]$$

30. $f(x) = \sqrt[3]{x}\left(\sqrt{x} + 3\right) = x^{1/3}(x^{1/2} + 3)$

$$f'(x) = x^{1/3}\left(\frac{1}{2}x^{-1/2}\right) + (x^{1/2} + 3)\left(\frac{1}{3}x^{-2/3}\right)$$

$$= \frac{5}{6}x^{-1/6} + x^{-2/3}$$

$$= \frac{5}{6x^{1/6}} + \frac{1}{x^{2/3}}$$

Alternate solution:

$$f(x) = \sqrt[3]{x}\left(\sqrt{x} + 3\right)$$

$$= x^{5/6} + 3x^{1/3}$$

$$f'(x) = \frac{5}{6}x^{-1/6} + x^{-2/3}$$

$$= \frac{5}{6x^{1/6}} + \frac{1}{x^{2/3}}$$

32. $h(x) = (x^2 - 1)^2 = x^4 - 2x^2 + 1$

$$h'(x) = 4x^3 - 4x = 4x(x^2 - 1)$$

34. $g(x) = x^2\left(\frac{2}{x} - \frac{1}{x + 1}\right) = 2x - \frac{x^2}{x + 1}$

$$g'(x) = 2 - \frac{(x + 1)2x - x^2(1)}{(x + 1)^2} = \frac{2(x^2 + 2x + 1) - x^2 - 2x}{(x + 1)^2} = \frac{x^2 + 2x + 2}{(x + 1)^2}$$

36. $f(x) = (x^2 - x)(x^2 + 1)(x^2 + x + 1)$

$$f'(x) = (2x - 1)(x^2 + 1)(x^2 + x + 1) + (x^2 - x)(2x)(x^2 + x + 1) + (x^2 - x)(x^2 + 1)(2x + 1)$$

$$= (2x - 1)(x^4 + x^3 + 2x^2 + x + 1) + (x^2 - x)(2x^3 + 2x^2 + 2x) + (x^2 - x)(2x^3 + x^2 + 2x + 1)$$

$$= 2x^5 + x^4 + 3x^3 + x - 1 + 2x^5 - 2x^2 + 2x^5 - x^4 + x^3 - x^2 - x$$

$$= 6x^5 + 4x^3 - 3x^2 - 1$$

38. $f(x) = \frac{c^2 - x^2}{c^2 + x^2}$

$$f'(x) = \frac{(c^2 + x^2)(-2x) - (c^2 - x^2)(2x)}{(c^2 + x^2)^2}$$

$$= \frac{-4xc^2}{(c^2 + x^2)^2}$$

40. $f(\theta) = (\theta + 1)\cos\theta$

$$f'(\theta) = (\theta + 1)(-\sin\theta) + (\cos\theta)(1)$$

$$= \cos\theta - (\theta + 1)\sin\theta$$

42. $f(x) = \dfrac{\sin x}{x}$

$f'(x) = \dfrac{x\cos x - \sin x}{x^2}$

44. $y = x + \cot x$

$y' = 1 - \csc^2 x = -\cot^2 x$

46. $h(s) = \dfrac{1}{s} - 10\csc s$

$h'(s) = -\dfrac{1}{s^2} + 10\csc s\cot s$

48. $y = \dfrac{\sec x}{x}$

$y' = \dfrac{x\sec x\tan x - \sec x}{x^2}$

$= \dfrac{\sec x(x\tan x - 1)}{x^2}$

50. $y = x\sin x + \cos x$

$y' = x\cos x + \sin x - \sin x = x\cos x$

52. $f(x) = \sin x\cos x$

$f'(x) = \sin x(-\sin x) + \cos x(\cos x)$

$= \cos 2x$

54. $h(\theta) = 5\theta\sec\theta + \theta\tan\theta$

$h'(\theta) = 5\theta\sec\theta\tan\theta + 5\sec\theta + \theta\sec^2\theta + \tan\theta$

56. $f(x) = \left(\dfrac{x^2 - x - 3}{x^2 + 1}\right)(x^2 + x + 1)$

$f'(x) = 2\dfrac{x^5 + 2x^3 + 2x^2 - 2}{(x^2 + 1)^2}$ (form of answer may vary)

58. $f(\theta) = \dfrac{\sin\theta}{1 - \cos\theta}$

$f'(\theta) = \dfrac{1}{\cos\theta - 1} = \dfrac{\cos\theta - 1}{(1 - \cos\theta)^2}$

(form of answer may vary)

60. $f(x) = \tan x\ \cot x = 1$

$f'(x) = 0$

$f'(1) = 0$

62. $f(x) = \sin x(\sin x + \cos x)$

$f'(x) = \sin x(\cos x - \sin x) + (\sin x + \cos x)\cos x$

$= \sin x\cos x - \sin^2 x + \sin x\cos x + \cos^2 x$

$= \sin 2x + \cos 2x$

$f'\left(\dfrac{\pi}{4}\right) = \sin\dfrac{\pi}{2} + \cos\dfrac{\pi}{2} = 1$

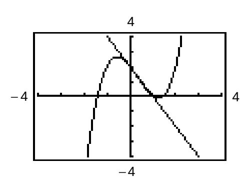
64. (a) $f(x) = (x - 1)(x^2 - 2)$, $(0, 2)$

$f'(x) = (x - 1)(2x) + (x^2 - 2)(1) = 3x^2 - 2x - 2$

$f'(0) = -2 =$ slope at $(0, 2)$.

Tangent line: $y - 2 = -2x \Longrightarrow y = -2x + 2$

(b)



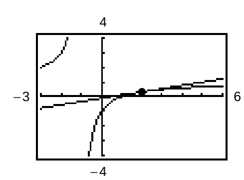
66. (a) $f(x) = \dfrac{x - 1}{x + 1}$, $\left(2, \dfrac{1}{3}\right)$

$f'(x) = \dfrac{(x + 1)(1) - (x - 1)(1)}{(x + 1)^2} = \dfrac{2}{(x + 1)^2}$

$f'(2) = \dfrac{2}{9} =$ slope at $\left(2, \dfrac{1}{3}\right)$.

Tangent line: $y - \dfrac{1}{3} = \dfrac{2}{9}(x - 2) \Longrightarrow y = \dfrac{2}{9}x - \dfrac{1}{9}$

(b)



68. (a) $f(x) = \sec x, \quad \left(\frac{\pi}{3}, 2\right)$

$f'(x) = \sec x \tan x$

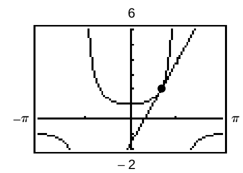
$f'\left(\frac{\pi}{3}\right) = 2\sqrt{3} = \text{slope at } \left(\frac{\pi}{3}, 2\right).$

Tangent line:

$y - 2 = 2\sqrt{3}\left(x - \frac{\pi}{3}\right)$

$6\sqrt{3}x - 3y + 6 - 2\sqrt{3}\pi = 0$

(b)



70. $f(x) = \frac{x^2}{x^2 + 1}$

$f'(x) = \frac{(x^2 + 1)(2x) - (x^2)(2x)}{(x^2 + 1)^2} = \frac{2x}{(x^2 + 1)^2}$

$f'(x) = 0$ when $x = 0$.

Horizontal tangent is at $(0, 0)$.

72. $f'(x) = \frac{x(\cos x - 3) - (\sin x - 3x)(1)}{x^2} = \frac{x \cos x - \sin x}{x^2}$

$g'(x) = \frac{x(\cos x + 2) - (\sin x + 2x)(1)}{x^2} = \frac{x \cos x - \sin x}{x^2}$

$g(x) = \frac{\sin x + 2x}{x} = \frac{\sin x - 3x + 5x}{x} = f(x) + 5$

f and g differ by a constant.

74. $f(x) = \frac{\cos x}{x^n} = x^{-n} \cos x$

$f'(x) = -x^{-n} \sin x - nx^{-n-1} \cos x$

$= -x^{-n-1}(x \sin x + n \cos x)$

$= -\frac{x \sin x + n \cos x}{x^{n+1}}$

When $n = 1$: $f'(x) = -\frac{x \sin x + \cos x}{x^2}$.

When $n = 2$: $f'(x) = -\frac{x \sin x + 2 \cos x}{x^3}$.

When $n = 3$: $f'(x) = -\frac{x \sin x + 3 \cos x}{x^4}$.

When $n = 4$: $f'(x) = -\frac{x \sin x + 4 \cos x}{x^5}$.

For general n, $f'(x) = -\frac{x \sin x + n \cos x}{x^{n+1}}$.

