

# CHAPTER 3

## Applications of Differentiation

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# CHAPTER 3

## Applications of Differentiation

### Section 3.1 Extrema on an Interval

Solutions to Odd-Numbered Exercises

1.  $f(x) = \frac{x^2}{x^2 + 4}$

$$f'(x) = \frac{(x^2 + 4)(2x) - (x^2)(2x)}{(x^2 + 4)^2} = \frac{8x}{(x^2 + 4)^2}$$

$$f'(0) = 0$$

5.  $f(x) = (x + 2)^{2/3}$

$$f'(x) = \frac{2}{3}(x + 2)^{-1/3}$$

$f'(-2)$  is undefined.

9. Critical numbers:  $x = 1, 2, 3$

$x = 1, 3$ : absolute maximum

$x = 2$ : absolute minimum

13.  $g(t) = t\sqrt{4-t}, t < 3$

$$g'(t) = t \left[ \frac{1}{2}(4-t)^{-1/2}(-1) \right] + (4-t)^{1/2}$$

$$= \frac{1}{2}(4-t)^{-1/2}[-t + 2(4-t)]$$

$$= \frac{8-3t}{2\sqrt{4-t}}$$

Critical number is  $t = \frac{8}{3}$ .

17.  $f(x) = 2(3-x), [-1, 2]$

$$f'(x) = -2 \implies \text{No critical numbers}$$

Left endpoint:  $(-1, 8)$  Maximum

Right endpoint:  $(2, 2)$  Minimum

3.  $f(x) = x + \frac{27}{2x^2} = x + \frac{27}{x}x^{-2}$

$$f'(x) = 1 - 27x^{-3} = 1 - \frac{27}{x^3}$$

$$f'(3) = 1 - \frac{27}{3^3} = 1 - 1 = 0$$

7. Critical numbers:  $x = 2$

$x = 2$ : absolute maximum

11.  $f(x) = x^2(x-3) = x^3 - 3x^2$

$$f'(x) = 3x^2 - 6x = 3x(x-2)$$

Critical numbers:  $x = 0, x = 2$

15.  $h(x) = \sin^2 x + \cos x, 0 < x < 2\pi$

$$h'(x) = 2 \sin x \cos x - \sin x = \sin x(2 \cos x - 1)$$

On  $(0, 2\pi)$ , critical numbers:  $x = \frac{\pi}{3}, x = \pi, x = \frac{5\pi}{3}$

19.  $f(x) = -x^2 + 3x, [0, 3]$

$$f'(x) = -2x + 3$$

Left endpoint:  $(0, 0)$  Minimum

Critical number:  $(\frac{3}{2}, \frac{9}{4})$  Maximum

Right endpoint:  $(3, 0)$  Minimum

21.  $f(x) = x^3 - \frac{3}{2}x^2, [-1, 2]$

$$f'(x) = 3x^2 - 3x = 3x(x - 1)$$

Left endpoint:  $(-1, -\frac{5}{2})$  Minimum

Right endpoint: (2, 2) Maximum

Critical number: (0, 0)

Critical number:  $(1, -\frac{1}{2})$

25.  $g(t) = \frac{t^2}{t^2 + 3}, [-1, 1]$

$$g'(t) = \frac{6t}{(t^2 + 3)^2}$$

Left endpoint:  $(-1, \frac{1}{4})$  Maximum

Critical number: (0, 0) Minimum

Right endpoint:  $(1, \frac{1}{4})$  Maximum

29.  $f(x) = \cos \pi x, [0, \frac{1}{6}]$

$$f'(x) = -\pi \sin \pi x$$

Left endpoint: (0, 1) Maximum

Right endpoint:  $(\frac{1}{6}, \frac{\sqrt{3}}{2})$  Minimum

33. (a) Minimum: (0, -3)

Maximum: (2, 1)

(b) Minimum: (0, -3)

(c) Maximum: (2, 1)

(d) No extrema

23.  $f(x) = 3x^{2/3} - 2x, [-1, 1]$

$$f'(x) = 2x^{-1/3} - 2 = \frac{2(1 - \sqrt[3]{x})}{\sqrt[3]{x}}$$

Left endpoint: (-1, 5) Maximum

Critical number: (0, 0) Minimum

Right endpoint: (1, 1)

27.  $h(s) = \frac{1}{s - 2}, [0, 1]$

$$h'(s) = \frac{-1}{(s - 2)^2}$$

Left endpoint:  $(0, -\frac{1}{2})$  Maximum

Right endpoint: (1, -1) Minimum

31.  $y = \frac{4}{x} + \tan \frac{\pi x}{8}, [1, 2]$

$$y' = \frac{-4}{x^2} + \frac{\pi}{8} \sec^2 \frac{\pi x}{8} = 0$$

$$\frac{\pi}{8} \sec^2 \frac{\pi x}{8} = \frac{4}{x^2}$$

On the interval [1, 2], this equation has no solutions. Thus, there are no critical numbers.

Left endpoint:  $(1, \sqrt{2} + 3) \approx (1, 4.4142)$  Maximum

Right endpoint: (2, 3) Minimum

35.  $f(x) = x^2 - 2x$

(a) Minimum: (1, -1)

Maximum: (-1, 3)

(b) Maximum: (3, 3)

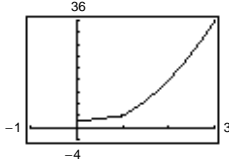
(c) Minimum: (1, -1)

(d) Minimum: (1, -1)

$$37. f(x) = \begin{cases} 2x + 2, & 0 \leq x \leq 1 \\ 4x^2, & 1 < x \leq 3 \end{cases}$$

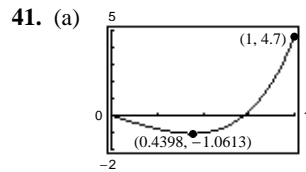
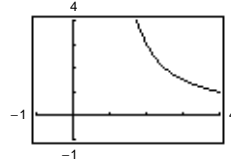
Left endpoint: (0, 2) Minimum

Right endpoint: (3, 36) Maximum



$$39. f(x) = \frac{3}{x-1}, (1, 4]$$

Right endpoint: (4, 1) Minimum



Maximum: (1, 4.7) (endpoint)

Minimum: (0.4398, -1.0613)

(b)  $f(x) = 3.2x^5 + 5x^3 - 3.5x, [0, 1]$

$$f'(x) = 16x^4 + 15x^2 - 3.5$$

$$16x^4 + 15x^2 - 3.5 = 0$$

$$x^2 = \frac{-15 \pm \sqrt{(15)^2 - 4(16)(-3.5)}}{2(16)}$$

$$= \frac{-15 \pm \sqrt{449}}{32}$$

$$x = \sqrt{\frac{-15 + \sqrt{449}}{32}} \approx 0.4398$$

$$f(0) = 0$$

$$f(1) = 4.7 \text{ Maximum (endpoint)}$$

$$f\left(\sqrt{\frac{-15 + \sqrt{449}}{32}}\right) \approx -1.0613$$

Minimum: (0.4398, -1.0613)

43.  $f(x) = (1 + x^3)^{1/2}, [0, 2]$

$$f'(x) = \frac{3}{2}x^2(1 + x^3)^{-1/2}$$

$$f''(x) = \frac{3}{4}(x^4 + 4x)(1 + x^3)^{-3/2}$$

$$f'''(x) = -\frac{3}{8}(x^6 + 20x^3 - 8)(1 + x^3)^{-5/2}$$

Setting  $f''' = 0$ , we have  $x^6 + 20x^3 - 8 = 0$ .

$$x^3 = \frac{-20 \pm \sqrt{400 - 4(1)(-8)}}{2}$$

$$x = \sqrt[3]{-10 \pm \sqrt{108}} = \sqrt{3} - 1$$

In the interval  $[0, 2]$ , choose

$$x = \sqrt[3]{-10 + \sqrt{108}} = \sqrt{3} - 1 \approx 0.732.$$

$$\left| f''\left(\sqrt[3]{-10 + \sqrt{108}}\right) \right| \approx 1.47 \text{ is the maximum value.}$$

45.  $f(x) = (x + 1)^{2/3}, [0, 2]$

$$f'(x) = \frac{2}{3}(x + 1)^{-1/3}$$

$$f''(x) = -\frac{2}{9}(x + 1)^{-4/3}$$

$$f'''(x) = \frac{8}{27}(x + 1)^{-7/3}$$

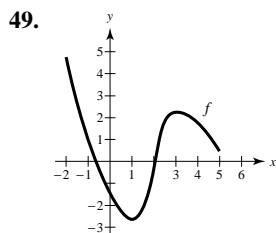
$$f^{(4)}(x) = -\frac{56}{81}(x + 1)^{-10/3}$$

$$f^{(5)}(x) = \frac{560}{243}(x + 1)^{-13/3}$$

$$|f^{(4)}(0)| = \frac{56}{81} \text{ is the maximum value.}$$

47.  $f(x) = \tan x$

$f$  is continuous on  $[0, \pi/4]$  but not on  $[0, \pi]$ .  $\lim_{x \rightarrow \pi/2^-} \tan x = \infty$ .



51. (a) Yes
- 
- (b) No

53. (a) No
- 
- (b) Yes

55.  $P = VI - RI^2 = 12I - 0.5I^2, 0 \leq I \leq 15$

$P = 0$  when  $I = 0$ .

$P = 67.5$  when  $I = 15$ .

$P' = 12 - I = 0$

Critical number:  $I = 12$  amps

When  $I = 12$  amps,  $P = 72$ , the maximum output.

No, a 20-amp fuse would not increase the power output.  $P$  is decreasing for  $I > 12$ .

57.  $S = 6hs + \frac{3s^2}{2} \left( \frac{\sqrt{3} - \cos \theta}{\sin \theta} \right), \frac{\pi}{6} \leq \theta \leq \frac{\pi}{2}$

$$\begin{aligned} \frac{dS}{d\theta} &= \frac{3s^2}{2} (-\sqrt{3} \csc \theta \cot \theta + \csc^2 \theta) \\ &= \frac{3s^2}{2} \csc \theta (-\sqrt{3} \cot \theta + \csc \theta) = 0 \end{aligned}$$

$\csc \theta = \sqrt{3} \cot \theta$

$\sec \theta = \sqrt{3}$

$\theta = \operatorname{arcsec} \sqrt{3} \approx 0.9553$  radians

$S\left(\frac{\pi}{6}\right) = 6hs + \frac{3s^2}{2}(\sqrt{3})$

$S\left(\frac{\pi}{2}\right) = 6hs + \frac{3s^2}{2}(\sqrt{3})$

$S(\operatorname{arcsec} \sqrt{3}) = 6hs + \frac{3s^2}{2}(\sqrt{2})$

$S$  is minimum when  $\theta = \operatorname{arcsec} \sqrt{3} \approx 0.9553$  radians.

59. (a)  $y = ax^2 + bx + c$

$y' = 2ax + b$

The coordinates of  $B$  are  $(500, 30)$ , and those of  $A$  are  $(-500, 45)$ .

From the slopes at  $A$  and  $B$ ,

$-1000a + b = -0.09$

$1000a + b = 0.06$ .

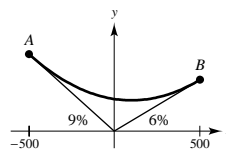
Solving these two equations, you obtain  $a = 3/40000$  and  $b = -3/200$ . From the points  $(500, 30)$  and  $(-500, 45)$ , you obtain

$30 = \frac{3}{40000} 500^2 + 500 \left( \frac{-3}{200} \right) + c$

$45 = \frac{3}{40000} 500^2 - 500 \left( \frac{-3}{200} \right) + c$ .

In both cases,  $c = 18.75 = \frac{75}{4}$ . Thus,

$$y = \frac{3}{40000} x^2 - \frac{3}{200} x + \frac{75}{4}$$
.



—CONTINUED—

## 59. —CONTINUED—

(b)	$x$	-500	-400	-300	-200	-100	0	100	200	300	400	500
	$d$	0	.75	3	6.75	12	18.75	12	6.75	3	.75	0

$$\text{For } -500 \leq x \leq 0, d = (ax^2 + bx + c) - (-0.09x).$$

$$\text{For } 0 \leq x \leq 500, d = (ax^2 + bx + c) - (0.06x).$$

(c) The lowest point on the highway is (100, 18), which is not directly over the point where the two hillsides come together.

61. True. See Exercise 25.

63. True.

## Section 3.2 Rolle's Theorem and the Mean Value Theorem

1. Rolle's Theorem does not apply to  $f(x) = 1 - |x - 1|$  over  $[0, 2]$  since  $f$  is not differentiable at  $x = 1$ .

$$3. f(x) = x^2 - x - 2 = (x - 2)(x + 1)$$

$x$ -intercepts:  $(-1, 0), (2, 0)$

$$f'(x) = 2x - 1 = 0 \text{ at } x = \frac{1}{2}.$$

$$5. f(x) = x\sqrt{x+4}$$

$x$ -intercepts:  $(-4, 0), (0, 0)$

$$f'(x) = x \frac{1}{2}(x+4)^{-1/2} + (x+4)^{1/2}$$

$$= (x+4)^{-1/2} \left( \frac{x}{2} + (x+4) \right)$$

$$f'(x) = \left( \frac{3}{2}x + 4 \right) (x+4)^{-1/2} = 0 \text{ at } x = -\frac{8}{3}$$

$$7. f(x) = x^2 - 2x, [0, 2]$$

$$f(0) = f(2) = 0$$

$f$  is continuous on  $[0, 2]$ .  $f$  is differentiable on  $(0, 2)$ . Rolle's Theorem applies.

$$f'(x) = 2x - 2$$

$$2x - 2 = 0 \Rightarrow x = 1$$

$c$  value: 1

$$9. f(x) = (x-1)(x-2)(x-3), [1, 3]$$

$$f(1) = f(3) = 0$$

$f$  is continuous on  $[1, 3]$ .  $f$  is differentiable on  $(1, 3)$ . Rolle's Theorem applies.

$$f(x) = x^3 - 6x^2 + 11x - 6$$

$$f'(x) = 3x^2 - 12x + 11$$

$$3x^2 - 12x + 11 = 0 \Rightarrow x = \frac{6 \pm \sqrt{3}}{3}$$

$$c = \frac{6 - \sqrt{3}}{3}, c = \frac{6 + \sqrt{3}}{3}$$

$$11. f(x) = x^{2/3} - 1, [-8, 8]$$

$$f(-8) = f(8) = 3$$

$f$  is continuous on  $[-8, 8]$ .  $f$  is not differentiable on  $(-8, 8)$  since  $f'(0)$  does not exist. Rolle's Theorem does not apply.

$$13. f(x) = \frac{x^2 - 2x - 3}{x + 2}, [-1, 3]$$

$$f(-1) = f(3) = 0$$

$f$  is continuous on  $[-1, 3]$ . (**Note:** The discontinuity,  $x = -2$ , is not in the interval.)  $f$  is differentiable on  $(-1, 3)$ . Rolle's Theorem applies.

$$f'(x) = \frac{(x+2)(2x-2) - (x^2-2x-3)(1)}{(x+2)^2} = 0$$

$$\frac{x^2 + 4x - 1}{(x+2)^2} = 0$$

$$x = \frac{-4 \pm 2\sqrt{5}}{2} = -2 \pm \sqrt{5}$$

$$c \text{ value: } -2 + \sqrt{5}$$

$$15. f(x) = \sin x, [0, 2\pi]$$

$$f(0) = f(2\pi) = 0$$

$f$  is continuous on  $[0, 2\pi]$ .  $f$  is differentiable on  $(0, 2\pi)$ . Rolle's Theorem applies.

$$f'(x) = \cos x$$

$$c \text{ values: } \frac{\pi}{2}, \frac{3\pi}{2}$$

$$17. f(x) = \frac{6x}{\pi} - 4 \sin^2 x, \left[0, \frac{\pi}{6}\right]$$

$$f(0) = f\left(\frac{\pi}{6}\right) = 0$$

$f$  is continuous on  $[0, \pi/6]$ .  $f$  is differentiable on  $(0, \pi/6)$ . Rolle's Theorem applies.

$$f'(x) = \frac{6}{\pi} - 8 \sin x \cos x = 0$$

$$\frac{6}{\pi} = 8 \sin x \cos x$$

$$\frac{3}{4\pi} = \frac{1}{2} \sin 2x$$

$$\frac{3}{2\pi} = \sin 2x$$

$$\frac{1}{2} \arcsin\left(\frac{3}{2\pi}\right) = x$$

$$x \approx 0.2489$$

$$c \text{ value: } 0.2489$$

$$19. f(x) = \tan x, [0, \pi]$$

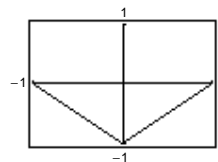
$$f(0) = f(\pi) = 0$$

$f$  is not continuous on  $[0, \pi]$  since  $f(\pi/2)$  does not exist. Rolle's Theorem does not apply.

$$21. f(x) = |x| - 1, [-1, 1]$$

$$f(-1) = f(1) = 0$$

$f$  is continuous on  $[-1, 1]$ .  $f$  is not differentiable on  $(-1, 1)$  since  $f'(0)$  does not exist. Rolle's Theorem does not apply.



$$23. \quad f(x) = 4x - \tan \pi x, \quad \left[-\frac{1}{4}, \frac{1}{4}\right]$$

$$f\left(-\frac{1}{4}\right) = f\left(\frac{1}{4}\right) = 0$$

$f$  is continuous on  $[-1/4, 1/4]$ .  $f$  is differentiable on  $(-1/4, 1/4)$ . Rolle's Theorem applies.

$$f'(x) = 4 - \pi \sec^2 \pi x = 0$$

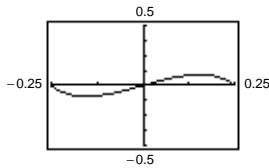
$$\sec^2 \pi x = \frac{4}{\pi}$$

$$\sec \pi x = \pm \frac{2}{\sqrt{\pi}}$$

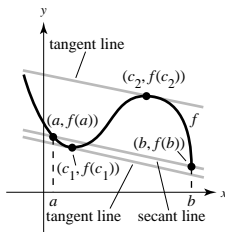
$$x = \pm \frac{1}{\pi} \operatorname{arcsec} \frac{2}{\sqrt{\pi}} = \pm \frac{1}{\pi} \arccos \frac{\sqrt{\pi}}{2}$$

$$\approx \pm 0.1533 \text{ radian}$$

$c$  values:  $\pm 0.1533$  radian



27.



31.  $f(x) = x^2$  is continuous on  $[-2, 1]$  and differentiable on  $(-2, 1)$ .

$$\frac{f(1) - f(-2)}{1 - (-2)} = \frac{1 - 4}{3} = -1$$

$f'(x) = 2x = -1$  when  $x = -\frac{1}{2}$ . Therefore,

$$c = -\frac{1}{2}$$

$$25. \quad f(t) = -16t^2 + 48t + 32$$

$$(a) \quad f(1) = f(2) = 64$$

(b)  $v = f'(t)$  must be 0 at some time in  $(1, 2)$ .

$$f'(t) = -32t + 48 = 0$$

$$t = \frac{3}{2} \text{ seconds}$$

$$29. \quad f(x) = \frac{1}{x-3}, \quad [0, 6]$$

$f$  has a discontinuity at  $x = 3$ .

33.  $f(x) = x^{2/3}$  is continuous on  $[0, 1]$  and differentiable on  $(0, 1)$ .

$$\frac{f(1) - f(0)}{1 - 0} = 1$$

$$f'(x) = \frac{2}{3}x^{-1/3} = 1$$

$$x = \left(\frac{2}{3}\right)^3 = \frac{8}{27}$$

$$c = \frac{8}{27}$$



35.  $f(x) = \sqrt{2-x}$  is continuous on  $[-7, 2]$  and differentiable on  $(-7, 2)$ .

$$\frac{f(2) - f(-7)}{2 - (-7)} = \frac{0 - 3}{9} = -\frac{1}{3}$$

$$f'(x) = \frac{-1}{2\sqrt{2-x}} = -\frac{1}{3}$$

$$2\sqrt{2-x} = 3$$

$$\sqrt{2-x} = \frac{3}{2}$$

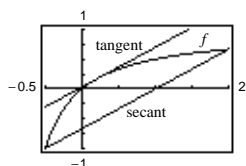
$$2-x = \frac{9}{4}$$

$$x = -\frac{1}{4}$$

$$c = -\frac{1}{4}$$

39.  $f(x) = \frac{x}{x+1}$  on  $[-\frac{1}{2}, 2]$ .

(a)



(b) Secant line:

$$\text{slope} = \frac{f(2) - f(-1/2)}{2 - (-1/2)} = \frac{2/3 - (-1)}{5/2} = \frac{2}{3}$$

$$y - \frac{2}{3} = \frac{2}{3}(x - 2)$$

$$3y - 2 = 2x - 4$$

$$3y - 2x + 2 = 0$$

37.  $f(x) = \sin x$  is continuous on  $[0, \pi]$  and differentiable on  $(0, \pi)$ .

$$\frac{f(\pi) - f(0)}{\pi - 0} = \frac{0 - 0}{\pi} = 0$$

$$f'(x) = \cos x = 0$$

$$c = \frac{\pi}{2}$$

(c)  $f'(x) = \frac{1}{(x+1)^2} = \frac{2}{3}$

$$(x+1)^2 = \frac{3}{2}$$

$$x = -1 \pm \sqrt{\frac{3}{2}} = -1 \pm \frac{\sqrt{6}}{2}$$

In the interval  $[-1/2, 2]$ ,  $c = -1 + (\sqrt{6}/2)$ .

$$f(c) = \frac{-1 + (\sqrt{6}/2)}{[-1 + (\sqrt{6}/2)] + 1} = \frac{-2 + \sqrt{6}}{\sqrt{6}} = \frac{-2}{\sqrt{6}} + 1$$

Tangent line:  $y - 1 + \frac{2}{\sqrt{6}} = \frac{2}{3}\left(x - \frac{\sqrt{6}}{2} + 1\right)$

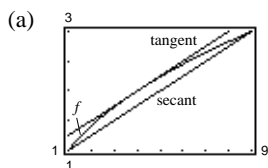
$$y - 1 + \frac{\sqrt{6}}{3} = \frac{2}{3}x - \frac{\sqrt{6}}{3} + \frac{2}{3}$$

$$3y - 2x - 5 + 2\sqrt{6} = 0$$

41.  $f(x) = \sqrt{x}$ ,  $[1, 9]$

$(1, 1), (9, 3)$

$$m = \frac{3 - 1}{9 - 1} = \frac{1}{4}$$



(b) Secant line:  $y - 1 = \frac{1}{4}(x - 1)$

$$y = \frac{1}{4}x + \frac{3}{4}$$

$$0 = x - 4y + 3$$

(c)  $f'(x) = \frac{1}{2\sqrt{x}}$

$$\frac{f(9) - f(1)}{9 - 1} = \frac{1}{4}$$

$$\frac{1}{2\sqrt{c}} = \frac{1}{4}$$

$$\sqrt{c} = 2$$

$$c = 4$$

$$(c, f(c)) = (4, 2)$$

$$m = f'(4) = \frac{1}{4}$$

Tangent line:  $y - 2 = \frac{1}{4}(x - 4)$

$$y = \frac{1}{4}x + 1$$

$$0 = x - 4y + 4$$

43.  $s(t) = -4.9t^2 + 500$

(a)  $V_{\text{avg}} = \frac{s(3) - s(0)}{3 - 0} = \frac{455.9 - 500}{3} = -14.7$  m/sec

(b)  $s(t)$  is continuous on  $[0, 3]$  and differentiable on  $(0, 3)$ .  
Therefore, the Mean Value Theorem applies.

$$v(t) = s'(t) = -9.8t = -14.7 \text{ m/sec}$$

$$t = \frac{-14.7}{-9.8} = 1.5 \text{ seconds}$$

45. No. Let  $f(x) = x^2$  on  $[-1, 2]$ .

$$f'(x) = 2x$$

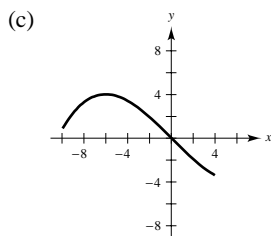
$f'(0) = 0$  and zero is in the interval  $(-1, 2)$  but  
 $f(-1) \neq f(2)$ .

47. Let  $S(t)$  be the position function of the plane. If  $t = 0$  corresponds to 2 P.M.,  $S(0) = 0$ ,  $S(5.5) = 2500$  and the Mean Value Theorem says that there exists a time  $t_0$ ,  $0 < t_0 < 5.5$ , such that

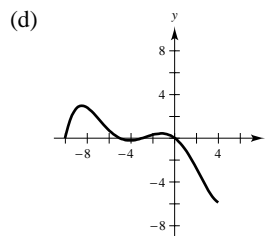
$$S'(t_0) = v(t_0) = \frac{2500 - 0}{5.5 - 0} \approx 454.54.$$

Applying the Intermediate Value Theorem to the velocity function on the intervals  $[0, t_0]$  and  $[t_0, 5.5]$ , you see that there are at least two times during the flight when the speed was 400 miles per hour. ( $0 < 400 < 454.54$ )

49. (a)  $f$  is continuous on  $[-10, 4]$  and changes sign, ( $f(-8) > 0$ ,  $f(3) < 0$ ). By the Intermediate Value Theorem, there exists at least one value of  $x$  in  $[-10, 4]$  satisfying  $f(x) = 0$ .



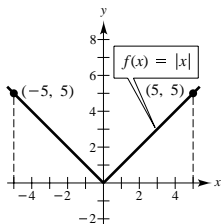
- (b) There exist real numbers  $a$  and  $b$  such that  $-10 < a < b < 4$  and  $f(a) = f(b) = 2$ . Therefore, by Rolle's Theorem there exists at least one number  $c$  in  $(-10, 4)$  such that  $f'(c) = 0$ . This is called a critical number.



- (e) No,  $f'$  did not have to be continuous on  $[-10, 4]$ .

51.  $f$  is continuous on  $[-5, 5]$  and does not satisfy the conditions of the Mean Value Theorem.  
 $\Rightarrow f$  is not differentiable on  $(-5, 5)$ .

Example:  $f(x) = |x|$



53. False.  $f(x) = 1/x$  has a discontinuity at  $x = 0$ .

55. True. A polynomial is continuous and differentiable everywhere.

57. Suppose that  $p(x) = x^{2n+1} + ax + b$  has two real roots  $x_1$  and  $x_2$ . Then by Rolle's Theorem, since  $p(x_1) = p(x_2) = 0$ , there exists  $c$  in  $(x_1, x_2)$  such that  $p'(c) = 0$ . But  $p'(x) = (2n+1)x^{2n} + a \neq 0$ , since  $n > 0$ ,  $a > 0$ . Therefore,  $p(x)$  cannot have two real roots.

59. If  $p(x) = Ax^2 + Bx + C$ , then

$$\begin{aligned} p'(x) &= 2Ax + B = \frac{f(b) - f(a)}{b - a} = \frac{(Ab^2 + Bb + C) - (Aa^2 + Ba + C)}{b - a} \\ &= \frac{A(b^2 - a^2) + B(b - a)}{b - a} \\ &= \frac{(b - a)[A(b + a) + B]}{b - a} \\ &= A(b + a) + B. \end{aligned}$$

Thus,  $2Ax = A(b + a)$  and  $x = (b + a)/2$  which is the midpoint of  $[a, b]$ .