76. $V = \pi r^2 h = \pi(t + 2)\left(\frac{1}{2}\sqrt{t}\right)$

$= \frac{1}{2}(t^{3/2} + 2t^{1/2})\pi$

$V'(t) = \frac{1}{2}\left(\frac{3}{2}t^{1/2} + t^{-1/2}\right)\pi = \frac{3t + 2}{4t^{1/2}}\pi$ cubic inches/sec

78. $P = \frac{k}{V}$

$\frac{dP}{dV} = -\frac{k}{V^2}$

80. $f(x) = \sec x$

$g(x) = \csc x, \quad [0, 2\pi)$

$f'(x) = g'(x)$

$$\sec x \tan x = -\csc x \cot x \Rightarrow \frac{\sec x \tan x}{\csc x \cot x} = -1 \Rightarrow \frac{\frac{1}{\cos x} \cdot \frac{\sin x}{\cos x}}{\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x}} = -1 \Rightarrow$$

$$\frac{\sin^3 x}{\cos^3 x} = -1 \Rightarrow \tan^3 x = -1 \Rightarrow \tan x = -1$$

$$x = \frac{3\pi}{4}, \frac{7\pi}{4}$$

82. (a) $n(t) = -9.6643t^2 + 90.7414t + 77.5029$

$v(t) = -276.4643t^2 + 2987.6929t + 1809.9714$

(b) $A = \dfrac{v(t)}{n(t)} \approx \dfrac{-276.46t^2 + 2987.69t + 1809.97}{-9.66t^2 + 90.74t + 77.50}$

A represents the average retail value (in millions of dollars) per 1000 motor homes.

(c) $A'(t) \approx \dfrac{40.46(x^2 - 2.09x + 17.83)}{(x^2 - 9.39x - 8.02)^2}$

84. $f(x) = x + \dfrac{32}{x^2}$

$f'(x) = 1 - \dfrac{64}{x^3}$

$f''(x) = \dfrac{192}{x^4}$

86. $f(x) = \dfrac{x^2 + 2x - 1}{x} = x + 2 - \dfrac{1}{x}$

$f'(x) = 1 + \dfrac{1}{x^2}$

$f''(x) = -\dfrac{2}{x^3}$

88. $f(x) = \sec x$

$f'(x) = \sec x \tan x$

$f''(x) = \sec x(\sec^2 x) + \tan x(\sec x \tan x)$

$= \sec x(\sec^2 x + \tan^2 x)$

90. $f''(x) = 2 - 2x^{-1}$

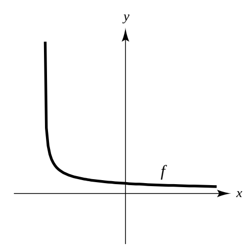
$f'''(x) = 2x^{-2} = \dfrac{2}{x^2}$

92. $f^{(4)}(x) = 2x + 1$

$f^{(5)}(x) = 2$

$f^{(6)}(x) = 0$

94. The graph of a differentiable function f such that $f > 0$ and $f' < 0$ for all real numbers x would in general look like the graph below.



96. $f(x) = 4 - h(x)$

$f'(x) = -h'(x)$

$f'(2) = -h'(2) = -4$

98. $f(x) = g(x)h(x)$

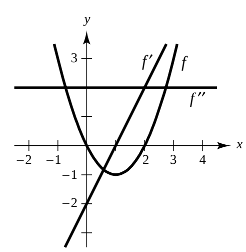
$f'(x) = g(x)h'(x) + h(x)g'(x)$

$f'(2) = g(2)h'(2) + h(2)g'(2)$

$= (3)(4) + (-1)(-2)$

$= 14$

100.



It appears that f is quadratic; so f' would be linear and f'' would be constant.

102. $s(t) = -8.25t^2 + 66t$

$v(t) = -16.50t + 66$

$a(t) = -16.50$

t(sec)	0	1	2	3	4
$s(t)$ (ft)	0	57.75	99	123.75	132
$v(t) = s'(t)$ (ft/sec)	66	49.5	33	16.5	0
$a(t) = v'(t)$ (ft/sec²)	−16.5	−16.5	−16.5	−16.5	−16.5

Average velocity on:

$[0, 1]$ is $\dfrac{57.75 - 0}{1 - 0} = 57.75$.

$[1, 2]$ is $\dfrac{99 - 57.75}{2 - 1} = 41.25$.

$[2, 3]$ is $\dfrac{123.75 - 99}{3 - 2} = 24.75$.

$[3, 4]$ is $\dfrac{132 - 123.75}{4 - 3} = 8.25$.

104. (a) $f(x) = x^n$

$f^n(x) = n(n-1)(n-2)\cdots(2)(1) = n!$

(b) $f(x) = \dfrac{1}{x}$

$$f^{(n)}(x) = \frac{(-1)^n(n)(n-1)(n-2)\cdots(2)(1)}{x^{n+1}} = \frac{(-1)^n n!}{x^{n+1}}$$

Note: $n! = n(n-1)\cdots 3 \cdot 2 \cdot 1$ (read "n factorial.")

106. $[xf(x)]' = xf'(x) + f(x)$

$[xf(x)]'' = xf''(x) + f'(x) + f'(x) = xf''(x) + 2f'(x)$

$[xf(x)]''' = xf'''(x) + f''(x) + 2f''(x) = xf'''(x) + 3f''(x)$

In general, $[xf(x)]^{(n)} = xf^{(n)}(x) + nf^{(n-1)}(x)$.

108. $f(x) = \sin x$ $\qquad f\left(\dfrac{\pi}{2}\right) = 1$

$f'(x) = \cos x$ $\qquad f'\left(\dfrac{\pi}{2}\right) = 0$

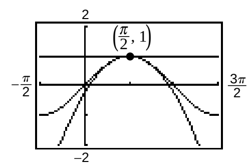
$f''(x) = -\sin x$ $\qquad f''\left(\dfrac{\pi}{2}\right) = -1$

(a) $P_1(x) = f'(a)(x-a) + f(a) = 0\left(x - \dfrac{\pi}{2}\right) + 1 = 1$

$$P_2(x) = \frac{1}{2}f''(a)(x-a)^2 + f'(a)(x-a) + f(a) = \frac{1}{2}(-1)\left(x - \frac{\pi}{2}\right)^2 + 1$$

$$= 1 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2$$

(b)



(c) P_2 is a better approximation than P_1.

(d) The accuracy worsens as you move farther away from $x = a = \dfrac{\pi}{2}$.

110. True. y is a fourth-degree polynomial.

$\dfrac{d^n y}{dx^n} = 0$ when $n > 4$.

112. True

114. True. If $v(t) = c$ then

$a(t) = v'(t) = 0$.

116. (a) $(fg' - f'g)' = fg'' + f'g' - f'g' - f''g$

$= fg'' - f''g$ True

(b) $(fg)'' = (fg' + f'g)'$

$= fg'' + f'g' + f'g' + f''g$

$= fg'' + 2f'g' + f''g$

$\neq fg'' + f''g$ False

Section 2.4 The Chain Rule

$y = f(g(x))$	$u = g(x)$	$y = f(u)$
2. $y = \dfrac{1}{\sqrt{x+1}}$	$u = x + 1$	$y = u^{-1/2}$
4. $y = 3\tan(\pi x^2)$	$u = \pi x^2$	$y = 3\tan u$
6. $y = \cos\dfrac{3x}{2}$	$u = \dfrac{3x}{2}$	$y = \cos u$

8. $y = (2x^3 + 1)^2$

$y' = 2(2x^3 + 1)(6x^2) = 12x^2(2x^3 + 1)$

10. $y = 3(4 - x^2)^5$

$y' = 15(4 - x^2)(-2x) = -30x(4 - x^2)$

12. $f(t) = (9t + 2)^{2/3}$

$f'(t) = \dfrac{2}{3}(9t + 2)^{-1/3}(9) = \dfrac{6}{\sqrt[3]{9t + 2}}$

14. $g(x) = \sqrt{5 - 3x} = (5 - 3x)^{1/2}$

$g'(x) = \dfrac{1}{2}(5 - 3x)^{-1/2}(-3) = \dfrac{-3}{2\sqrt{5 - 3x}}$

16. $g(x) = \sqrt{x^2 - 2x + 1} = \sqrt{(x - 1)^2} = |x - 1|$

$g'(x) = \begin{cases} 1, & x > 1 \\ -1, & x < 1 \end{cases}$

18. $f(x) = -3(2 - 9x)^{1/4}$

$f'(x) = -\dfrac{3}{4}(2 - 9x)^{-3/4}(-9) = \dfrac{27}{4(2 - 9x)^{3/4}}$

20. $s(t) = (t^2 + 3t - 1)^{-1}$

$s'(t) = -1(t^2 + 3t - 1)^{-2}(2t + 3)$

$= \dfrac{-(2t + 3)}{(t^2 + 3t - 1)^2}$

22. $y = -5(t + 3)^{-3}$

$y' = 15(t + 3)^{-4} = \dfrac{15}{(t + 3)^4}$

24. $g(t) = (t^2 - 2)^{-1/2}$

$g'(t) = -\dfrac{1}{2}(t^2 - 2)^{-3/2}(2t) = -\dfrac{t}{(t^2 - 2)^{3/2}}$

26. $f(x) = x(3x - 9)^3$

$f'(x) = x[3(3x - 9)^2(3)] + (3x - 9)^3(1)$

$= (3x - 9)^2[9x + 3x - 9]$

$= 27(x - 3)^2(4x - 3)$

28. $y = \dfrac{1}{2}x^2\sqrt{16 - x^2}$

$y' = \dfrac{1}{2}x^2\left(\dfrac{1}{2}(16 - x^2)^{-1/2}(-2x)\right) + x(16 - x^2)^{1/2}$