61.  $f(x) = \frac{1}{2} \cos x$  differentiable on  $(-\infty, \infty)$ .

$$f'(x) = -\frac{1}{2} \sin x$$

$$-\frac{1}{2} \leq f'(x) \leq \frac{1}{2} \Rightarrow f'(x) < 1 \text{ for all real numbers.}$$

Thus, from Exercise 60,  $f$  has, at most, one fixed point. ( $x \approx 0.4502$ )

## Section 3.3 Increasing and Decreasing Functions and the First Derivative Test

1.  $f(x) = x^2 - 6x + 8$

Increasing on:  $(3, \infty)$ Decreasing on:  $(-\infty, 3)$ 

5.  $f(x) = \frac{1}{x^2} = x^{-2}$

$$f'(x) = \frac{-2}{x^3}$$

Discontinuity:  $x = 0$ 

Test intervals:	$-\infty < x < 0$	$0 < x < \infty$
Sign of $f'(x)$ :	$f' > 0$	$f' < 0$
Conclusion:	Increasing	Decreasing

Increasing on  $(-\infty, 0)$ Decreasing on  $(0, \infty)$ 

9.  $y = x\sqrt{16 - x^2}$  Domain:  $[-4, 4]$

$$y' = \frac{-2(x^2 - 8)}{\sqrt{16 - x^2}} = \frac{-2}{\sqrt{16 - x^2}}(x - 2\sqrt{2})(x + 2\sqrt{2})$$

Critical numbers:  $x = \pm 2\sqrt{2}$ 

Test intervals:	$-4 < x < -2\sqrt{2}$	$-2\sqrt{2} < x < 2\sqrt{2}$	$2\sqrt{2} < x < 4$
Sign of $y'$ :	$y' < 0$	$y' > 0$	$y' < 0$
Conclusion:	Decreasing	Increasing	Decreasing

Increasing on  $(-2\sqrt{2}, 2\sqrt{2})$ Decreasing on  $(-4, -2\sqrt{2}), (2\sqrt{2}, 4)$ 

11.  $f(x) = x^2 - 6x$

$$f'(x) = 2x - 6 = 0$$

Critical number:  $x = 3$ 

Test intervals:	$-\infty < x < 3$	$3 < x < \infty$
Sign of $f'(x)$ :	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing

Increasing on:  $(3, \infty)$ Decreasing on:  $(-\infty, 3)$ Relative minimum:  $(3, -9)$ 

3.  $y = \frac{x^3}{4} - 3x$

Increasing on:  $(-\infty, -2), (2, \infty)$ Decreasing on:  $(-2, 2)$ 

7.  $g(x) = x^2 - 2x - 8$

$$g'(x) = 2x - 2$$

Critical number:  $x = 1$ 

Test intervals:	$-\infty < x < 1$	$1 < x < \infty$
Sign of $g'(x)$ :	$g' < 0$	$g' > 0$
Conclusion:	Decreasing	Increasing

Increasing on:  $(1, \infty)$ Decreasing on:  $(-\infty, 1)$ 

13.  $f(x) = -2x^2 + 4x + 3$

$$f'(x) = -4x + 4 = 0$$

Critical number:  $x = 1$ 

Test intervals:	$-\infty < x < 1$	$1 < x < \infty$
Sign of $f'(x)$ :	$f' > 0$	$f' < 0$
Conclusion:	Increasing	Decreasing

Increasing on:  $(-\infty, 1)$ Decreasing on:  $(1, \infty)$ Relative maximum:  $(1, 5)$

15.  $f(x) = 2x^3 + 3x^2 - 12x$

$f'(x) = 6x^2 + 6x - 12 = 6(x + 2)(x - 1) = 0$

Critical numbers:  $x = -2, 1$ 

Test intervals:	$-\infty < x < -2$	$-2 < x < 1$	$1 < x < \infty$
Sign of $f'(x)$ :	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

Increasing on:  $(-\infty, -2), (1, \infty)$ Decreasing on:  $(-2, 1)$ Relative maximum:  $(-2, 20)$ Relative minimum:  $(1, -7)$ 

17.  $f(x) = x^2(3 - x) = 3x^2 - x^3$

$f'(x) = 6x - 3x^2 = 3x(2 - x)$

Critical numbers:  $x = 0, 2$ 

Test intervals:	$-\infty < x < 0$	$0 < x < 2$	$2 < x < \infty$
Sign of $f'(x)$ :	$f' < 0$	$f' > 0$	$f' < 0$
Conclusion:	Decreasing	Increasing	Decreasing

Increasing on:  $(0, 2)$ Decreasing on:  $(-\infty, 0), (2, \infty)$ Relative maximum:  $(2, 4)$ Relative minimum:  $(0, 0)$ 

19.  $f(x) = \frac{x^5 - 5x}{5}$

$f'(x) = x^4 - 1$

Critical numbers:  $x = -1, 1$ 

Test intervals:	$-\infty < x < -1$	$-1 < x < 1$	$1 < x < \infty$
Sign of $f'(x)$ :	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

Increasing on:  $(-\infty, -1), (1, \infty)$ Decreasing on:  $(-1, 1)$ Relative maximum:  $(-1, \frac{4}{5})$ Relative minimum:  $(1, -\frac{4}{5})$

21.  $f(x) = x^{1/3} + 1$

$$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$$

Critical number:  $x = 0$ 

Test intervals:	$-\infty < x < 0$	$0 < x < \infty$
Sign of $f'(x)$ :	$f' > 0$	$f' > 0$
Conclusion:	Increasing	Increasing

Increasing on:  $(-\infty, \infty)$ 

No relative extrema

23.  $f(x) = (x - 1)^{2/3}$

$$f'(x) = \frac{2}{3(x - 1)^{1/3}}$$

Critical number:  $x = 1$ 

Test intervals:	$-\infty < x < 1$	$1 < x < \infty$
Sign of $f'(x)$ :	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing

Increasing on:  $(1, \infty)$ Decreasing on:  $(-\infty, 1)$ Relative minimum:  $(1, 0)$ 

25.  $f(x) = 5 - |x - 5|$

$$f'(x) = -\frac{x - 5}{|x - 5|} = \begin{cases} 1, & x < 5 \\ -1, & x > 5 \end{cases}$$

Critical number:  $x = 5$ 

Test intervals:	$-\infty < x < 5$	$5 < x < \infty$
Sign of $f'(x)$ :	$f' > 0$	$f' < 0$
Conclusion:	Increasing	Decreasing

Increasing on:  $(-\infty, 5)$ Decreasing on:  $(5, \infty)$ Relative maximum:  $(5, 5)$ 

27.  $f(x) = x + \frac{1}{x}$

$$f'(x) = 1 - \frac{1}{x^2} = \frac{x^2 - 1}{x^2}$$

Critical numbers:  $x = -1, 1$ Discontinuity:  $x = 0$ 

Test intervals:	$-\infty < x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x < \infty$
Sign of $f'(x)$ :	$f' > 0$	$f' < 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Decreasing	Increasing

Increasing on:  $(-\infty, -1), (1, \infty)$ Decreasing on:  $(-1, 0), (0, 1)$ Relative maximum:  $(-1, -2)$ Relative minimum:  $(1, 2)$

29.  $f(x) = \frac{x^2}{x^2 - 9}$

$$f'(x) = \frac{(x^2 - 9)(2x) - (x^2)(2x)}{(x^2 - 9)^2} = \frac{-18x}{(x^2 - 9)^2}$$

Critical number:  $x = 0$

Discontinuities:  $x = -3, 3$

Test intervals:	$-\infty < x < -3$	$-3 < x < 0$	$0 < x < 3$	$3 < x < \infty$
Sign of $f'(x)$ :	$f' > 0$	$f' > 0$	$f' < 0$	$f' < 0$
Conclusion:	Increasing	Increasing	Decreasing	Decreasing

Increasing on:  $(-\infty, -3), (-3, 0)$

Decreasing on:  $(0, 3), (3, \infty)$

Relative maximum:  $(0, 0)$

31.  $f(x) = \frac{x^2 - 2x + 1}{x + 1}$

$$f'(x) = \frac{(x + 1)(2x - 2) - (x^2 - 2x + 1)(1)}{(x + 1)^2} = \frac{x^2 + 2x - 3}{(x + 1)^2} = \frac{(x + 3)(x - 1)}{(x + 1)^2}$$

Critical numbers:  $x = -3, 1$

Discontinuity:  $x = -1$

Test intervals:	$-\infty < x < -3$	$-3 < x < -1$	$-1 < x < 1$	$1 < x < \infty$
Sign of $f'(x)$ :	$f' > 0$	$f' < 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Decreasing	Increasing

Increasing on:  $(-\infty, -3), (1, \infty)$

Decreasing on:  $(-3, -1), (-1, 1)$

Relative maximum:  $(-3, -8)$

Relative minimum:  $(1, 0)$

33.  $f(x) = \frac{x}{2} + \cos x, 0 < x < 2\pi$

$$f'(x) = \frac{1}{2} - \sin x = 0$$

Critical numbers:  $x = \frac{\pi}{6}, \frac{5\pi}{6}$

Test intervals:	$0 < x < \frac{\pi}{6}$	$\frac{\pi}{6} < x < \frac{5\pi}{6}$	$\frac{5\pi}{6} < x < 2\pi$
Sign of $f'(x)$ :	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

Increasing on:  $(0, \frac{\pi}{6}), (\frac{5\pi}{6}, 2\pi)$

Relative maximum:  $(\frac{\pi}{6}, \frac{\pi + 6\sqrt{3}}{12})$

Decreasing on:  $(\frac{\pi}{6}, \frac{5\pi}{6})$

Relative minimum:  $(\frac{5\pi}{6}, \frac{5\pi - 6\sqrt{3}}{12})$

35.  $f(x) = \sin^2 x + \sin x, 0 < x < 2\pi$

$$f'(x) = 2 \sin x \cos x + \cos x = \cos x(2 \sin x + 1) = 0$$

Critical numbers:  $x = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$

Test intervals:	$0 < x < \frac{\pi}{2}$	$\frac{\pi}{2} < x < \frac{7\pi}{6}$	$\frac{7\pi}{6} < x < \frac{3\pi}{2}$	$\frac{3\pi}{2} < x < \frac{11\pi}{6}$	$\frac{11\pi}{6} < x < 2\pi$
Sign of $f'(x)$ :	$f' > 0$	$f' < 0$	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing	Decreasing	Increasing

Increasing on:  $\left(0, \frac{\pi}{2}\right), \left(\frac{7\pi}{6}, \frac{3\pi}{2}\right), \left(\frac{11\pi}{6}, 2\pi\right)$

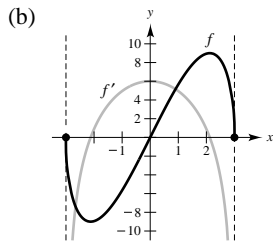
Decreasing on:  $\left(\frac{\pi}{2}, \frac{7\pi}{6}\right), \left(\frac{3\pi}{2}, \frac{11\pi}{6}\right)$

Relative minima:  $\left(\frac{7\pi}{6}, -\frac{1}{4}\right), \left(\frac{11\pi}{6}, -\frac{1}{4}\right)$

Relative maxima:  $\left(\frac{\pi}{2}, 2\right), \left(\frac{3\pi}{2}, 0\right)$

37.  $f(x) = 2x\sqrt{9-x^2}, [-3, 3]$

(a)  $f'(x) = \frac{2(9-2x^2)}{\sqrt{9-x^2}}$



(c)  $\frac{2(9-2x^2)}{\sqrt{9-x^2}} = 0$

Critical numbers:  $x = \pm \frac{3}{\sqrt{2}} = \pm \frac{3\sqrt{2}}{2}$

(d) Intervals:

$$\left(-3, -\frac{3\sqrt{2}}{2}\right) \quad \left(-\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right) \quad \left(\frac{3\sqrt{2}}{2}, 3\right)$$

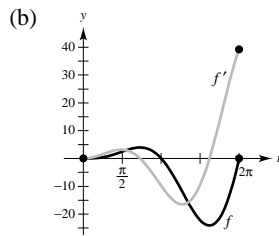
$$f'(x) < 0 \quad f'(x) > 0 \quad f'(x) < 0$$

Decreasing      Increasing      Decreasing

$f$  is increasing when  $f'$  is positive and decreasing when  $f'$  is negative.

39.  $f(t) = t^2 \sin t, [0, 2\pi]$

(a)  $f'(t) = t^2 \cos t + 2t \sin t$   
 $= t(t \cos t + 2 \sin t)$



(c)  $t(t \cos t + 2 \sin t) = 0$

$$t = 0 \text{ or } t = -2 \tan t$$

$$t \cot t = -2$$

$$t \approx 2.2889, 5.0870 \text{ (graphing utility)}$$

Critical numbers:  $t = 2.2889, t = 5.0870$

(d) Intervals:

$$(0, 2.2889) \quad (2.2889, 5.0870) \quad (5.0870, 2\pi)$$

$$f'(t) > 0 \quad f'(t) < 0 \quad f'(t) > 0$$

Increasing      Decreasing      Increasing

$f$  is increasing when  $f'$  is positive and decreasing when  $f'$  is negative.



41.  $f(x) = \frac{x^5 - 4x^3 + 3x}{x^2 - 1} = \frac{(x^2 - 1)(x^3 - 3x)}{x^2 - 1} = x^3 - 3x, x \neq \pm 1$

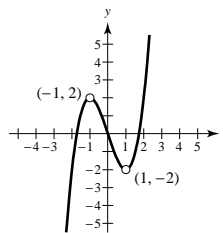
$f(x) = g(x) = x^3 - 3x$  for all  $x \neq \pm 1$ .

$f'(x) = 3x^2 - 3 = 3(x^2 - 1), x \neq \pm 1 \quad f'(x) \neq 0$

$f$  symmetric about origin

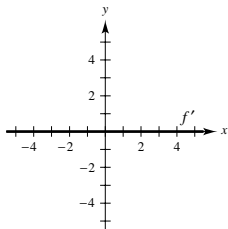
zeros of  $f$ :  $(0, 0), (\pm\sqrt{3}, 0)$

No relative extrema

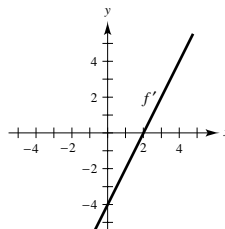


Holes at  $(-1, 2)$  and  $(1, -2)$

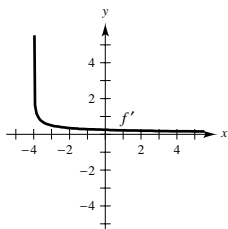
43.  $f(x) = c$  is constant  $\implies f'(x) = 0$



45.  $f$  is quadratic  $\implies f'$  is a line.



47.  $f$  has positive, but decreasing slope



In Exercises 49–53,  $f'(x) > 0$  on  $(-\infty, -4)$ ,  $f'(x) < 0$  on  $(-4, 6)$  and  $f'(x) > 0$  on  $(6, \infty)$ .

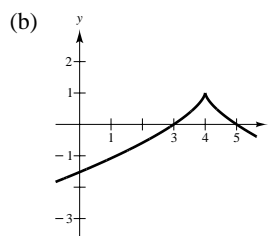
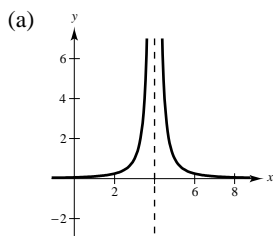
49.  $g(x) = f(x) + 5$   
 $g'(x) = f'(x)$   
 $g'(0) = f'(0) < 0$

51.  $g(x) = -f(x)$   
 $g'(x) = -f'(x)$   
 $g'(-6) = -f'(-6) < 0$

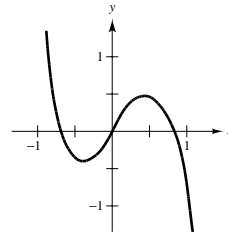
53.  $g(x) = f(x - 10)$   
 $g'(x) = f'(x - 10)$   
 $g'(0) = f'(-10) > 0$

55.  $f'(x) = \begin{cases} > 0, & x < 4 \implies f \text{ is increasing on } (-\infty, 4). \\ \text{undefined}, & x = 4 \\ < 0, & x > 4 \implies f \text{ is decreasing on } (4, \infty). \end{cases}$

Two possibilities for  $f(x)$  are given below.