$= \dfrac{-x^3}{2\sqrt{16 - x^2}} + x\sqrt{16 - x^2}$

$= \dfrac{-x(3x^2 - 32)}{2\sqrt{16 - x^2}}$

30. $y = \dfrac{x}{\sqrt{x^4 + 4}}$

$y' = \dfrac{(x^4 + 4)^{1/2}(1) - x\dfrac{1}{2}(x^4 + 4)^{-1/2}(4x^3)}{x^4 + 4}$

$= \dfrac{x^4 + 4 - 2x^4}{(x^4 + 4)^{3/2}} = \dfrac{4 - x^4}{(x^4 + 4)^{3/2}}$

32. $h(t) = \left(\dfrac{t^2}{t^3 + 2}\right)^2$

$h'(t) = 2\left(\dfrac{t^2}{t^3 + 2}\right)\left(\dfrac{(t^3 + 2)(2t) - t^2(3t^2)}{(t^3 + 2)^2}\right)$

$= \dfrac{2t^2(4t - t^4)}{(t^3 + 2)^3} = \dfrac{2t^3(4 - t^3)}{(t^3 + 2)^3}$

34. $g(x) = \left(\dfrac{3x^2 - 2}{2x + 3}\right)^3$

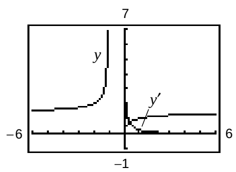
$$g'(x) = 3\left(\frac{3x^2 - 2}{2x + 3}\right)^2\left(\frac{(2x + 3)(6x) - (3x^2 - 2)(2)}{(2x + 3)^2}\right)$$

$$= \frac{3(3x^2 - 2)^2(6x^2 + 18x + 4)}{(2x + 3)^4} = \frac{6(3x^2 - 2)^2(3x^2 + 9x + 2)}{(2x + 3)^4}$$

36. $y = \sqrt{\dfrac{2x}{x + 1}}$

$$y' = \frac{1}{\sqrt{2x}(x + 1)^{3/2}}$$

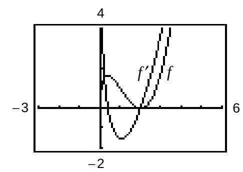
y' has no zeros.



38. $f(x) = \sqrt{x}(2 - x)^2$

$$f'(x) = \frac{(x - 2)(5x - 2)}{2\sqrt{x}}$$

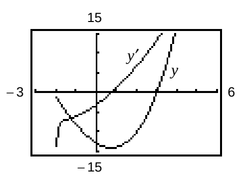
The zeros of f' correspond to the points on the graph of f where the tangent lines are horizontal.



40. $y = (t^2 - 9)\sqrt{t + 2}$

$$y' = \frac{5t^2 + 8t - 9}{2\sqrt{t + 2}}$$

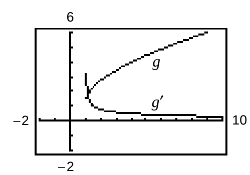
The zero of y' corresponds to the point on the graph of y where the tangent line is horizontal.



42. $g(x) = \sqrt{x - 1} + \sqrt{x + 1}$

$$g'(x) = \frac{1}{2\sqrt{x - 1}} + \frac{1}{2\sqrt{x + 1}}$$

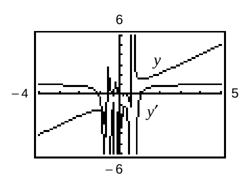
g' has no zeros.



44. $y = x^2 \tan \dfrac{1}{x}$

$$\frac{dy}{dx} = 2x \tan \frac{1}{x} - \sec^2 \frac{1}{x}$$

The zeros of y' correspond to the points on the graph of y where the tangent lines are horizontal.



46. (a) $y = \sin 3x$

$y' = 3 \cos 3x$

$y'(0) = 3$

3 cycles in $[0, 2\pi]$

(b) $y = \sin\left(\dfrac{x}{2}\right)$

$y' = \left(\dfrac{1}{2}\right)\cos\left(\dfrac{x}{2}\right)$

$y'(0) = \dfrac{1}{2}$

Half cycle in $[0, 2\pi]$

The slope of $\sin ax$ at the origin is a.

48. $y = \sin \pi x$

$$\frac{dy}{dx} = \pi \cos \pi x$$

50. $h(x) = \sec(x^2)$

$$h'(x) = 2x \sec(x^2) \tan(x^2)$$

52. $y = \cos(1 - 2x)^2 = \cos((1 - 2x)^2)$

$$y' = -\sin(1 - 2x)^2(2(1 - 2x)(-2)) = 4(1 - 2x)\sin(1 - 2x)^2$$

54. $g(\theta) = \sec\left(\frac{1}{2}\theta\right)\tan\left(\frac{1}{2}\theta\right)$

$$g'(\theta) = \sec\left(\frac{1}{2}\theta\right)\sec^2\left(\frac{1}{2}\theta\right)\frac{1}{2} + \tan\left(\frac{1}{2}\theta\right)\sec\left(\frac{1}{2}\theta\right)\tan\left(\frac{1}{2}\theta\right)\frac{1}{2}$$

$$= \frac{1}{2}\sec\left(\frac{1}{2}\theta\right)\left[\sec^2\left(\frac{1}{2}\theta\right) + \tan^2\left(\frac{1}{2}\theta\right)\right]$$

56. $g(v) = \dfrac{\cos v}{\csc v} = \cos v \cdot \sin v$

$$g'(v) = \cos v(\cos v) + \sin v(-\sin v) = \cos^2 v - \sin^2 v = \cos 2v$$

58. $y = 2\tan^3 x$

$$y' = 6\tan^2 x \cdot \sec^2 x$$

60. $g(t) = 5\cos^2 \pi t = 5(\cos \pi t)^2$

$$g'(t) = 10\cos \pi t(-\sin \pi t)(\pi)$$

$$= -10\pi(\sin \pi t)(\cos \pi t) = -5\pi \sin 2\pi t$$

62. $h(t) = 2\cot^2(\pi t + 2)$

$$h'(t) = 4\cot(\pi t + 2)(-\csc^2(\pi t + 2)(\pi))$$

$$= -4\pi \cot(\pi t + 2)\csc^2(\pi t + 2)$$

64. $y = 3x - 5\cos(\pi x)^2$

$$= 3x - 5\cos(\pi^2 x^2)$$

$$\frac{dy}{dx} = 3 + 5\sin(\pi^2 x^2)(2\pi^2 x)$$

$$= 3 + 10\pi^2 x \sin(\pi x)^2$$

66. $y = \sin x^{1/3} + (\sin x)^{1/3}$

$$y' = \cos x^{1/3}\left(\frac{1}{3}x^{-2/3}\right) + \frac{1}{3}(\sin x)^{-2/3}\cos x$$

$$= \frac{1}{3}\left[\frac{\cos x^{1/3}}{x^{2/3}} + \frac{\cos x}{(\sin x)^{2/3}}\right]$$

68. $y = (3x^3 + 4x)^{1/5}, \quad (2, 2)$

$$y' = \frac{1}{5}(3x^3 + 4x)^{-4/5}(9x^2 + 4)$$

$$= \frac{9x^2 + 4}{5(3x^3 + 4x)^{4/5}}$$

$$y'(2) = \frac{1}{2}$$

70. $f(x) = \dfrac{1}{(x^2 - 3x)^2} = (x^2 - 3x)^{-2}, \quad \left(4, \dfrac{1}{16}\right)$

$$f'(x) = -2(x^2 - 3x)^{-3}(2x - 3) = \frac{-2(2x - 3)}{(x^2 - 3x)^3}$$

$$f'(4) = -\frac{5}{32}$$

72. $f(x) = \dfrac{x + 1}{2x - 3}, \quad (2, 3)$

$$f'(x) = \frac{(2x - 3)(1) - (x + 1)(2)}{(2x - 3)^2} = \frac{-5}{(2x - 3)^2}$$

$$f'(2) = -5$$

74. $y = \dfrac{1}{x} + \sqrt{\cos x}, \quad \left(\dfrac{\pi}{2}, \dfrac{2}{\pi}\right)$

$$y' = -\frac{1}{x^2} - \frac{\sin x}{2\sqrt{\cos x}}$$

$y'(\pi/2)$ is undefined.

76. (a) $f(x) = \frac{1}{3}x\sqrt{x^2+5}, \quad (2, 2)$

$$f'(x) = \frac{1}{3}x\left[\frac{1}{2}(x^2+5)^{-1/2}(2x)\right] + \frac{1}{3}(x^2+5)^{1/2}$$

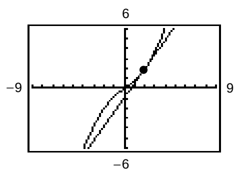
$$= \frac{x^2}{3\sqrt{x^2+5}} + \frac{1}{3}\sqrt{x^2+5}$$

$$f'(2) = \frac{4}{3(3)} + \frac{1}{3}(3) = \frac{13}{9}$$

Tangent line:

$$y - 2 = \frac{13}{9}(x-2) \Longrightarrow 13x - 9y - 8 = 0$$

(b)



78. (a) $f(x) = \tan^2 x, \quad \left(\frac{\pi}{4}, 1\right)$

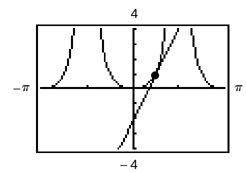
$$f'(x) = 2\tan x\sec^2 x$$

$$f'\left(\frac{\pi}{4}\right) = 2(1)(2) = 4$$

Tangent line:

$$y - 1 = 4\left(x - \frac{\pi}{4}\right) \Longrightarrow 4x - y + (1 - \pi) = 0$$

(b)



80. $f(x) = (x-2)^{-1}$

$$f'(x) = -(x-2)^{-2} = \frac{-1}{(x-2)^2}$$

$$f''(x) = 2(x-2)^{-3} = \frac{2}{(x-2)^3}$$

82. $f(x) = \sec^2 \pi x$

$$f'(x) = 2\sec \pi x(\pi\sec \pi x\tan \pi x)$$

$$= 2\pi\sec^2 \pi x\tan \pi x$$

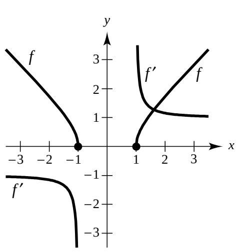
$$f''(x) = 2\pi\sec^2 \pi x(\sec^2 \pi x)(\pi) + 2\pi\tan \pi x(2\pi\sec^2 \pi x\tan \pi x)$$

$$= 2\pi^2\sec^4 \pi x + 4\pi^2\sec^2 \pi x\tan^2 \pi x$$

$$= 2\pi^2\sec^2 \pi x(\sec^2 \pi x + 2\tan^2 \pi x)$$

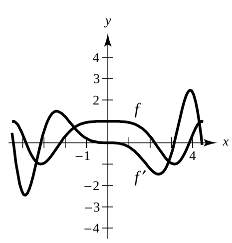
$$= 2\pi^2\sec^2 \pi x(3\sec^2 \pi x - 2)$$

84.



f is decreasing on $(-\infty, -1)$ so f' must be negative there. f is increasing on $(1, \infty)$ so f' must be postive there.

86.



The zeros of f' correspond to the points where the graph of f has horizontal tangents.

88. $g(x) = f(x^2)$

$$g'(x) = f'(x^2)(2x) \Longrightarrow g'(x) = 2x f'(x^2)$$

90. (a) $g(x) = \sin^2 x + \cos^2 x = 1 \Longrightarrow g'(x) = 0$

$g'(x) = 2 \sin x \cos x + 2 \cos x(-\sin x) = 0$

(b) $\tan^2 x + 1 = \sec^2 x$

$g(x) + 1 = f(x)$

Taking derivatives of both sides,

$g'(x) = f'(x)$.

Equivalently, $f'(x) = 2 \sec x \cdot \sec x \cdot \tan x$ and $g'(x) = 2 \tan x \cdot \sec^2 x$, which are the same.

92. $y = \frac{1}{3} \cos 12t - \frac{1}{4} \sin 12t$

$v = y' = \frac{1}{3}[-12 \sin 12t] - \frac{1}{4}[12 \cos 12t]$

$= -4 \sin 12t - 3 \cos 12t$

When $t = \pi/8$, $y = 0.25$ feet and $v = 4$ feet per second.

94. $y = A \cos \omega t$

(a) Amplitude: $A = \dfrac{3.5}{2} = 1.75$

$y = 1.75 \cos \omega t$

Period: $10 \Longrightarrow \omega = \dfrac{2\pi}{10} = \dfrac{\pi}{5}$

$y = 1.75 \cos \dfrac{\pi t}{5}$

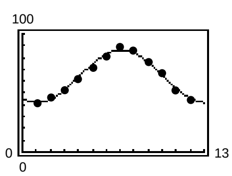
(b) $v = y' = 1.75\left[-\dfrac{\pi}{5} \sin \dfrac{\pi t}{5}\right]$

$= -0.35\,\pi \sin \dfrac{\pi t}{5}$

96. (a) Using a graphing utility, or by trial and error, you obtain a model of the form

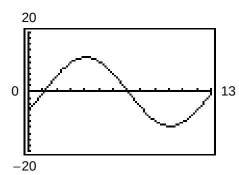
$T(t) = 64.18 - 22.15 \sin\left(\dfrac{\pi t}{6} + 1\right)$

(b)



(c) $T'(t) = -22.15 \cos\left(\dfrac{\pi t}{6} + 1\right)\left(\dfrac{\pi}{6}\right)$

$= -11.60 \cos\left(\dfrac{\pi t}{6} + 1\right)$



(d) The temperature changes most rapidly when $t \approx 4.1$ (April) and $t \approx 10.1$ (October). The temperature changes most slowly ($T'(t) = 0$) when $t \approx 1.1$ (January) and $t \approx 7.1$ (July).

98. (a) $g(x) = f(x) - 2 \Longrightarrow g'(x) = f'(x)$

(b) $h(x) = 2f(x) \Longrightarrow h'(x) = 2f'(x)$

(c) $r(x) = f(-3x) \Longrightarrow r'(x) = f'(-3x)(-3) = -3f'(-3x)$

Hence, you need to know $f'(-3x)$.

$r'(0) = -3f'(0) = (-3)\left(-\frac{1}{3}\right) = 1$

$r'(-1) = -3f'(3) = (-3)(-4) = 12$

(d) $s(x) = f(x + 2) \Longrightarrow s'(x) = f'(x + 2)$

Hence, you need to know $f'(x + 2)$.

$s'(-2) = f'(0) = -\frac{1}{3}$, etc.

x	-2	-1	0	1	2	3
$f'(x)$	4	$\frac{2}{3}$	$-\frac{1}{3}$	-1	-2	-4
$g'(x)$	4	$\frac{2}{3}$	$-\frac{1}{3}$	-1	-2	-4
$h'(x)$	8	$\frac{4}{3}$	$-\frac{2}{3}$	-2	-4	-8
$r'(x)$		12	1			
$s'(x)$	$-\frac{1}{3}$	-1	-2	-4		

100. $f(x + p) = f(x)$ for all x.

(a) Yes, $f'(x + p) = f'(x)$, which shows that f' is periodic as well.

(b) Yes, let $g(x) = f(2x)$, so $g'(x) = 2f'(2x)$. Since f' is periodic, so is g'.

102. If $f(-x) = -f(x)$, then

$$\frac{d}{dx}[f(-x)] = \frac{d}{x}[-f(x)]$$
$$f'(-x)(-1) = -f'(x)$$
$$f'(-x) = f'(x).$$

Thus, $f'(x)$ is even.

104. $|u| = \sqrt{u^2}$

$$\frac{d}{dx}[|u|] = \frac{d}{dx}\left[\sqrt{u^2}\right] = \frac{1}{2}(u^2)^{-1/2}(2uu') = \frac{uu'}{\sqrt{u^2}} = u'\frac{u}{|u|},\ u \neq 0$$

106. $f(x) = |x^2 - 4|$

$$f'(x) = 2x\left(\frac{x^2 - 4}{|x^2 - 4|}\right),\ x \neq \pm 2$$

108. $f(x) = |\sin x|$

$$f'(x) = \cos x\left(\frac{\sin x}{|\sin x|}\right),\ x \neq k\pi$$

110. (a) $f(x) = \sec(2x)$

$$f'(x) = 2(\sec 2x)(\tan 2x)$$
$$f''(x) = 2[2(\sec 2x)(\tan 2x)]\tan 2x + 2(\sec 2x)(\sec^2 2x)(2)$$
$$= 4[(\sec 2x)(\tan^2 2x) + \sec^3 2x]$$

$$f\left(\frac{\pi}{6}\right) = \sec\left(\frac{\pi}{3}\right) = 2$$

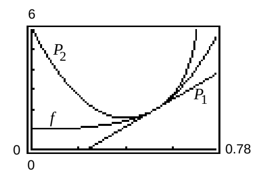
$$f'\left(\frac{\pi}{6}\right) = 2\sec\left(\frac{\pi}{3}\right)\tan\left(\frac{\pi}{3}\right) = 4\sqrt{3}$$

$$f''\left(\frac{\pi}{6}\right) = 4[2(3) + 2^3] = 56$$

$$P_1(x) = 4\sqrt{3}\left(x - \frac{\pi}{6}\right) + 2$$

$$P_2(x) = \frac{1}{2}(56)\left(x - \frac{\pi}{6}\right)^2 + 4\sqrt{3}\left(x - \frac{\pi}{6}\right) + 2$$
$$= 28\left(x - \frac{\pi}{6}\right)^2 + 4\sqrt{3}\left(x - \frac{\pi}{6}\right) + 2$$

(b)



(c) P_2 is a better approximation than P_1.