57. The critical numbers are in intervals  $(-0.50, -0.25)$  and  $(0.25, 0.50)$  since the sign of  $f'$  changes in these intervals.  $f$  is decreasing on approximately  $(-1, -0.40)$ ,  $(0.48, 1)$ , and increasing on  $(-0.40, 0.48)$ .



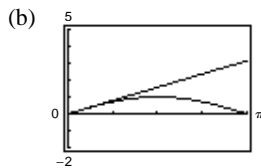
Relative minimum when  $x \approx -0.40$ .  
Relative maximum when  $x \approx 0.48$ .

59.  $f(x) = x, g(x) = \sin x, 0 < x < \pi$

(a)

$x$	0.5	1	1.5	2	2.5	3
$f(x)$	0.5	1	1.5	2	2.5	3
$g(x)$	0.479	0.841	0.997	0.909	0.598	0.141

$f(x)$  seems greater than  $g(x)$  on  $(0, \pi)$ .



$x > \sin x$  on  $(0, \pi)$

- (c) Let  $h(x) = f(x) - g(x) = x - \sin x$

$$h'(x) = 1 - \cos x > 0 \text{ on } (0, \pi).$$

Therefore,  $h(x)$  is increasing on  $(0, \pi)$ . Since  $h(0) = 0, h(x) > 0$  on  $(0, \pi)$ . Thus,

$$x - \sin x > 0$$

$$x > \sin x$$

$$f(x) > g(x) \text{ on } (0, \pi).$$

61.  $v = k(R - r)r^2 = k(Rr^2 - r^3)$

$$v' = k(2Rr - 3r^2)$$

$$= kr(2R - 3r) = 0$$

$$r = 0 \text{ or } \frac{2}{3}R$$

Maximum when  $r = \frac{2}{3}R$ .

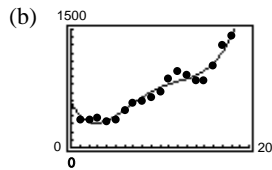
63.  $P = \frac{vR_1R_2}{(R_1 + R_2)^2}$ ,  $v$  and  $R_1$  are constant

$$\frac{dP}{dR_2} = \frac{(R_1 + R_2)^2(vR_1) - vR_1R_2[2(R_1 + R_2)(1)]}{(R_1 + R_2)^4}$$

$$= \frac{vR_1(R_1 - R_2)}{(R_1 + R_2)^3} = 0 \Rightarrow R_2 = R_1$$

Maximum when  $R_1 = R_2$ .

65. (a)  $B = 0.1198t^4 - 4.4879t^3 + 56.9909t^2 - 223.0222t + 579.9541$



- (c)  $B' = 0$  for  $t \approx 2.78$ , or 1983, (311.1 thousand bankruptcies)  
Actual minimum: 1984 (344.3 thousand bankruptcies)

67. (a) Use a cubic polynomial  
 $f(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ .

(b)  $f'(x) = 3a_3x^2 + 2a_2x + a_1$

$(0, 0): \quad 0 = a_0 \quad (f(0) = 0)$

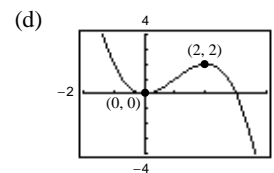
$0 = a_1 \quad (f'(0) = 0)$

$(2, 2): \quad 2 = 8a_3 + 4a_2 \quad (f(2) = 2)$

$0 = 12a_3 + 4a_2 \quad (f'(2) = 0)$

- (c) The solution is  $a_0 = a_1 = 0, a_2 = \frac{3}{2}, a_3 = -\frac{1}{2}$ :

$$f(x) = -\frac{1}{2}x^3 + \frac{3}{2}x^2.$$



69. (a) Use a fourth degree polynomial  $f(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ .

(b)  $f'(x) = 4a_4x^3 + 3a_3x^2 + 2a_2x + a_1$

(0, 0):  $0 = a_0$   $(f(0) = 0)$

$0 = a_1$   $(f'(0) = 0)$

(4, 0):  $0 = 256a_4 + 64a_3 + 16a_2$   $(f(4) = 0)$

$0 = 256a_4 + 48a_3 + 8a_2$   $(f'(4) = 0)$

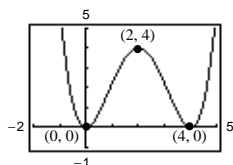
(2, 4):  $4 = 16a_4 + 8a_3 + 4a_2$   $(f(2) = 4)$

$0 = 32a_4 + 12a_3 + 4a_2$   $(f'(2) = 0)$

(c) The solution is  $a_0 = a_1 = 0$ ,  $a_2 = 4$ ,  $a_3 = -2$ ,  $a_4 = \frac{1}{4}$ .

$$f(x) = \frac{1}{4}x^4 - 2x^3 + 4x^2$$

(d)



71. True

Let  $h(x) = f(x) + g(x)$  where  $f$  and  $g$  are increasing. Then  $h'(x) = f'(x) + g'(x) > 0$  since  $f'(x) > 0$  and  $g'(x) > 0$ .

73. False

Let  $f(x) = x^3$ , then  $f'(x) = 3x^2$  and  $f$  only has one critical number. Or, let  $f(x) = x^3 + 3x + 1$ , then  $f'(x) = 3(x^2 + 1)$  has no critical numbers.

75. False. For example,  $f(x) = x^3$  does not have a relative extrema at the critical number  $x = 0$ .

77. Assume that  $f'(x) < 0$  for all  $x$  in the interval  $(a, b)$  and let  $x_1 < x_2$  be any two points in the interval. By the Mean Value Theorem, we know there exists a number  $c$  such that  $x_1 < c < x_2$ , and

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Since  $f'(c) < 0$  and  $x_2 - x_1 > 0$ , then  $f(x_2) - f(x_1) < 0$ , which implies that  $f(x_2) < f(x_1)$ . Thus,  $f$  is decreasing on the interval.

79. Let  $f(x) = (1 + x)^n - nx - 1$ . Then

$$f'(x) = n(1 + x)^{n-1} - n$$

$$= n[(1 + x)^{n-1} - 1] > 0 \text{ since } x > 0 \text{ and } n > 1.$$

Thus,  $f(x)$  is increasing on  $(0, \infty)$ . Since  $f(0) = 0 \Rightarrow f(x) > 0$  on  $(0, \infty)$

$$(1 + x)^n - nx - 1 > 0 \Rightarrow (1 + x)^n > 1 + nx.$$

## Section 3.4 Concavity and the Second Derivative Test

1.  $y = x^2 - x - 2, y'' = 2$

Concave upward:  $(-\infty, \infty)$

5.  $f(x) = \frac{x^2 + 1}{x^2 - 1}, y'' = \frac{4(3x^2 + 1)}{(x^2 - 1)^3}$

Concave upward:  $(-\infty, -1), (1, \infty)$

Concave downward:  $(-1, 1)$

9.  $y = 2x - \tan x, \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$y' = 2 - \sec^2 x$$

$$y'' = -2 \sec^2 x \tan x$$

Concave upward:  $\left(-\frac{\pi}{2}, 0\right)$

Concave downward:  $\left(0, \frac{\pi}{2}\right)$

13.  $f(x) = \frac{1}{4}x^4 - 2x^2$

$$f'(x) = x^3 - 4x$$

$$f''(x) = 3x^2 - 4$$

$$f''(x) = 3x^2 - 4 = 0 \text{ when } x = \pm \frac{2}{\sqrt{3}}.$$

Test interval:	$-\infty < x < -\frac{2}{\sqrt{3}}$	$-\frac{2}{\sqrt{3}} < x < \frac{2}{\sqrt{3}}$	$\frac{2}{\sqrt{3}} < x < \infty$
Sign of $f''(x)$ :	$f''(x) > 0$	$f''(x) < 0$	$f''(x) > 0$
Conclusion:	Concave upward	Concave downward	Concave upward

Points of inflection:  $\left(\pm \frac{2}{\sqrt{3}}, -\frac{20}{9}\right)$

3.  $f(x) = \frac{24}{x^2 + 12}, y'' = \frac{-144(4 - x^2)}{(x^2 + 12)^3}$

Concave upward:  $(-\infty, -2), (2, \infty)$

Concave downward:  $(-2, 2)$

7.  $f(x) = 3x^2 - x^3$

$$f'(x) = 6x - 3x^2$$

$$f''(x) = 6 - 6x$$

Concave upward:  $(-\infty, 1)$

Concave downward:  $(1, \infty)$

11.  $f(x) = x^3 - 6x^2 + 12x$

$$f'(x) = 3x^2 - 12x + 12$$

$$f''(x) = 6(x - 2) = 0 \text{ when } x = 2.$$

The concavity changes at  $x = 2$ .  $(2, 8)$  is a point of inflection.

Concave upward:  $(2, \infty)$

Concave downward:  $(-\infty, 2)$

15.  $f(x) = x(x - 4)^3$

$$f'(x) = x[3(x - 4)^2] + (x - 4)^3$$

$$= (x - 4)^2(4x - 4)$$

$$f''(x) = 4(x - 1)[2(x - 4)] + 4(x - 4)^2$$

$$= 4(x - 4)[2(x - 1) + (x - 4)]$$

$$= 4(x - 4)(3x - 6) = 12(x - 4)(x - 2)$$

$$f''(x) = 12(x - 4)(x - 2) = 0 \text{ when } x = 2, 4.$$

Test interval:	$-\infty < x < 2$	$2 < x < 4$	$4 < x < \infty$
Sign of $f''(x)$ :	$f''(x) > 0$	$f''(x) < 0$	$f''(x) > 0$
Conclusion:	Concave upward	Concave downward	Concave upward

Points of inflection:  $(2, -16), (4, 0)$ 

17.  $f(x) = x\sqrt{x+3}$ , Domain:  $[-3, \infty)$

$$f'(x) = x\left(\frac{1}{2}\right)(x+3)^{-1/2} + \sqrt{x+3} = \frac{3(x+2)}{2\sqrt{x+3}}$$

$$f''(x) = \frac{6\sqrt{x+3} - 3(x+2)(x+3)^{-1/2}}{4(x+3)} = \frac{3(x+4)}{4(x+3)^{3/2}}$$

 $f''(x) > 0$  on the entire domain of  $f$  (except for  $x = -3$ , for which  $f''(x)$  is undefined). There are no points of inflection.Concave upward on  $(-3, \infty)$ 

19.  $f(x) = \frac{x}{x^2 + 1}$

$$f'(x) = \frac{1 - x^2}{(x^2 + 1)^2}$$

$$f''(x) = \frac{2x(x^2 - 3)}{(x^2 + 1)^3} = 0 \text{ when } x = 0, \pm\sqrt{3}$$

Test intervals:	$-\infty < x < -\sqrt{3}$	$-\sqrt{3} < x < 0$	$0 < x < \sqrt{3}$	$\sqrt{3} < x < \infty$
Sign of $f''(x)$ :	$f'' < 0$	$f'' > 0$	$f'' < 0$	$f'' > 0$
Conclusion:	Concave downward	Concave upward	Concave downward	Concave upward

Points of inflection:  $\left(-\sqrt{3}, -\frac{\sqrt{3}}{4}\right), (0, 0), \left(\sqrt{3}, \frac{\sqrt{3}}{4}\right)$ 

21.  $f(x) = \sin\left(\frac{x}{2}\right), 0 \leq x \leq 4\pi$

$$f'(x) = \frac{1}{2}\cos\left(\frac{x}{2}\right)$$

$$f''(x) = -\frac{1}{4}\sin\left(\frac{x}{2}\right)$$

$$f''(x) = 0 \text{ when } x = 0, 2\pi, 4\pi.$$

Point of inflection:  $(2\pi, 0)$ 

Test interval:	$0 < x < 2\pi$	$2\pi < x < 4\pi$
Sign of $f''(x)$ :	$f'' < 0$	$f'' > 0$
Conclusion:	Concave downward	Concave upward

23.  $f(x) = \sec\left(x - \frac{\pi}{2}\right), 0 < x < 4\pi$

$$f'(x) = \sec\left(x - \frac{\pi}{2}\right) \tan\left(x - \frac{\pi}{2}\right)$$

$$f''(x) = \sec^3\left(x - \frac{\pi}{2}\right) + \sec\left(x - \frac{\pi}{2}\right) \tan^2\left(x - \frac{\pi}{2}\right) \neq 0 \text{ for any } x \text{ in the domain of } f.$$

Concave upward:  $(0, \pi), (2\pi, 3\pi)$

Concave downward:  $(\pi, 2\pi), (3\pi, 4\pi)$

No points of inflection

25.  $f(x) = 2 \sin x + \sin 2x, 0 \leq x \leq 2\pi$

$$f'(x) = 2 \cos x + 2 \cos 2x$$

$$f''(x) = -2 \sin x - 4 \sin 2x = -2 \sin x(1 + 4 \cos x)$$

$$f''(x) = 0 \text{ when } x = 0, 1.823, \pi, 4.460.$$

Test interval:	$0 < x < 1.823$	$1.823 < x < \pi$	$\pi < x < 4.460$	$4.460 < x < 2\pi$
Sign of $f''(x)$ :	$f'' < 0$	$f'' > 0$	$f'' < 0$	$f'' > 0$
Conclusion:	Concave downward	Concave upward	Concave downward	Concave upward

Points of inflection:  $(1.823, 1.452), (\pi, 0), (4.46, -1.452)$

27.  $f(x) = x^4 - 4x^3 + 2$

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$$

$$f''(x) = 12x^2 - 24x = 12x(x - 2)$$

Critical numbers:  $x = 0, x = 3$

However,  $f''(0) = 0$ , so we must use the First Derivative Test.  $f'(x) < 0$  on the intervals  $(-\infty, 0)$  and  $(0, 3)$ ; hence,  $(0, 2)$  is not an extremum.  $f''(3) > 0$  so  $(3, -25)$  is a relative minimum.

31.  $f(x) = x^3 - 3x^2 + 3$

$$f'(x) = 3x^2 - 6x = 3x(x - 2)$$

$$f''(x) = 6x - 6 = 6(x - 1)$$

Critical numbers:  $x = 0, x = 2$

$$f''(0) = -6 < 0$$

Therefore,  $(0, 3)$  is a relative maximum.

$$f''(2) = 6 > 0$$

Therefore,  $(2, -1)$  is a relative minimum.

29.  $f(x) = (x - 5)^2$

$$f'(x) = 2(x - 5)$$

$$f''(x) = 2$$

Critical number:  $x = 5$

$$f''(5) > 0$$

Therefore,  $(5, 0)$  is a relative minimum.