(d) The accuracy worsens as you move away from $x = \pi/6$.

112. False. If $f(x) = \sin^2 2x$, then $f'(x) = 2(\sin 2x)(2\cos 2x)$.

114. False. First apply the Product Rule.

Section 2.5 Implicit Differentiations

2. $x^2 - y^2 = 16$

$2x - 2yy' = 0$

$y' = \dfrac{x}{y}$

4. $x^3 + y^3 = 8$

$3x^2 + 3y^2y' = 0$

$y' = -\dfrac{x^2}{y^2}$

6. $x^2y + y^2x = -2$

$x^2y' + 2xy + y^2 + 2yxy' = 0$

$(x^2 + 2xy)y' = -(y^2 + 2xy)$

$y' = \dfrac{-y(y + 2x)}{x(x + 2y)}$

8. $(xy)^{1/2} - x + 2y = 0$

$\frac{1}{2}(xy)^{-1/2}(xy' + y) - 1 + 2y' = 0$

$\dfrac{x}{2\sqrt{xy}}y' + \dfrac{y}{2\sqrt{xy}} - 1 + 2y' = 0$

$xy' + y - 2\sqrt{xy} + 4\sqrt{xy}\,y' = 0$

$y' = \dfrac{2\sqrt{xy} - y}{4\sqrt{xy} + x}$

10. $2 \sin x \cos y = 1$

$2[\sin x(-\sin y)y' + \cos y(\cos x)] = 0$

$y' = \dfrac{\cos x \cos y}{\sin x \sin y}$

$= \cot x \cot y$

12. $(\sin \pi x + \cos \pi y)^2 = 2$

$2(\sin \pi x + \cos \pi y)[\pi \cos \pi x - \pi(\sin \pi y)y'] = 0$

$\pi \cos \pi x - \pi(\sin \pi y)y' = 0$

$y' = \dfrac{\cos \pi x}{\sin \pi y}$

14. $\cot y = x - y$

$(-\csc^2 y)y' = 1 - y'$

$y' = \dfrac{1}{1 - \csc^2 y}$

$= \dfrac{1}{-\cot^2 y} = -\tan^2 y$

16. $x = \sec\dfrac{1}{y}$

$1 = -\dfrac{y'}{y^2}\sec\dfrac{1}{y}\tan\dfrac{1}{y}$

$y' = \dfrac{-y^2}{\sec(1/y)\tan(1/y)} = -y^2\cos\left(\dfrac{1}{y}\right)\cot\left(\dfrac{1}{y}\right)$

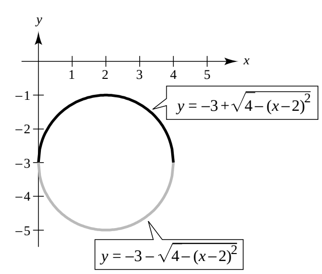
18. (a) $(x^2 - 4x + 4) + (y^2 + 6y + 9) = -9 + 4 + 9$

$(x - 2)^2 + (y + 3)^2 = 4$ (Circle)

$(y + 3)^2 = 4 - (x - 2)^2$

$y = -3 \pm \sqrt{4 - (x - 2)^2}$

(b)



(c) Explicitly:

$\dfrac{dy}{dx} = \pm\dfrac{1}{2}[4 - (x - 2)^2]^{-1/2}(-2)(x - 2)$

$= \dfrac{\mp(x - 2)}{(\sqrt{4 - (x - 2)^2}}$

$= \dfrac{-(x - 2)}{\pm\sqrt{4 - (x - 2)^2}}$

$= \dfrac{-(x - 2)}{-3 \pm \sqrt{4 - (x - 2)^2} + 3}$

$= \dfrac{-(x - 2)}{y + 3}$

(d) Implicitly:

$2x + 2yy' - 4 + 6y' = 0$

$(2y + 6)y' = -2(x - 2)$

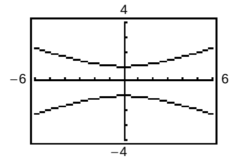
$y' = \dfrac{-(x - 2)}{y + 3}$

20. (a) $9y^2 = x^2 + 9$

$$y^2 = \frac{x^2}{9} + 1 = \frac{x^2 + 9}{9}$$

$$y = \frac{\pm\sqrt{x^2 + 9}}{3}$$

(b)



(c) Explicitly: $\dfrac{dy}{dx} = \dfrac{\pm\frac{1}{2}(x^2 + 9)^{-1/2}(2x)}{3} = \dfrac{\pm x}{3\sqrt{x^2 + 9}} = \dfrac{\pm x}{3(\pm 3y)} = \dfrac{x}{9y}$

(d) Implicitly: $9y^2 - x^2 = 9$

$$18yy' - 2x = 0$$

$$18yy' = 2x$$

$$y' = \frac{2x}{18y} = \frac{x}{9y}$$

22. $x^2 - y^3 = 0$

$$2x - 3y^2y' = 0$$

$$y' = \frac{2x}{3y^2}$$

At $(1, 1)$: $y' = \dfrac{2}{3}$.

24. $(x + y)^3 = x^3 + y^3$

$$x^3 + 3x^2y + 3xy^2 + y^3 = x^3 + y^3$$

$$3x^2y + 3xy^2 = 0$$

$$x^2y + xy^2 = 0$$

$$x^2y' + 2xy + 2xyy' + y^2 = 0$$

$$(x^2 + 2xy)y' = -(y^2 + 2xy)$$

$$y' = -\frac{y(y + 2x)}{x(x + 2y)}$$

At $(-1, 1)$: $y' = -1$.

26. $x^3 + y^3 = 4xy + 1$

$$3x^2 + 3y^2y' = 4xy' + 4y$$

$$(3y^2 - 4x)y' = 4y - 3x^2$$

$$y' = \frac{4y - 3x^2}{(3y^2 - 4x)}$$

At $(2, 1)$, $y' = \dfrac{4 - 12}{3 - 8} = \dfrac{8}{5}$

28. $x\cos y = 1$

$$x[-y'\sin y] + \cos y = 0$$

$$y' = \frac{\cos y}{x \sin y}$$

$$= \frac{1}{x}\cot y = \frac{\cot y}{x}$$

At $\left(2, \dfrac{\pi}{3}\right)$: $y' = \dfrac{1}{2\sqrt{3}}$.

30. $(4 - x)y^2 = x^3$

$$(4 - x)(2yy') + y^2(-1) = 3x^2$$

$$y' = \frac{3x^2 + y^2}{2y(4 - x)}$$

At $(2, 2)$: $y' = 2$.

32. $x^3 + y^3 - 6xy = 0$

$$3x^2 + 3y^2y' - 6xy' - 6y = 0$$

$$y'(3y^2 - 6x) = 6y - 3x^2$$

$$y' = \frac{6y - 3x^2}{3y^2 - 6x} = \frac{2y - x^2}{y^2 - 2x}$$

At $\left(\dfrac{4}{3}, \dfrac{8}{3}\right)$: $y' = \dfrac{(16/3) - (16/9)}{(64/9) - (8/3)} = \dfrac{32}{40} = \dfrac{4}{5}$.

34. $\cos y = x$

$-\sin y \cdot y' = 1$

$y' = \dfrac{-1}{\sin y},\ 0 < y < \pi$

$\sin^2 y + \cos^2 y = 1$

$\sin^2 y = 1 - \cos^2 y$

$\sin y = \sqrt{1 - \cos^2 y} = \sqrt{1 - x^2}$

$y' = \dfrac{-1}{\sqrt{1 - x^2}},\ -1 < x < 1$

36.
$$x^2y^2 - 2x = 3$$
$$2x^2yy' + 2xy^2 - 2 = 0$$
$$x^2yy' + xy^2 - 1 = 0$$
$$y' = \frac{1 - xy^2}{x^2y}$$
$$2xyy' + x^2(y')^2 + x^2yy'' + 2xyy' + y^2 = 0$$
$$4xyy' + x^2(y')^2 + x^2yy'' + y^2 = 0$$
$$\frac{4 - 4xy^2}{x} + \frac{(1 - xy^2)^2}{x^2y^2} + x^2yy'' + y^2 = 0$$
$$4xy^2 - 4x^2y^4 + 1 - 2xy^2 + x^2y^4 + x^4y^3y'' + x^2y^4 = 0$$
$$x^4y^3y'' = 2x^2y^4 - 2xy^2 - 1$$
$$y'' = \frac{2x^2y^4 - 2xy^2 - 1}{x^4y^3}$$

38. $1 - xy = x - y$

$y - xy = x - 1$

$y = \dfrac{x - 1}{1 - x} = -1$

$y' = 0$

$y'' = 0$

40. $y^2 = 4x$

$2yy' = 4$

$y' = \dfrac{2}{y}$

$y'' = -2y^{-2}y' = \left[\dfrac{-2}{y^2}\right] \cdot \dfrac{2}{y} = \dfrac{-4}{y^3}$

42. $y^2 = \dfrac{x - 1}{x^2 + 1}$

$2yy' = \dfrac{(x^2 + 1)(1) - (x - 1)(2x)}{(x^2 + 1)^2}$

$= \dfrac{x^2 + 1 - 2x^2 + 2x}{(x^2 + 1)^2}$

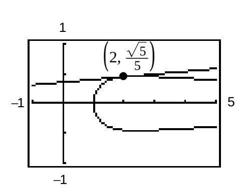
$y' = \dfrac{1 + 2x - x^2}{2y(x^2 + 1)^2}$

At $\left(2, \dfrac{\sqrt{5}}{5}\right)$: $y' = \dfrac{1 + 4 - 4}{\left[\left(2\sqrt{5}\right)/5\right](4 + 1)^2} = \dfrac{1}{10\sqrt{5}}$.