33.  $g(x) = x^2(6 - x)^3$

$$g'(x) = x(x - 6)^2(12 - 5x)$$

$$g''(x) = 4(6 - x)(5x^2 - 24x + 18)$$

Critical numbers:  $x = 0, \frac{12}{5}, 6$

$$g''(0) = 432 > 0$$

Therefore,  $(0, 0)$  is a relative minimum.

$$g''\left(\frac{12}{5}\right) = -155.52 < 0$$

Therefore,  $\left(\frac{12}{5}, 268.7\right)$  is a relative minimum.

$$g''(6) = 0$$

Test fails by the First Derivative Test,  $(6, 0)$  is not an extremum.

35.  $f(x) = x^{2/3} - 3$

$$f'(x) = \frac{2}{3x^{1/3}}$$

$$f''(x) = \frac{-2}{9x^{4/3}}$$

Critical number:  $x = 0$ 

However,  $f''(0)$  is undefined, so we must use the First Derivative Test. Since  $f'(x) < 0$  on  $(-\infty, 0)$  and  $f'(x) > 0$  on  $(0, \infty)$ ,  $(0, -3)$  is a relative minimum.

39.  $f(x) = \cos x - x$ ,  $0 \leq x \leq 4\pi$

$$f'(x) = -\sin x - 1 \leq 0$$

Therefore,  $f$  is non-increasing and there are no relative extrema.

41.  $f(x) = 0.2x^2(x - 3)^3$ ,  $[-1, 4]$

(a)  $f'(x) = 0.2x(5x - 6)(x - 3)^2$

$$f''(x) = (x - 3)(4x^2 - 9.6x + 3.6)$$

$$= 0.4(x - 3)(10x^2 - 24x + 9)$$

(b)  $f''(0) < 0 \Rightarrow (0, 0)$  is a relative maximum.

$f''\left(\frac{6}{5}\right) > 0 \Rightarrow (1.2, -1.6796)$  is a relative minimum.

Points of inflection:

$(3, 0), (0.4652, -0.7049), (1.9348, -0.9049)$

43.  $f(x) = \sin x - \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x$ ,  $[0, \pi]$

(a)  $f'(x) = \cos x - \cos 3x + \cos 5x$

$f'(x) = 0$  when  $x = \frac{\pi}{6}, x = \frac{\pi}{2}, x = \frac{5\pi}{6}$ .

$f''(x) = -\sin x + 3\sin 3x - 5\sin 5x$

$f''(x) = 0$  when  $x = \frac{\pi}{6}, x = \frac{5\pi}{6}, x \approx 1.1731, x \approx 1.9685$

(b)  $f''\left(\frac{\pi}{2}\right) < 0 \Rightarrow \left(\frac{\pi}{2}, 1.53333\right)$  is a relative maximum.

Points of inflection:  $\left(\frac{\pi}{6}, 0.2667\right), (1.1731, 0.9638),$

$(1.9685, 0.9637), \left(\frac{5\pi}{6}, 0.2667\right)$

**Note:**  $(0, 0)$  and  $(\pi, 0)$  are not points of inflection since they are endpoints.

37.  $f(x) = x + \frac{4}{x}$

$$f'(x) = 1 - \frac{4}{x^2} = \frac{x^2 - 4}{x^2}$$

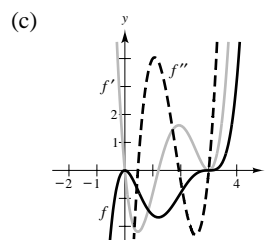
$$f''(x) = \frac{8}{x^3}$$

Critical numbers:  $x = \pm 2$ 

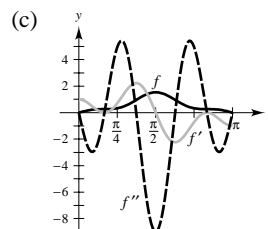
$f''(-2) < 0$

Therefore,  $(-2, -4)$  is a relative maximum.

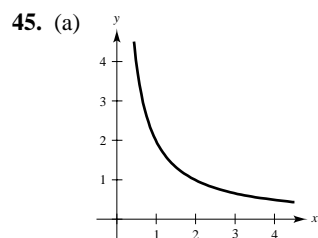
$f''(2) > 0$

Therefore,  $(2, 4)$  is a relative minimum.

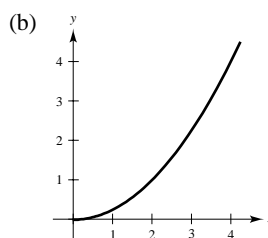
$f$  is increasing when  $f' > 0$  and decreasing when  $f' < 0$ .  $f$  is concave upward when  $f'' > 0$  and concave downward when  $f'' < 0$ .



The graph of  $f$  is increasing when  $f' > 0$  and decreasing when  $f' < 0$ .  $f$  is concave upward when  $f'' > 0$  and concave downward when  $f'' < 0$ .



$f' < 0$  means  $f$  decreasing  
 $f'$  increasing means concave upward

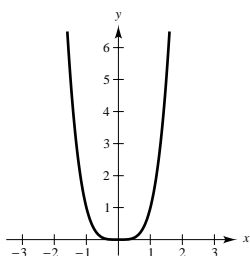


$f' > 0$  means  $f$  increasing  
 $f'$  increasing means concave upward

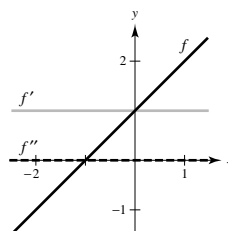
47. Let  $f(x) = x^4$ .

$$f''(x) = 12x^2$$

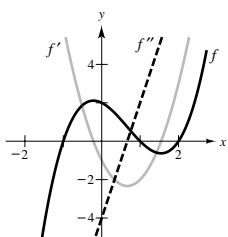
$f''(0) = 0$ , but  $(0, 0)$  is not a point of inflection.



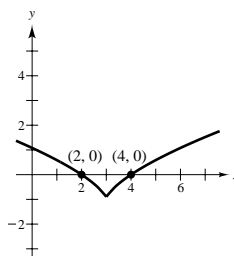
49.



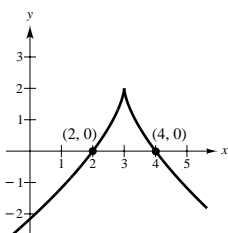
51.



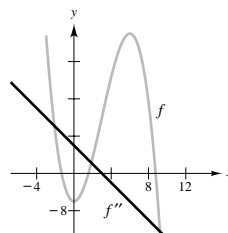
53.



55.



57.



$f''$  is linear.

$f'$  is quadratic.

$f$  is cubic.

$f$  concave upwards on  $(-\infty, 3)$ , downward on  $(3, \infty)$ .



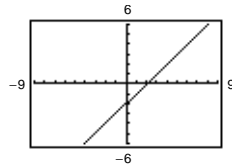
59. (a)  $n = 1$ :

$$f(x) = x - 2$$

$$f'(x) = 1$$

$$f''(x) = 0$$

No inflection points



$n = 2$ :

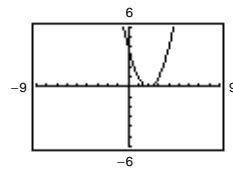
$$f(x) = (x - 2)^2$$

$$f'(x) = 2(x - 2)$$

$$f''(x) = 2$$

No inflection points

Relative minimum:  
(2, 0)



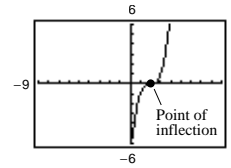
$n = 3$ :

$$f(x) = (x - 2)^3$$

$$f'(x) = 3(x - 2)^2$$

$$f''(x) = 6(x - 2)$$

Inflection point: (2, 0)



$n = 4$ :

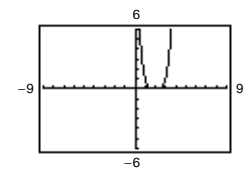
$$f(x) = (x - 2)^4$$

$$f'(x) = 4(x - 2)^3$$

$$f''(x) = 12(x - 2)^2$$

No inflection points:

Relative minimum:  
(2, 0)



**Conclusion:** If  $n \geq 3$  and  $n$  is odd, then (2, 0) is an inflection point. If  $n \geq 2$  and  $n$  is even, then (2, 0) is a relative minimum.

(b) Let  $f(x) = (x - 2)^n$ ,  $f'(x) = n(x - 2)^{n-1}$ ,  $f''(x) = n(n - 1)(x - 2)^{n-2}$ .

For  $n \geq 3$  and odd,  $n - 2$  is also odd and the concavity changes at  $x = 2$ .

For  $n \geq 4$  and even,  $n - 2$  is also even and the concavity does not change at  $x = 2$ .

Thus,  $x = 2$  is an inflection point if and only if  $n \geq 3$  is odd.

61.  $f(x) = ax^3 + bx^2 + cx + d$

Relative maximum: (3, 3)

Relative minimum: (5, 1)

Point of inflection: (4, 2)

$$f'(x) = 3ax^2 + 2bx + c, f''(x) = 6ax + 2b$$

$$\left. \begin{aligned} f(3) &= 27a + 9b + 3c + d = 3 \\ f(5) &= 125a + 25b + 5c + d = 1 \end{aligned} \right\} 98a + 16b + 2c = -2 \Rightarrow 49a + 8b + c = -1$$

$$f'(3) = 27a + 6b + c = 0, f''(4) = 24a + 2b = 0$$

$$49a + 8b + c = -1 \quad 24a + 2b = 0$$

$$\underline{27a + 6b + c = 0} \quad \underline{22a + 2b = -1}$$

$$22a + 2b = -1 \quad 2a = 1$$

$$a = \frac{1}{2}, b = -6, c = \frac{45}{2}, d = -24$$

$$f(x) = \frac{1}{2}x^3 - 6x^2 + \frac{45}{2}x - 24$$

63.  $f(x) = ax^3 + bx^2 + cx + d$

Maximum:  $(-4, 1)$

Minimum:  $(0, 0)$

(a)  $f'(x) = 3ax^2 + 2bx + c$ ,  $f''(x) = 6ax + 2b$

$$f(0) = 0 \Rightarrow d = 0$$

$$f(-4) = 1 \Rightarrow -64a + 16b - 4c = 1$$

$$f'(-4) = 0 \Rightarrow 48a - 8b + c = 0$$

$$f''(0) = 0 \Rightarrow c = 0$$

Solving this system yields  $a = \frac{1}{32}$  and  $b = 6a = \frac{3}{16}$ .

$$f(x) = \frac{1}{32}x^3 + \frac{3}{16}x^2$$

(b) The plane would be descending at the greatest rate at the point of inflection.

$$f''(x) = 6ax + 2b = \frac{3}{16}x + \frac{3}{8} = 0 \Rightarrow x = -2.$$

Two miles from touchdown.

65.  $D = 2x^4 - 5Lx^3 + 3L^2x^2$

$$D' = 8x^3 - 15Lx^2 + 6L^2x = x(8x^2 - 15Lx + 6L^2) = 0$$

$$x = 0 \text{ or } x = \frac{15L \pm \sqrt{33}L}{16} = \left(\frac{15 \pm \sqrt{33}}{16}\right)L$$

By the Second Derivative Test, the deflection is maximum when

$$x = \left(\frac{15 - \sqrt{33}}{16}\right)L \approx 0.578L.$$

67.  $C = 0.5x^2 + 15x + 5000$

$$\bar{C} = \frac{C}{x} = 0.5x + 15 + \frac{5000}{x}$$

$\bar{C}$  = average cost per unit

$$\frac{d\bar{C}}{dx} = 0.5 - \frac{5000}{x^2} = 0 \text{ when } x = 100$$

By the First Derivative Test,  $\bar{C}$  is minimized when  $x = 100$  units.

69.  $S = \frac{5000t^2}{8 + t^2}$

$$S'(t) = \frac{80,000t}{(8 + t^2)^2}$$

$$S''(t) = \frac{80,000(8 - 3t^2)}{(8 + t^2)^3}$$

$$S''(t) = 0 \text{ for } t = \sqrt{8/3} \approx 1.633.$$

Sales are increasing at the greatest rate at  $t = 1.633$  years.

71.  $f(x) = 2(\sin x + \cos x)$ ,  $f\left(\frac{\pi}{4}\right) = 2\sqrt{2}$

$$f'(x) = 2(\cos x - \sin x), \quad f'\left(\frac{\pi}{4}\right) = 0$$

$$f''(x) = 2(-\sin x - \cos x), \quad f''\left(\frac{\pi}{4}\right) = -2\sqrt{2}$$

$$P_1(x) = 2\sqrt{2} + 0\left(x - \frac{\pi}{4}\right) = 2\sqrt{2}$$

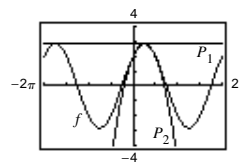
$$P_1'(x) = 0$$

$$P_2(x) = 2\sqrt{2} + 0\left(x - \frac{\pi}{4}\right) + \frac{1}{2}(-2\sqrt{2})\left(x - \frac{\pi}{4}\right)^2 = 2\sqrt{2} - \sqrt{2}\left(x - \frac{\pi}{4}\right)^2$$

$$P_2'(x) = -2\sqrt{2}\left(x - \frac{\pi}{4}\right)$$

$$P_2''(x) = -2\sqrt{2}$$

The values of  $f$ ,  $P_1$ ,  $P_2$ , and their first derivatives are equal at  $x = \pi/4$ . The values of the second derivatives of  $f$  and  $P_2$  are equal at  $x = \pi/4$ . The approximations worsen as you move away from  $x = \pi/4$ .



$$73. f(x) = \sqrt{1-x}, \quad f(0) = 1$$

$$f'(x) = -\frac{1}{2\sqrt{1-x}}, \quad f'(0) = -\frac{1}{2}$$

$$f''(x) = -\frac{1}{4(1-x)^{3/2}}, \quad f''(0) = -\frac{1}{4}$$

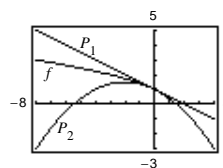
$$P_1(x) = 1 + \left(-\frac{1}{2}\right)(x-0) = 1 - \frac{x}{2}$$

$$P_1'(x) = -\frac{1}{2}$$

$$P_2(x) = 1 + \left(-\frac{1}{2}\right)(x-0) + \frac{1}{2}\left(-\frac{1}{4}\right)(x-0)^2 = 1 - \frac{x}{2} - \frac{x^2}{8}$$

$$P_2'(x) = -\frac{1}{2} - \frac{x}{4}$$

$$P_2''(x) = -\frac{1}{4}$$



The values of  $f$ ,  $P_1$ ,  $P_2$ , and their first derivatives are equal at  $x = 0$ . The values of the second derivatives of  $f$  and  $P_2$  are equal at  $x = 0$ . The approximations worsen as you move away from  $x = 0$ .