Tangent line: $y - \dfrac{\sqrt{5}}{5} = \dfrac{1}{10\sqrt{5}}(x - 2)$

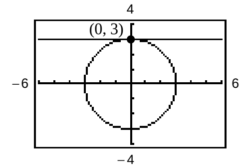
$10\sqrt{5}y - 10 = x - 2$

$x - 10\sqrt{5}y + 8 = 0$



44. $x^2 + y^2 = 9$

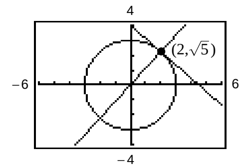
$$y' = \frac{-x}{y}$$



At $(0, 3)$:

Tangent line: $y = 3$

Normal line: $x = 0$.



At $(2, \sqrt{5})$:

Tangent line: $y - \sqrt{5} = \dfrac{-2}{\sqrt{5}}(x - 2) \Longrightarrow 2x + \sqrt{5}y - 9 = 0$

Normal line: $y - \sqrt{5} = \dfrac{\sqrt{5}}{2}(x - 2) \Longrightarrow \sqrt{5}x - 2y = 0$.

46. $y^2 = 4x$

$2yy' = 4$

$$y' = \frac{2}{y} = 1 \text{ at } (1, 2)$$

Equation of normal at $(1, 2)$ is $y - 2 = -1(x - 1)$, $y = 3 - x$. The centers of the circles must be on the normal and at a distance of 4 units from $(1, 2)$. Therefore,

$$(x - 1)^2 + [(3 - x) - 2]^2 = 16$$

$$2(x - 1)^2 = 16$$

$$x = 1 \pm 2\sqrt{2}.$$

Centers of the circles: $(1 + 2\sqrt{2}, 2 - 2\sqrt{2})$ and $(1 - 2\sqrt{2}, 2 + 2\sqrt{2})$

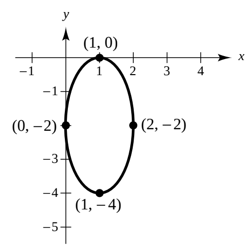
Equations: $(x - 1 - 2\sqrt{2})^2 + (y - 2 + 2\sqrt{2})^2 = 16$

$(x - 1 + 2\sqrt{2})^2 + (y - 2 - 2\sqrt{2})^2 = 16$

48. $4x^2 + y^2 - 8x + 4y + 4 = 0$

$8x + 2yy' - 8 + 4y' = 0$

$$y' = \frac{8 - 8x}{2y + 4} = \frac{4 - 4x}{y + 2}$$



Horizontal tangents occur when $x = 1$:

$4(1)^2 + y^2 - 8(1) + 4y + 4 = 0$

$$y^2 + 4y = y(y + 4) = 0 \Longrightarrow y = 0, -4$$

Horizontal tangents: $(1, 0), (1, -4)$.

Vertical tangents occur when $y = -2$:

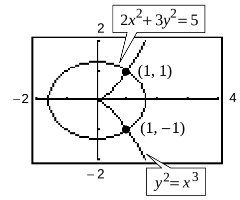
$4x^2 + (-2)^2 - 8x + 4(-2) + 4 = 0$

$$4x^2 - 8x = 4x(x - 2) = 0 \Longrightarrow x = 0, 2$$

Vertical tangents: $(0, -2), (2, -2)$.

50. Find the points of intersection by letting $y^2 = x^3$ in the equation $2x^2 + 3y^2 = 5$.

$2x^2 + 3x^3 = 5$ and $3x^3 + 2x^2 - 5 = 0$



Intersect when $x = 1$.

Points of intersection: $(1, \pm 1)$

$\underline{y^2 = x^3}$: $\quad\quad$ $\underline{2x^2 + 3y^2 = 5}$:

$2yy' = 3x^2$ $\quad\quad$ $4x + 6yy' = 0$

$y' = \dfrac{3x^2}{2y}$ $\quad\quad$ $y' = -\dfrac{2x}{3y}$

At $(1, 1)$, the slopes are:

$y' = \dfrac{3}{2}$ $\quad\quad$ $y' = -\dfrac{2}{3}$.

At $(1, -1)$, the slopes are:

$y' = -\dfrac{3}{2}$ $\quad\quad$ $y' = \dfrac{2}{3}$.

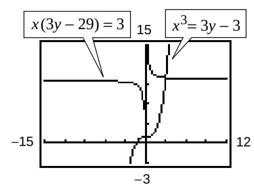
Tangents are perpendicular.

52. Rewriting each equation and differentiating,

$x^3 = 3(y - 1)$ $\quad\quad$ $x(3y - 29) = 3$

$y = \dfrac{x^3}{3} + 1$ $\quad\quad$ $y = \dfrac{1}{3}\left(\dfrac{3}{x} + 29\right)$

$y' = x^2$ $\quad\quad$ $y' = -\dfrac{1}{x^2}$.

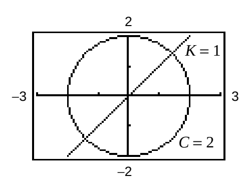


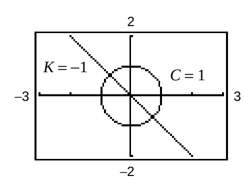
For each value of x, the derivatives are negative reciprocals of each other. Thus, the tangent lines are orthogonal at both points of intersection.

54. $x^2 + y^2 = C^2$ $\quad\quad$ $y = Kx$

$2x + 2yy' = 0$ $\quad\quad$ $y' = K$

$y' = -\dfrac{x}{y}$





At the point of intersection (x, y) the product of the slopes is $(-x/y)(K) = (-x/Kx)(K) = -1$. The curves are orthogonal.

56. $x^2 - 3xy^2 + y^3 = 10$

(a) $2x - 3y^2 - 6xyy' + 3y^2y' = 0$

$(-6xy + 3y^2)y' = 3y^2 - 2x$

$y' = \dfrac{3y^2 - 2x}{3y^2 - 6xy}$

(b) $2x\dfrac{dx}{dt} - 3y^2\dfrac{dx}{dt} - 6xy\dfrac{dy}{dt} + 3y^2\dfrac{dy}{dt} = 0$

$(2x - 3y^2)\dfrac{dx}{dt} = (6xy - 3y^2)\dfrac{dy}{dt}$

58. (a) $4 \sin x \cos y = 1$

$4 \sin x(-\sin y)y' + 4 \cos x \cos y = 0$

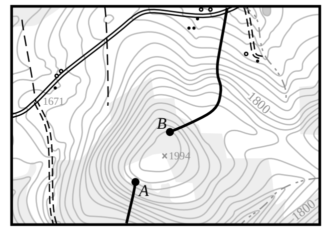
$y' = \dfrac{\cos x \cos y}{\sin x \sin y}$

(b) $4 \sin x(-\sin y)\dfrac{dy}{dt} + 4 \cos x\dfrac{dx}{dt}\cos y = 0$

$\cos x \cos y\dfrac{dx}{dt} = \sin x \sin y\dfrac{dy}{dt}$

60. Given an implicit equation, first differentiate both sides with respect to x. Collect all terms involving y' on the left, and all other terms to the right. Factor out y' on the left side. Finally, divide both sides by the left-hand factor that does not contain y'.

62.



Use starting point B.