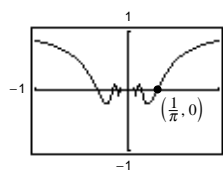
$$75. f(x) = x \sin\left(\frac{1}{x}\right)$$

$$f'(x) = x \left[ -\frac{1}{x^2} \cos\left(\frac{1}{x}\right) \right] + \sin\left(\frac{1}{x}\right) = -\frac{1}{x} \cos\left(\frac{1}{x}\right) + \sin\left(\frac{1}{x}\right)$$

$$f''(x) = -\frac{1}{x} \left[ \frac{1}{x^2} \sin\left(\frac{1}{x}\right) \right] + \frac{1}{x^2} \cos\left(\frac{1}{x}\right) - \frac{1}{x^2} \cos\left(\frac{1}{x}\right) = -\frac{1}{x^3} \sin\left(\frac{1}{x}\right) = 0$$

$$x = \frac{1}{\pi}$$

$$\text{Point of inflection: } \left(\frac{1}{\pi}, 0\right)$$



When  $x > 1/\pi$ ,  $f'' < 0$ , so the graph is concave downward.

77. Assume the zeros of  $f$  are all real. Then express the function as  $f(x) = a(x - r_1)(x - r_2)(x - r_3)$  where  $r_1$ ,  $r_2$ , and  $r_3$  are the distinct zeros of  $f$ . From the Product Rule for a function involving three factors, we have

$$f'(x) = a[(x - r_1)(x - r_2) + (x - r_1)(x - r_3) + (x - r_2)(x - r_3)]$$

$$f''(x) = a[(x - r_1) + (x - r_2) + (x - r_1) + (x - r_3) + (x - r_2) + (x - r_3)]$$

$$= a[6x - 2(r_1 + r_2 + r_3)].$$

Consequently,  $f''(x) = 0$  if

$$x = \frac{2(r_1 + r_2 + r_3)}{6} = \frac{r_1 + r_2 + r_3}{3} = (\text{Average of } r_1, r_2, \text{ and } r_3).$$

79. True. Let  $y = ax^3 + bx^2 + cx + d$ ,  $a \neq 0$ . Then  $y'' = 6ax + 2b = 0$  when  $x = -(b/3a)$ , and the concavity changes at this point.

81. False.

$$f(x) = 3 \sin x + 2 \cos x$$

$$f'(x) = 3 \cos x - 2 \sin x$$

$$3 \cos x - 2 \sin x = 0$$

$$3 \cos x = 2 \sin x$$

$$\frac{3}{2} = \tan x$$

$$\text{Critical number: } x = \tan^{-1}\left(\frac{3}{2}\right)$$

$$f\left(\tan^{-1}\left(\frac{3}{2}\right)\right) \approx 3.60555 \text{ is the maximum value of } y.$$

83. False. Concavity is determined by  $f''$ .

## Section 3.5 Limits at Infinity

$$1. f(x) = \frac{3x^2}{x^2 + 2}$$

No vertical asymptotes

Horizontal asymptote:  $y = 3$ 

Matches (f)

$$3. f(x) = \frac{x}{x^2 + 2}$$

No vertical asymptotes

Horizontal asymptote:  $y = 0$ 

Matches (d)

$$5. f(x) = \frac{4 \sin x}{x^2 + 1}$$

No vertical asymptotes

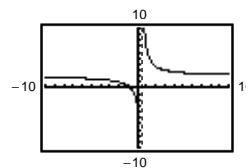
Horizontal asymptotes:  $y = 0$ 

Matches (b)

$$7. f(x) = \frac{4x + 3}{2x - 1}$$

$x$	$10^0$	$10^1$	$10^2$	$10^3$	$10^4$	$10^5$	$10^6$
$f(x)$	7	2.26	2.025	2.0025	2.0003	2	2

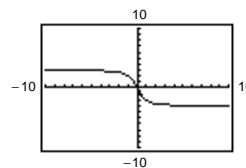
$$\lim_{x \rightarrow \infty} f(x) = 2$$



$$9. f(x) = \frac{-6x}{\sqrt{4x^2 + 5}}$$

$x$	$10^0$	$10^1$	$10^2$	$10^3$	$10^4$	$10^5$	$10^6$
$f(x)$	-2	-2.98	-2.9998	-3	-3	-3	-3

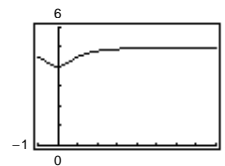
$$\lim_{x \rightarrow \infty} f(x) = -3$$



$$11. f(x) = 5 - \frac{1}{x^2 + 1}$$

$x$	$10^0$	$10^1$	$10^2$	$10^3$	$10^4$	$10^5$	$10^6$
$f(x)$	4.5	4.99	4.9999	4.999999	5	5	5

$$\lim_{x \rightarrow \infty} f(x) = 5$$



$$13. (a) h(x) = \frac{f(x)}{x^2} = \frac{5x^3 - 3x^2 + 10}{x^2} = 5x - 3 + \frac{10}{x^2}$$

$$\lim_{x \rightarrow \infty} h(x) = \infty \quad (\text{Limit does not exist})$$

$$(b) h(x) = \frac{f(x)}{x^3} = \frac{5x^3 - 3x^2 + 10}{x^3} = 5 - \frac{3}{x} + \frac{10}{x^3}$$

$$\lim_{x \rightarrow \infty} h(x) = 5$$

$$(c) h(x) = \frac{f(x)}{x^4} = \frac{5x^3 - 3x^2 + 10}{x^4} = \frac{5}{x} - \frac{3}{x^2} + \frac{10}{x^4}$$

$$\lim_{x \rightarrow \infty} h(x) = 0$$

$$17. (a) \lim_{x \rightarrow \infty} \frac{5 - 2x^{3/2}}{3x^2 - 4} = 0$$

$$(b) \lim_{x \rightarrow \infty} \frac{5 - 2x^{3/2}}{3x^{3/2} - 4} = -\frac{2}{3}$$

$$(c) \lim_{x \rightarrow \infty} \frac{5 - 2x^{3/2}}{3x - 4} = -\infty \quad (\text{Limit does not exist})$$

$$21. \lim_{x \rightarrow \infty} \frac{x}{x^2 - 1} = \lim_{x \rightarrow \infty} \frac{1/x}{1 - (1/x^2)} = \frac{0}{1} = 0$$

$$25. \lim_{x \rightarrow -\infty} \frac{x}{\sqrt{x^2 - x}} = \lim_{x \rightarrow -\infty} \frac{1}{\frac{\sqrt{x^2 - x}}{-\sqrt{x^2}}}, \quad (\text{for } x < 0 \text{ we have } x = -\sqrt{x^2})$$

$$= \lim_{x \rightarrow -\infty} \frac{-1}{\sqrt{1 - (1/x)}} = -1$$

$$27. \lim_{x \rightarrow -\infty} \frac{2x + 1}{\sqrt{x^2 - x}} = \lim_{x \rightarrow -\infty} \frac{2 + \frac{1}{x}}{\left( \frac{\sqrt{x^2 - x}}{-\sqrt{x^2}} \right)} \quad (\text{for } x < 0, x = -\sqrt{x^2})$$

$$= \lim_{x \rightarrow -\infty} \frac{-2 - (1/x)}{\sqrt{x + (1/x)}} = -2$$

29. Since  $(-1/x) \leq (\sin(2x))/x \leq (1/x)$  for all  $x \neq 0$ , we have by the Squeeze Theorem,

$$\lim_{x \rightarrow \infty} -\frac{1}{x} \leq \lim_{x \rightarrow \infty} \frac{\sin(2x)}{x} \leq \lim_{x \rightarrow \infty} \frac{1}{x}$$

$$0 \leq \lim_{x \rightarrow \infty} \frac{\sin(2x)}{x} \leq 0.$$

$$\text{Therefore, } \lim_{x \rightarrow \infty} \frac{\sin(2x)}{x} = 0.$$

$$15. (a) \lim_{x \rightarrow \infty} \frac{x^2 + 2}{x^3 - 1} = 0$$

$$(b) \lim_{x \rightarrow \infty} \frac{x^2 + 2}{x^2 - 1} = 1$$

$$(c) \lim_{x \rightarrow \infty} \frac{x^2 + 2}{x - 1} = \infty \quad (\text{Limit does not exist})$$

$$19. \lim_{x \rightarrow \infty} \frac{2x - 1}{3x + 2} = \lim_{x \rightarrow \infty} \frac{2 - (1/x)}{3 + (2/x)} = \frac{2 - 0}{3 + 0} = \frac{2}{3}$$

$$23. \lim_{x \rightarrow -\infty} \frac{5x^2}{x + 3} = \lim_{x \rightarrow -\infty} \frac{5x}{1 + (3/x)} = -\infty$$

Limit does not exist.

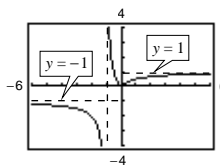
$$31. \lim_{x \rightarrow \infty} \frac{1}{2x + \sin x} = 0$$

$$33. (a) f(x) = \frac{|x|}{x+1}$$

$$\lim_{x \rightarrow \infty} \frac{|x|}{x+1} = 1$$

$$\lim_{x \rightarrow -\infty} \frac{|x|}{x+1} = -1$$

Therefore,  $y = 1$  and  $y = -1$  are both horizontal asymptotes.



$$35. \lim_{x \rightarrow \infty} x \sin \frac{1}{x} = \lim_{t \rightarrow 0^+} \frac{\sin t}{t} = 1$$

(Let  $x = 1/t$ .)

$$37. \lim_{x \rightarrow -\infty} (x + \sqrt{x^2 + 3}) = \lim_{x \rightarrow -\infty} \left[ (x + \sqrt{x^2 + 3}) \cdot \frac{x - \sqrt{x^2 + 3}}{x - \sqrt{x^2 + 3}} \right] = \lim_{x \rightarrow -\infty} \frac{-3}{x - \sqrt{x^2 + 3}} = 0$$

$$39. \lim_{x \rightarrow \infty} (x - \sqrt{x^2 + x}) = \lim_{x \rightarrow \infty} \left[ (x - \sqrt{x^2 + x}) \cdot \frac{x + \sqrt{x^2 + x}}{x + \sqrt{x^2 + x}} \right]$$

$$= \lim_{x \rightarrow \infty} \frac{-x}{x + \sqrt{x^2 + x}} = \lim_{x \rightarrow \infty} \frac{-1}{1 + \sqrt{1 + (1/x)}} = -\frac{1}{2}$$

41.

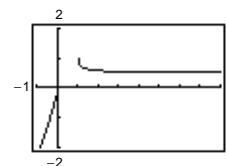
$x$	$10^0$	$10^1$	$10^2$	$10^3$	$10^4$	$10^5$	$10^6$
$f(x)$	1	0.513	0.501	0.500	0.500	0.500	0.500

$$\lim_{x \rightarrow \infty} (x - \sqrt{x(x-1)}) = \lim_{x \rightarrow \infty} \frac{x - \sqrt{x^2 - x}}{1} \cdot \frac{x + \sqrt{x^2 - x}}{x + \sqrt{x^2 - x}}$$

$$= \lim_{x \rightarrow \infty} \frac{x}{x + \sqrt{x^2 - x}}$$

$$= \lim_{x \rightarrow \infty} \frac{1}{1 + \sqrt{1 - (1/x)}}$$

$$= \frac{1}{2}$$

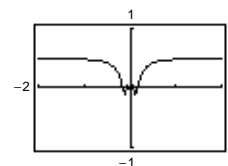


43.

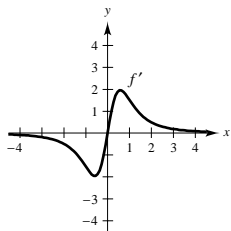
$x$	$10^0$	$10^1$	$10^2$	$10^3$	$10^4$	$10^5$	$10^6$
$f(x)$	0.479	0.500	0.500	0.500	0.500	0.500	0.500

Let  $x = 1/t$ .

$$\lim_{x \rightarrow \infty} x \sin \left( \frac{1}{2x} \right) = \lim_{t \rightarrow 0^+} \frac{\sin(t/2)}{t} = \lim_{t \rightarrow 0^+} \frac{1}{2} \frac{\sin(t/2)}{t/2} = \frac{1}{2}$$



45. (a)



(b)  $\lim_{x \rightarrow \infty} f(x) = 3 \quad \lim_{x \rightarrow \infty} f'(x) = 0$

 (c) Since  $\lim_{x \rightarrow \infty} f(x) = 3$ , the graph approaches that of a horizontal line,  $\lim_{x \rightarrow \infty} f'(x) = 0$ .

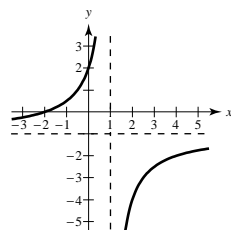
49.  $y = \frac{2+x}{1-x}$

 Intercepts:  $(-2, 0), (0, 2)$ 

Symmetry: none

 Horizontal asymptote:  $y = -1$  since

$$\lim_{x \rightarrow -\infty} \frac{2+x}{1-x} = -1 = \lim_{x \rightarrow \infty} \frac{2+x}{1-x}.$$

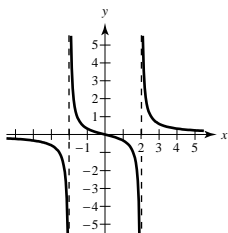
 Discontinuity:  $x = 1$  (Vertical asymptote)


51.  $y = \frac{x}{x^2 - 4}$

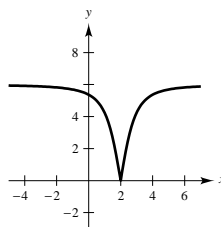
 Intercept:  $(0, 0)$ 

Symmetry: origin

 Horizontal asymptote:  $y = 0$ 

 Vertical asymptote:  $x = \pm 2$ 


47. Yes. For example, let  $f(x) = \frac{6|x-2|}{\sqrt{(x-2)^2 + 1}}$ .



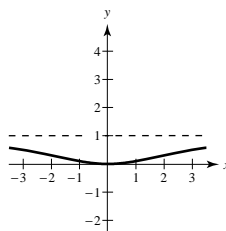
53.  $y = \frac{x^2}{x^2 + 9}$

 Intercept:  $(0, 0)$ 

Symmetry: y-axis

 Horizontal asymptote:  $y = 1$  since

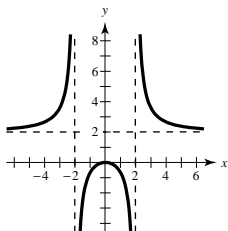
$$\lim_{x \rightarrow -\infty} \frac{x^2}{x^2 + 9} = 1 = \lim_{x \rightarrow \infty} \frac{x^2}{x^2 + 9}.$$

 Relative minimum:  $(0, 0)$ 


55.  $y = \frac{2x^2}{x^2 - 4}$

Intercept: (0, 0)

Symmetry: y-axis

Horizontal asymptote:  $y = 2$ Vertical asymptote:  $x = \pm 2$ 

57.  $xy^2 = 4$

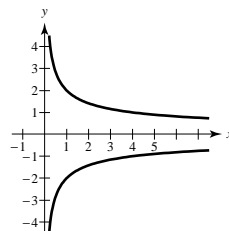
Domain:  $x > 0$ 

Intercepts: none

Symmetry: x-axis

Horizontal asymptote:  $y = 0$  since

$$\lim_{x \rightarrow \infty} \frac{2}{\sqrt{x}} = 0 = \lim_{x \rightarrow \infty} -\frac{2}{\sqrt{x}}.$$

Discontinuity:  $x = 0$  (Vertical asymptote)

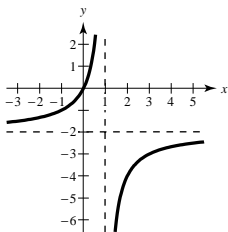
59.  $y = \frac{2x}{1 - x}$

Intercept: (0, 0)

Symmetry: none

Horizontal asymptote:  $y = -2$  since

$$\lim_{x \rightarrow -\infty} \frac{2x}{1 - x} = -2 = \lim_{x \rightarrow \infty} \frac{2x}{1 - x}.$$

Discontinuity:  $x = 1$  (Vertical asymptote)

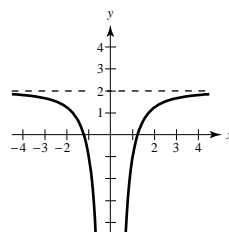
61.  $y = 2 - \frac{3}{x^2}$

Intercepts:  $(\pm\sqrt{3/2}, 0)$ 

Symmetry: y-axis

Horizontal asymptote:  $y = 2$  since

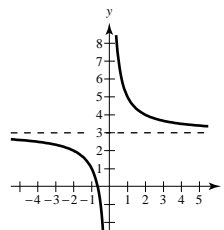
$$\lim_{x \rightarrow -\infty} \left(2 - \frac{3}{x^2}\right) = 2 = \lim_{x \rightarrow \infty} \left(2 - \frac{3}{x^2}\right).$$

Discontinuity:  $x = 0$  (Vertical asymptote)

63.  $y = 3 + \frac{2}{x}$

Intercept:  $y = 0 = 3 + \frac{2}{x} \Rightarrow \frac{2}{x} = -3 \Rightarrow x = -\frac{2}{3} \left(-\frac{2}{3}, 0\right)$ 

Symmetry: none

Horizontal asymptote:  $y = 3$ Vertical asymptote:  $x = 0$ 



$$65. y = \frac{x^3}{\sqrt{x^2 - 4}}$$

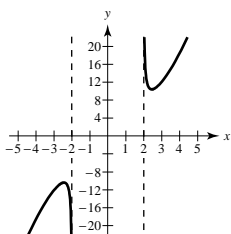
Domain:  $(-\infty, -2), (2, \infty)$

Intercepts: none

Symmetry: origin

Horizontal asymptote: none

Vertical asymptotes:  $x = \pm 2$  (discontinuities)



$$67. f(x) = 5 - \frac{1}{x^2} = \frac{5x^2 - 1}{x^2}$$

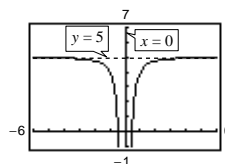
Domain:  $(-\infty, 0), (0, \infty)$

$$f'(x) = \frac{2}{x^3} \Rightarrow \text{No relative extrema}$$

$$f''(x) = -\frac{6}{x^4} \Rightarrow \text{No points of inflection}$$

Vertical asymptote:  $x = 0$

Horizontal asymptote:  $y = 5$



$$69. f(x) = \frac{x}{x^2 - 4}$$

$$f'(x) = \frac{(x^2 - 4) - x(2x)}{(x^2 - 4)^2}$$

$$= \frac{-(x^2 + 4)}{(x^2 - 4)^2} \neq 0 \text{ for any } x \text{ in the domain of } f.$$

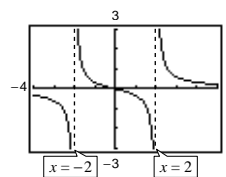
$$f''(x) = \frac{(x^2 - 4)^2(-2x) + (x^2 + 4)(2)(x^2 - 4)(2x)}{(x^2 - 4)^4}$$

$$= \frac{2x(x^2 + 12)}{(x^2 - 4)^3} = 0 \text{ when } x = 0.$$

Since  $f''(x) > 0$  on  $(-2, 0)$  and  $f''(x) < 0$  on  $(0, 2)$ , then  $(0, 0)$  is a point of inflection.

Vertical asymptotes:  $x = \pm 2$

Horizontal asymptote:  $y = 0$



$$71. f(x) = \frac{x - 2}{x^2 - 4x + 3} = \frac{x - 2}{(x - 1)(x - 3)}$$

$$f'(x) = \frac{(x^2 - 4x + 3) - (x - 2)(2x - 4)}{(x^2 - 4x + 3)^2} = \frac{-x^2 + 4x - 5}{(x^2 - 4x + 3)^2} \neq 0$$

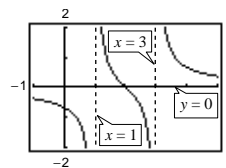
$$f''(x) = \frac{(x^2 - 4x + 3)^2(-2x + 4) - (-x^2 + 4x - 5)(2)(x^2 - 4x + 3)(2x - 4)}{(x^2 - 4x + 3)^4}$$

$$= \frac{2(x^3 - 6x^2 + 15x - 14)}{(x^2 - 4x + 3)^3} = 0 \text{ when } x = 2.$$

Since  $f''(x) > 0$  on  $(1, 2)$  and  $f''(x) < 0$  on  $(2, 3)$ , then  $(2, 0)$  is a point of inflection.

Vertical asymptote:  $x = 1, x = 3$

Horizontal asymptote:  $y = 0$



$$73. f(x) = \frac{3x}{\sqrt{4x^2 + 1}}$$

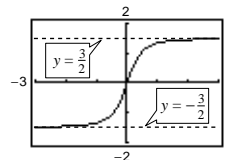
$$f'(x) = \frac{3}{(4x^2 + 1)^{3/2}} \Rightarrow \text{No relative extrema}$$

$$f''(x) = \frac{-36x}{(4x^2 + 1)^{5/2}} = 0 \text{ when } x = 0.$$

Point of inflection:  $(0, 0)$

Horizontal asymptotes:  $y = \pm \frac{3}{2}$

No vertical asymptotes



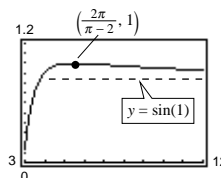
$$75. g(x) = \sin\left(\frac{x}{x-2}\right), \quad 3 < x < \infty$$

$$g'(x) = \frac{-2 \cos\left(\frac{x}{x-2}\right)}{(x-2)^2}$$

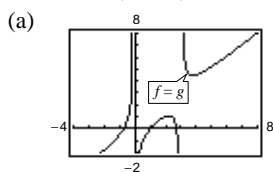
Horizontal asymptote:  $y = 1$

$$\text{Relative maximum: } \frac{x}{x-2} = \frac{\pi}{2} \Rightarrow x = \frac{2\pi}{\pi-2} \approx 5.5039$$

No vertical asymptotes

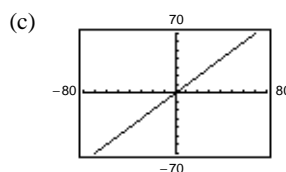


$$77. f(x) = \frac{x^3 - 3x^2 + 2}{x(x-3)}, \quad g(x) = x + \frac{2}{x(x-3)}$$



(b)

$$\begin{aligned} f(x) &= \frac{x^3 - 3x^2 + 2}{x(x-3)} \\ &= \frac{x^2(x-3)}{x(x-3)} + \frac{2}{x(x-3)} \\ &= x + \frac{2}{x(x-3)} = g(x) \end{aligned}$$



The graph appears as the slant asymptote  $y = x$ .

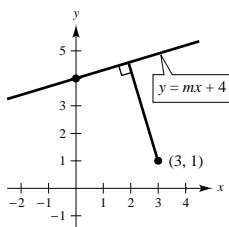
$$79. C = 0.5x + 500$$

$$\bar{C} = \frac{C}{x}$$

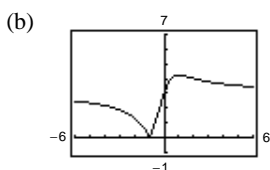
$$\bar{C} = 0.5 + \frac{500}{x}$$

$$\lim_{x \rightarrow \infty} \left(0.5 + \frac{500}{x}\right) = 0.5$$

81. line:  $mx - y + 4 = 0$



$$\begin{aligned} \text{(a) } d &= \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} = \frac{|m(3) - 1(1) + 4|}{\sqrt{m^2 + 1}} \\ &= \frac{|3m + 3|}{\sqrt{m^2 + 1}} \end{aligned}$$



$$\text{(c) } \lim_{m \rightarrow \infty} d(m) = 3 = \lim_{m \rightarrow -\infty} d(m)$$

The line approaches the vertical line  $x = 0$ . Hence, the distance approaches 3.

85. Answers will vary. See page 195.

## Section 3.6 A Summary of Curve Sketching

1.  $f$  has constant negative slope. Matches (D)

5. (a)  $f'(x) = 0$  for  $x = -2$  and  $x = 2$

$f'$  is negative for  $-2 < x < 2$  (decreasing function).

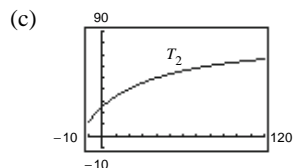
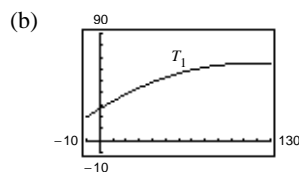
$f'$  is positive for  $x > 2$  and  $x < -2$  (increasing function).

(b)  $f''(x) = 0$  at  $x = 0$  (Inflection point).

$f''$  is positive for  $x > 0$  (Concave upwards).

$f''$  is negative for  $x < 0$  (Concave downward).

83. (a)  $T_1(t) = -0.003t^2 + 0.677t + 26.564$



$$T_2 = \frac{1451 + 86t}{58 + t}$$

(d)  $T_1(0) \approx 26.6$

$T_2(0) \approx 25.0$

(e)  $\lim_{t \rightarrow \infty} T_2 = \frac{86}{1} = 86$

(f) The limiting temperature is 86.  
 $T_1$  has no horizontal asymptote.

87. False. Let  $f(x) = \frac{2x}{\sqrt{x^2 + 2}}$ . (See Exercise 2.)

3. The slope is periodic, and zero at  $x = 0$ . Matches (A)

(c)  $f'$  is increasing on  $(0, \infty)$ . ( $f'' > 0$ )

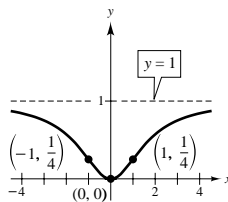
(d)  $f'(x)$  is minimum at  $x = 0$ . The rate of change of  $f$  at  $x = 0$  is less than the rate of change of  $f$  for all other values of  $x$ .

$$7. y = \frac{x^2}{x^2 + 3}$$

$$y' = \frac{6x}{(x^2 + 3)^2} = 0 \text{ when } x = 0.$$

$$y'' = \frac{18(1 - x^2)}{(x^2 + 3)^3} = 0 \text{ when } x = \pm 1.$$

Horizontal asymptote:  $y = 1$



	$y$	$y'$	$y''$	Conclusion
$-\infty < x < -1$		-	-	Decreasing, concave down
$x = -1$	$\frac{1}{4}$	-	0	Point of inflection
$-1 < x < 0$		-	+	Decreasing, concave up
$x = 0$	0	0	+	Relative minimum
$0 < x < 1$		+	+	Increasing, concave up
$x = 1$	$\frac{1}{4}$	+	0	Point of inflection
$1 < x < \infty$		+	-	Increasing, concave down

$$9. y = \frac{1}{x-2} - 3$$

$$y' = -\frac{1}{(x-2)^2} < 0 \text{ when } x \neq 2.$$

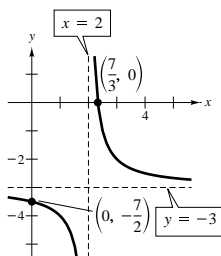
$$y'' = \frac{2}{(x-2)^3}$$

No relative extrema, no points of inflection

Intercepts:  $(\frac{7}{3}, 0), (0, -\frac{7}{2})$

Vertical asymptote:  $x = 2$

Horizontal asymptote:  $y = -3$



$$11. y = \frac{2x}{x^2 - 1}$$

$$y' = \frac{-2(x^2 + 1)}{(x^2 - 1)^2} < 0 \text{ if } x \neq \pm 1.$$

$$y'' = \frac{4x(x^2 + 3)}{(x^2 - 1)^3} = 0 \text{ if } x = 0.$$

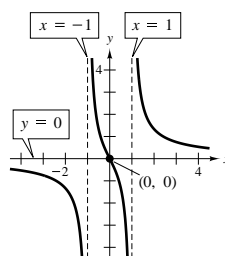
Inflection point:  $(0, 0)$

Intercept:  $(0, 0)$

Vertical asymptote:  $x = \pm 1$

Horizontal asymptote:  $y = 0$

Symmetry with respect to the origin



$$13. \quad g(x) = x + \frac{4}{x^2 + 1}$$

$$g'(x) = 1 - \frac{8x}{(x^2 + 1)^2} = \frac{x^4 + 2x^2 - 8x + 1}{(x^2 + 1)^2} = 0 \text{ when } x \approx 0.1292, 1.6085$$

$$g''(x) = \frac{8(3x^2 - 1)}{(x^2 + 1)^3} = 0 \text{ when } x = \pm \frac{\sqrt{3}}{3}$$

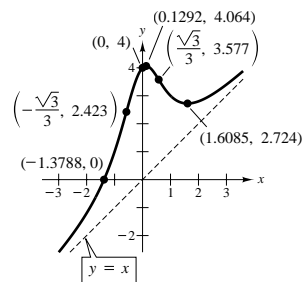
$g''(0.1292) < 0$ , therefore,  $(0.1292, 4.064)$  is relative maximum.