64. $$\sqrt{x} + \sqrt{y} = \sqrt{c}$$

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$$

Tangent line at (x_0, y_0):

$$y - y_0 = -\frac{\sqrt{y_0}}{\sqrt{x_0}}(x - x_0)$$

x-intercept: $\left(x_0 + \sqrt{x_0}\sqrt{y_0}, 0\right)$

y-intercept: $\left(0, y_0 + \sqrt{x_0}\sqrt{y_0}\right)$

Sum of intercepts:

$$\left(x_0 + \sqrt{x_0}\sqrt{y_0}\right) + \left(y_0 + \sqrt{x_0}\sqrt{y_0}\right) = x_0 + 2\sqrt{x_0}\sqrt{y_0} + y_0 = \left(\sqrt{x_0} + \sqrt{y_0}\right)^2 = \left(\sqrt{c}\right)^2 = c.$$

Section 2.6 Related Rates

2. $y = 2(x^2 - 3x)$

$$\frac{dy}{dt} = (4x - 6)\frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{1}{4x - 6}\frac{dy}{dt}$$

(a) When $x = 3$ and $\frac{dx}{dt} = 2$, $\frac{dy}{dt} = [4(3) - 6](2) = 12$

(b) When $x = 1$ and $\frac{dy}{dt} = 5$, $\frac{dx}{dt} = \frac{1}{4(1) - 6}(5) = -\frac{5}{2}$

4. $$x^2 + y^2 = 25$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = \left(-\frac{x}{y}\right)\frac{dx}{dt}$$

$$\frac{dx}{dt} = \left(-\frac{y}{x}\right)\frac{dy}{dt}$$

(a) When $x = 3$, $y = 4$, and $dx/dt = 8$,

$$\frac{dy}{dt} = -\frac{3}{4}(8) = -6$$

(b) When $x = 4$, $y = 3$, and $dy/dt = -2$,

$$\frac{dx}{dt} = -\frac{3}{4}(-2) = \frac{3}{2}.$$

6. $y = \dfrac{1}{1 + x^2}$

$\dfrac{dx}{dt} = 2$

$\dfrac{dy}{dt} = \left[\dfrac{-2x}{(1 + x^2)^2}\right]\dfrac{dx}{dt}$

(a) When $x = -2$,

$$\frac{dy}{dt} = \frac{-2(-2)(2)}{25} = \frac{8}{25} \text{ cm/sec.}$$

(b) When $x = 0$,

$$\frac{dy}{dt} = 0 \text{ cm/sec.}$$

(c) When $x = 2$,

$$\frac{dy}{dt} = \frac{-2(2)(2)}{25} = \frac{-8}{25} \text{ cm/sec.}$$

8. $y = \sin x$

$\dfrac{dx}{dt} = 2$

$\dfrac{dy}{dt} = \cos x \dfrac{dx}{dt}$

(a) When $x = \pi/6$,

$$\frac{dy}{dt} = \left(\cos\frac{\pi}{6}\right)(2) = \sqrt{3} \text{ cm/sec.}$$

(b) When $x = \pi/4$,

$$\frac{dy}{dt} = \left(\cos\frac{\pi}{4}\right)(2) = \sqrt{2} \text{ cm/sec.}$$

(c) When $x = \pi/3$,

$$\frac{dy}{dt} = \left(\cos\frac{\pi}{3}\right)(2) = 1 \text{ cm/sec.}$$

10. (a) $\dfrac{dx}{dt}$ negative $\Longrightarrow \dfrac{dy}{dt}$ negative

(b) $\dfrac{dy}{dt}$ positive $\Longrightarrow \dfrac{dx}{dt}$ positive

12. Answers will vary. See page 145.

14. $D = \sqrt{x^2 + y^2} = \sqrt{x^2 + \sin^2 x}$

$\dfrac{dx}{dt} = 2$

$$\frac{dD}{dt} = \frac{1}{2}(x^2 + \sin^2 x)^{-1/2}(2x + 2\sin x\cos x)\frac{dx}{dt} = \frac{x + \sin x\cos x}{\sqrt{x^2 + \sin^2 x}}\frac{dx}{dt} = \frac{2 + 2\sin x\cos x}{\sqrt{x^2 + \sin^2 x}}$$

16. $A = \pi r^2$

$\dfrac{dA}{dt} = 2\pi r\dfrac{dr}{dt}$

If dr/dt is constant, dA/dt is not constant.

$\dfrac{dA}{dt}$ depends on r and $\dfrac{dr}{dt}$.

18. $V = \dfrac{4}{3}\pi r^3$

$\dfrac{dr}{dt} = 2$

$\dfrac{dV}{dt} = 4\pi r^2 \dfrac{dr}{dt}$

(a) When $r = 6$, $\dfrac{dV}{dt} = 4\pi(6)^2(2) = 288\pi \text{ in}^3\text{/min.}$

When $r = 24$, $\dfrac{dV}{dt} = 4\pi(24)^2(2) = 4608\pi \text{ in}^3\text{/min.}$

(b) If dr/dt is constant, dV/dt is proportional to r^2.

20. $V = x^3$

$\dfrac{dx}{dt} = 3$

$\dfrac{dV}{dt} = 3x^2 \dfrac{dx}{dt}$

(a) When $x = 1$,

$$\frac{dV}{dt} = 3(1)^2(3) = 9 \text{ cm}^3\text{/sec.}$$

(b) When $x = 10$,

$$\frac{dV}{dt} = 3(10)^2(3) = 900 \text{ cm}^3\text{/sec.}$$

22. $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2(3r) = \pi r^3$

$\frac{dr}{dt} = 2$

$\frac{dV}{dt} = 3\pi r^2 \frac{dr}{dt}$

(a) When $r = 6$,

$$\frac{dV}{dt} = 3\pi(6)^2(2) = 216\pi \text{ in}^3/\text{min}.$$

(b) When $r = 24$,

$$\frac{dV}{dt} = 3\pi(24)^2(2) = 3456\pi \text{ in}^3/\text{min}.$$

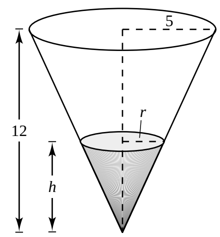
24. $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \frac{25}{144}h^3 = \frac{25\pi}{3(144)}h^3$

$\left(\text{By similar triangles, } \frac{r}{5} = \frac{h}{12} \Rightarrow r = \frac{5}{12}h.\right)$

$\frac{dV}{dt} = 10$

$\frac{dV}{dt} = \frac{25\pi}{144}h^2 \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \left(\frac{144}{25\pi h^2}\right)\frac{dV}{dt}$

When $h = 8$, $\frac{dh}{dt} = \frac{144}{25\pi(64)}(10) = \frac{9}{10\pi}$ ft/min.

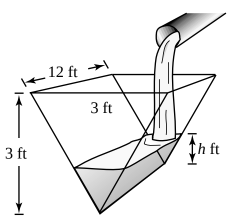


26. $V = \frac{1}{2}bh(12) = 6bh = 6h^2$ (since $b = h$)

(a) $\frac{dV}{dt} = 12h\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{1}{12h}\frac{dV}{dt}$

When $h = 1$ and $\frac{dV}{dt} = 2$, $\frac{dh}{dt} = \frac{1}{12(1)}(2) = \frac{1}{6}$ ft/min

(b) If $\frac{dh}{dt} = \frac{3}{8}$ and $h = 2$, then $\frac{dV}{dt} = 12(2)\left(\frac{3}{8}\right) = 9 \text{ ft}^3/\text{min}$.



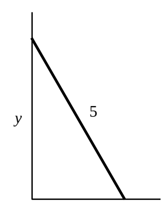
28. $x^2 + y^2 = 25$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$\frac{dx}{dt} = -\frac{y}{x} \cdot \frac{dy}{dt} = -\frac{0.15y}{x}$ since $\frac{dy}{dt} = 0.15$.

When $x = 2.5$,

$y = \sqrt{18.75}$, $\frac{dx}{dt} = -\frac{\sqrt{18.75}}{2.5}0.15 \approx -0.26$ m/sec



30. Let L be the length of the rope.

(a) $L^2 = 144 + x^2$

$2L\frac{dL}{dt} = 2x\frac{dx}{dt}$

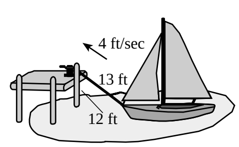
$\frac{dx}{dt} = \frac{L}{x} \cdot \frac{dL}{dt} = -\frac{4L}{x}$ since $\frac{dL}{dt} = -4$ ft/sec.

When $L = 13$,

$x = \sqrt{L^2 - 144} = \sqrt{169 - 144} = 5$

$\frac{dx}{dt} = -\frac{4(13)}{5} = -\frac{52}{5} = -10.4$ ft/sec.

Speed of the boat increases as it approaches the dock.



(b) If $\frac{dx}{dt} = -4$, and $L = 13$,

$\frac{dL}{dt} = \frac{x}{L}\frac{dx}{dt}$

$= \frac{5}{13}(-4)$

$= \frac{-20}{13}$ ft/sec

As $L \to 0$, $\frac{dL}{dt}$ increases.

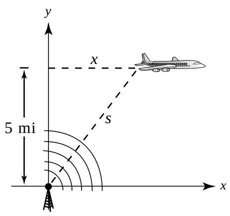
32. $x^2 + y^2 = s^2$

$$2x\frac{dx}{dt} + 0 = 2s\frac{ds}{dt} \qquad \left(\text{since } \frac{dy}{dt} = 0\right)$$

$$\frac{dx}{dt} = \frac{s}{x}\frac{ds}{dt}$$

When $s = 10$, $x = \sqrt{100 - 25} = \sqrt{75} = 5\sqrt{3}$

$$\frac{dx}{dt} = \frac{10}{5\sqrt{3}}(-240) = \frac{-480}{\sqrt{3}} = -160\sqrt{3} \approx -277.13 \text{ mph.}$$

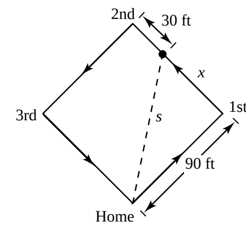


34. $s^2 = 90^2 + x^2$

$x = 60$

$$\frac{dx}{dt} = 28$$

$$\frac{ds}{dt} = \frac{x}{s} \cdot \frac{dx}{dt}$$



When $x = 60$,

$$s = \sqrt{90^2 + 60^2} = 30\sqrt{13}$$

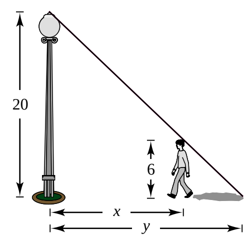
$$\frac{ds}{dt} = \frac{60}{30\sqrt{13}}(28) = \frac{56}{\sqrt{13}} \approx 15.53 \text{ ft/sec.}$$

36. (a) $$\frac{20}{6} = \frac{y}{y - x}$$

$$20y - 20x = 6y$$

$$14y = 20x$$

$$y = \frac{10}{7}x$$



$$\frac{dx}{dt} = -5$$

$$\frac{dy}{dt} = \frac{10}{7}\frac{dx}{dt} = \frac{10}{7}(-5) = \frac{-50}{7} \text{ ft/sec}$$

(b) $$\frac{d(y - x)}{dt} = \frac{dy}{dt} - \frac{dx}{dt} = \frac{-50}{7} - (-5) = \frac{-50}{7} + \frac{35}{7} = \frac{-15}{7} \text{ ft/sec}$$

38. $x(t) = \frac{3}{5}\sin \pi t$, $x^2 + y^2 = 1$

(a) Period: $\frac{2\pi}{\pi} = 2$ seconds

(b) When $x = \frac{3}{5}$, $y = \sqrt{1 - \left(\frac{3}{5}\right)^2} = \frac{4}{5}$ m.

Lowest point: $\left(0, \frac{4}{5}\right)$

(c) When $x = \frac{3}{10}$, $y = \sqrt{1 - \left(\frac{1}{4}\right)^2} = \frac{\sqrt{15}}{4}$ and

$$\frac{3}{10} = \frac{3}{5}\sin \pi t \Longrightarrow \sin \pi t = \frac{1}{2} \Longrightarrow t = \frac{1}{6}$$

$$\frac{dx}{dt} = \frac{3}{5}\pi \cos \pi t$$

$$x^2 + y^2 = 1$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \Longrightarrow \frac{dy}{dt} = \frac{-x}{y}\frac{dx}{dt}.$$

Thus, $$\frac{dy}{dt} = \frac{-3/10}{\sqrt{15/4}} \cdot \frac{3}{5}\pi \cos\left(\frac{\pi}{6}\right)$$

$$= \frac{-9\pi}{25\sqrt{5}} = \frac{-9\sqrt{5}\pi}{125}.$$

Speed $= \left|\frac{-9\sqrt{5}\pi}{125}\right| \approx 0.5058$ m/sec

40. $$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

$$\frac{dR_1}{dt} = 1$$

$$\frac{dR_2}{dt} = 1.5$$

$$\frac{1}{R^2} \cdot \frac{dR}{dt} = \frac{1}{{R_1}^2} \cdot \frac{dR_1}{dt} + \frac{1}{{R_2}^2} \cdot \frac{dR_2}{dt}$$

When $R_1 = 50$ and $R_2 = 75$,

$$R = 30$$

$$\frac{dR}{dt} = (30)^2\left[\frac{1}{(50)^2}(1) + \frac{1}{(75)^2}(1.5)\right]$$

$$= 0.6 \text{ ohms/sec.}$$

42. $$rg \tan \theta = v^2$$

$32r \tan \theta = v^2$, r is a constant.

$$32r \sec^2\theta \frac{d\theta}{dt} = 2v\frac{dv}{dt}$$

$$\frac{dv}{dt} = \frac{16r}{v} \sec^2 \theta \; \frac{d\theta}{dt}$$

Likewise, $\dfrac{d\theta}{dt} = \dfrac{v}{16r} \cos^2 \theta \; \dfrac{dv}{dt}$.

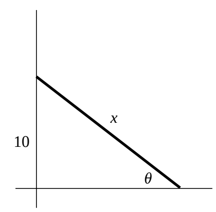
44. $$\sin \theta = \frac{10}{x}$$

$$\frac{dx}{dt} = (-1)\text{ft/sec}$$

$$\cos \theta\left(\frac{d\theta}{dt}\right) = \frac{-10}{x^2} \cdot \frac{dx}{dt}$$

$$\frac{d\theta}{dt} = \frac{-10}{x^2} \frac{dx}{dt}(\sec \theta)$$

$$= \frac{-10}{25^2}(-1)\frac{25}{\sqrt{25^2 - 10^2}} = \frac{10}{25}\frac{1}{5\sqrt{21}} = \frac{2}{25\sqrt{21}} = \frac{2\sqrt{21}}{525} \approx 0.017 \text{ rad/sec}$$

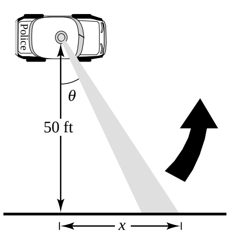


46. $$\tan \theta = \frac{x}{50}$$

$$\frac{d\theta}{dt} = 30(2\pi) = 60\pi \text{ rad/min} = \pi \text{ rad/sec}$$

$$\sec^2 \theta\left(\frac{d\theta}{dt}\right) = \frac{1}{50}\left(\frac{dx}{dt}\right)$$

$$\frac{dx}{dt} = 50 \sec^2 \theta\left(\frac{d\theta}{dt}\right)$$



(a) When $\theta = 30°$, $\dfrac{dx}{dt} = \dfrac{200\pi}{3}$ ft/sec.

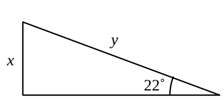
(b) When $\theta = 60°$, $\dfrac{dx}{dt} = 200\pi$ ft/sec.

(c) When $\theta = 70°$, $\dfrac{dx}{dt} \approx 427.43\pi$ ft/sec.

48. $\sin 22° = \dfrac{x}{y}$

$$0 = -\frac{x}{y^2} \cdot \frac{dy}{dt} + \frac{1}{y} \cdot \frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{x}{y} \cdot \frac{dy}{dt} = (\sin 22°)(240) \approx 89.9056 \text{ mi/hr}$$



50. (a) $dy/dt = 3(dx/dt)$ means that y changes three times as fast as x changes.

(b) y changes slowly when $x \approx 0$ or $x \approx L$. y changes more rapidly when x is near the middle of the interval.

52. $L^2 = 144 + x^2$; acceleration of the boat $= \dfrac{d^2x}{dt^2}$.

First derivative: $2L\dfrac{dL}{dt} = 2x\dfrac{dx}{dt}$

$$L\frac{dL}{dt} = x\frac{dx}{dt}$$

Second derivative: $L\dfrac{d^2L}{dt^2} + \dfrac{dL}{dt}\cdot\dfrac{dL}{dt} = x\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt}\cdot\dfrac{dx}{dt}$

$$\frac{d^2x}{dt^2} = \left(\frac{1}{x}\right)\left[L\frac{d^2L}{dt^2} + \left(\frac{dL}{dt}\right)^2 - \left(\frac{dx}{dt}\right)^2\right]$$

When $L = 13$, $x = 5$, $\dfrac{dx}{dt} = -10.4$, and $\dfrac{dL}{dt} = -4$ (see Exercise 30). Since $\dfrac{dL}{dt}$ is constant, $\dfrac{d^2L}{dt^2} = 0$.

$$\frac{d^2x}{dt^2} = \frac{1}{5}[13(0) + (-4)^2 - (-10.4)^2]$$

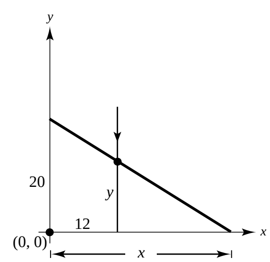
$$= \frac{1}{5}[16 - 108.16] = \frac{1}{5}[-92.16] = -18.432 \text{ ft/sec}^2$$

54. $y(t) = -4.9t^2 + 20$

$$\frac{dy}{dt} = -9.8t$$

$y(1) = -4.9 + 20 = 15.1$

$y'(1) = -9.8$



By similar triangles, $\dfrac{20}{x} = \dfrac{y}{x - 12}$

$$20x - 240 = xy.$$

When $y = 15.1$, $20x - 240 = x(15.1)$

$$(20 - 15.1)x = 240$$

$$x = \frac{240}{4.9}.$$

$$20x - 240 = xy$$

$$20\frac{dx}{dt} = x\frac{dy}{dt} + y\frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{x}{20 - y}\frac{dy}{dt}$$

At $t = 1$, $\dfrac{dx}{dt} = \dfrac{240/4.9}{20 - 15.1}(-9.8) \approx -97.96$ m/sec.