$g''(1.6085) > 0$ , therefore,  $(1.6085, 2.724)$  is a relative minimum.

Points of inflection:  $\left(-\frac{\sqrt{3}}{3}, 2.423\right), \left(\frac{\sqrt{3}}{3}, 3.577\right)$

Intercepts:  $(0, 4), (-1.3788, 0)$

Slant asymptote:  $y = x$



$$15. \quad f(x) = \frac{x^2 + 1}{x} = x + \frac{1}{x}$$

$$f'(x) = 1 - \frac{1}{x^2} = 0 \text{ when } x = \pm 1.$$

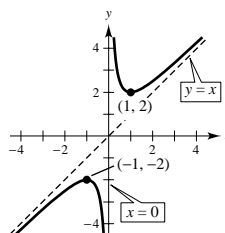
$$f''(x) = \frac{2}{x^3} \neq 0$$

Relative maximum:  $(-1, -2)$

Relative minimum:  $(1, 2)$

Vertical asymptote:  $x = 0$

Slant asymptote:  $y = x$



$$17. \quad y = \frac{x^2 - 6x + 12}{x - 4} = x - 2 + \frac{4}{x - 4}$$

$$y' = 1 - \frac{4}{(x - 4)^2}$$

$$= \frac{(x - 2)(x - 6)}{(x - 4)^2} = 0 \text{ when } x = 2, 6.$$

$$y'' = \frac{8}{(x - 4)^3}$$

$y'' < 0$  when  $x = 2$ .

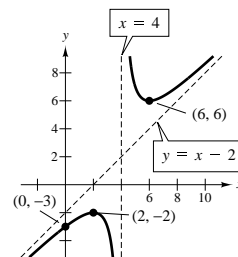
Therefore,  $(2, -2)$  is a relative maximum.

$y'' > 0$  when  $x = 6$ .

Therefore,  $(6, 6)$  is a relative minimum.

Vertical asymptote:  $x = 4$

Slant asymptote:  $y = x - 2$



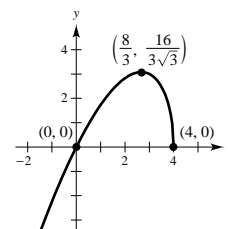
19.  $y = x\sqrt{x-4}$ ,

Domain:  $(-\infty, 4]$

$$y' = \frac{8-3x}{2\sqrt{4-x}} = 0 \text{ when } x = \frac{8}{3} \text{ and undefined when } x = 4.$$

$$y'' = \frac{3x-16}{4(4-x)^{3/2}} = 0 \text{ when } x = \frac{16}{3} \text{ and undefined when } x = 4.$$

**Note:**  $x = \frac{16}{3}$  is not in the domain.



	$y$	$y'$	$y''$	Conclusion
$-\infty < x < \frac{8}{3}$		+	-	Increasing, concave down
$x = \frac{8}{3}$	$\frac{16}{3\sqrt{3}}$	0	-	Relative maximum
$\frac{8}{3} < x < 4$		-	-	Decreasing, concave down
$x = 4$	0	Undefined	Undefined	Endpoint

21.  $h(x) = x\sqrt{9-x^2}$  Domain:  $-3 \leq x \leq 3$

$$h'(x) = \frac{9-2x^2}{\sqrt{9-x^2}} = 0 \text{ when } x = \pm \frac{3}{\sqrt{2}} = \pm \frac{3\sqrt{2}}{2}$$

$$h''(x) = \frac{x(2x^2-27)}{(9-x^2)^{3/2}} = 0 \text{ when } x = 0$$

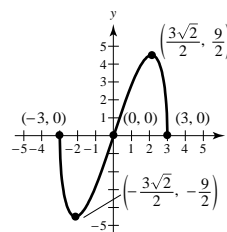
Relative maximum:  $\left(\frac{3\sqrt{2}}{2}, \frac{9}{2}\right)$

Relative minimum:  $\left(-\frac{3\sqrt{2}}{2}, -\frac{9}{2}\right)$

Intercepts:  $(0, 0), (\pm 3, 0)$

Symmetric with respect to the origin

Point of inflection:  $(0, 0)$

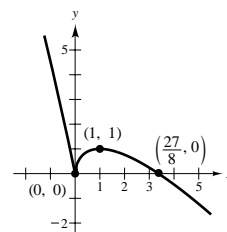


23.  $y = 3x^{2/3} - 2x$

$$y' = 2x^{-1/3} - 2 = \frac{2(1-x^{1/3})}{x^{1/3}}$$

$$= 0 \text{ when } x = 1 \text{ and undefined when } x = 0.$$

$$y'' = \frac{-2}{3x^{4/3}} < 0 \text{ when } x \neq 0.$$



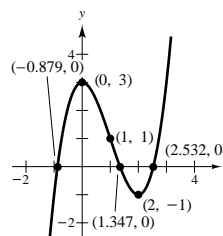
	$y$	$y'$	$y''$	Conclusion
$-\infty < x < 0$		-	-	Decreasing, concave down
$x = 0$	0	Undefined	Undefined	Relative minimum
$0 < x < 1$		+	-	Increasing, concave down
$x = 1$	1	0	-	Relative maximum
$1 < x < \infty$		-	-	Decreasing, concave down

25.  $y = x^3 - 3x^2 + 3$

$$y' = 3x^2 - 6x = 3x(x - 2) = 0 \text{ when } x = 0, x = 2$$

$$y'' = 6x - 6 = 6(x - 1) = 0 \text{ when } x = 1$$

	$y$	$y'$	$y''$	Conclusion
$-\infty < x < 0$		+	-	Increasing, concave down
$x = 0$	3	0	-	Relative maximum
$0 < x < 1$		-	-	Decreasing, concave down
$x = 1$	1	-	0	Point of inflection
$1 < x < 2$		-	+	Decreasing, concave up
$x = 2$	-1	0	+	Relative minimum
$2 < x < \infty$		+	+	Increasing, concave up



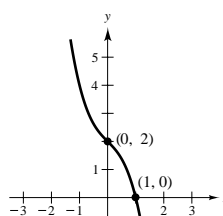
27.  $y = 2 - x - x^3$

$$y' = -1 - 3x^2$$

No critical numbers

$$y'' = -6x = 0 \text{ when } x = 0.$$

	$y$	$y'$	$y''$	Conclusion
$-\infty < x < 0$		-	+	Decreasing, concave up
$x = 0$	2	-	0	Point of inflection
$0 < x < \infty$		-	-	Decreasing, concave down

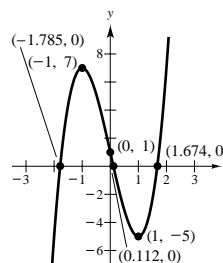


29.  $f(x) = 3x^3 - 9x + 1$

$$f'(x) = 9x^2 - 9 = 9(x^2 - 1) = 0 \text{ when } x = \pm 1$$

$$f''(x) = 18x = 0 \text{ when } x = 0$$

	$f(x)$	$f'(x)$	$f''(x)$	Conclusion
$-\infty < x < -1$		+	-	Increasing, concave down
$x = -1$	7	0	-	Relative maximum
$-1 < x < 0$		-	-	Decreasing, concave down
$x = 0$	1	-	0	Point of inflection
$0 < x < 1$		-	+	Decreasing, concave up
$x = 1$	-5	0	+	Relative minimum
$1 < x < \infty$		+	+	Increasing, concave up

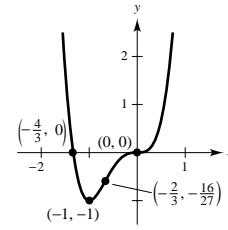


31.  $y = 3x^4 + 4x^3$

$$y' = 12x^3 + 12x^2 = 12x^2(x + 1) = 0 \text{ when } x = 0, x = -1.$$

$$y'' = 36x^2 + 24x = 12x(3x + 2) = 0 \text{ when } x = 0, x = -\frac{2}{3}.$$

	$y$	$y'$	$y''$	Conclusion
$-\infty < x < -1$		-	+	Decreasing, concave up
$x = -1$	-1	0	+	Relative minimum
$-1 < x < -\frac{2}{3}$		+	+	Increasing, concave up
$x = -\frac{2}{3}$	$-\frac{16}{27}$	+	0	Point of inflection
$-\frac{2}{3} < x < 0$		+	-	Increasing, concave down
$x = 0$	0	0	0	Point of inflection
$0 < x < \infty$		+	+	Increasing, concave up

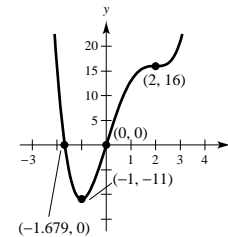


33.  $f(x) = x^4 - 4x^3 + 16x$

$$f'(x) = 4x^3 - 12x^2 + 16 = 4(x + 1)(x - 2)^2 = 0 \text{ when } x = -1, x = 2.$$

$$f''(x) = 12x^2 - 24x = 12x(x - 2) = 0 \text{ when } x = 0, x = 2.$$

	$f(x)$	$f'(x)$	$f''(x)$	Conclusion
$-\infty < x < -1$		-	+	Decreasing, concave up
$x = -1$	-11	0	+	Relative minimum
$-1 < x < 0$		+	+	Increasing, concave up
$x = 0$	0	+	0	Point of inflection
$0 < x < 2$		+	-	Increasing, concave down
$x = 2$	16	0	0	Point of inflection
$2 < x < \infty$		+	+	Increasing, concave up

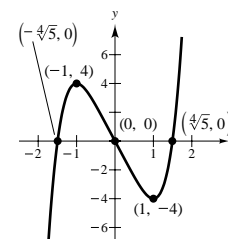


35.  $y = x^5 - 5x$

$$y' = 5x^4 - 5 = 5(x^4 - 1) = 0 \text{ when } x = \pm 1.$$

$$y'' = 20x^3 = 0 \text{ when } x = 0.$$

	$y$	$y'$	$y''$	Conclusion
$-\infty < x < -1$		+	-	Increasing, concave down
$x = -1$	4	0	-	Relative maximum
$-1 < x < 0$		-	-	Decreasing, concave down
$x = 0$	0	-	0	Point of inflection
$0 < x < 1$		-	+	Decreasing, concave up
$x = 1$	-4	0	+	Relative minimum
$1 < x < \infty$		+	+	Increasing, concave up



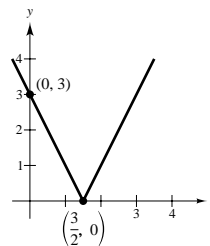


37.  $y = |2x - 3|$

$$y' = \frac{2(2x - 3)}{|2x - 3|} \text{ undefined at } x = \frac{3}{2}.$$

$$y'' = 0$$

	$y$	$y'$	Conclusion
$-\infty < x < \frac{3}{2}$		-	Decreasing
$x = \frac{3}{2}$	0	Undefined	Relative minimum
$\frac{3}{2} < x < \infty$		+	Increasing



39.  $y = \sin x - \frac{1}{18} \sin 3x, 0 \leq x \leq 2\pi$

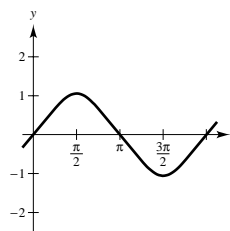
$$y' = \cos x - \frac{1}{6} \cos 3x = 0 \text{ when } x = \frac{\pi}{2}, \frac{3\pi}{2}.$$

$$y'' = -\sin x + \frac{1}{2} \sin 3x = 0 \text{ when } x = 0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}.$$

Relative maximum:  $\left(\frac{\pi}{2}, \frac{19}{18}\right)$

Relative minimum:  $\left(\frac{3\pi}{2}, -\frac{19}{18}\right)$

Inflection points:  $\left(\frac{\pi}{6}, \frac{4}{9}\right), \left(\frac{5\pi}{6}, \frac{4}{9}\right), (\pi, 0), \left(\frac{7\pi}{6}, -\frac{4}{9}\right), \left(\frac{11\pi}{6}, -\frac{4}{9}\right)$



41.  $y = 2x - \tan x, -\frac{\pi}{2} < x < \frac{\pi}{2}$

$$y' = 2 - \sec^2 x = 0 \text{ when } x = \pm \frac{\pi}{4}.$$

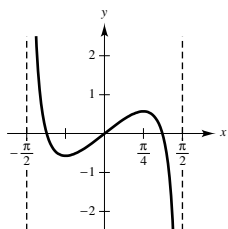
$$y'' = -2\sec^2 x \tan x = 0 \text{ when } x = 0.$$

Relative maximum:  $\left(\frac{\pi}{4}, \frac{\pi}{2} - 1\right)$

Relative minimum:  $\left(-\frac{\pi}{4}, 1 - \frac{\pi}{2}\right)$

Inflection point:  $(0, 0)$

Vertical asymptotes:  $x = \pm \frac{\pi}{2}$

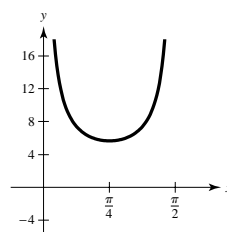


43.  $y = 2(\csc x + \sec x), 0 < x < \frac{\pi}{2}$

$$y' = 2(\sec x \tan x - \csc x \cot x) = 0 \Rightarrow x = \pi/4$$

Relative minimum:  $\left(\frac{\pi}{4}, 4\sqrt{2}\right)$

Vertical asymptotes:  $x = 0, x = \frac{\pi}{2}$



45.  $g(x) = x \tan x, -\frac{3\pi}{2} < x < \frac{3\pi}{2}$

$$g'(x) = \frac{x + \sin x \cos x}{\cos^2 x} = 0 \text{ when } x = 0$$

$$g''(x) = \frac{2(\cos x + x \sin x)}{\cos^3 x}$$

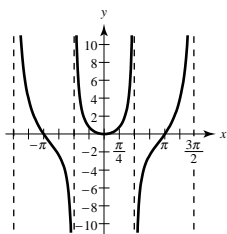
Vertical asymptotes:  $x = -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$

Intercepts:  $(-\pi, 0), (0, 0), (\pi, 0)$

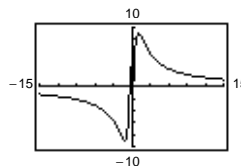
Symmetric with respect to y-axis.

Increasing on  $(0, \frac{\pi}{2})$  and  $(\frac{\pi}{2}, \frac{3\pi}{2})$

Points of inflection:  $(\pm 2.80, 0)$



47.  $f(x) = \frac{20x}{x^2 + 1} - \frac{1}{x} = \frac{19x^2 - 1}{x(x^2 + 1)}$


 $x = 0$  vertical asymptote

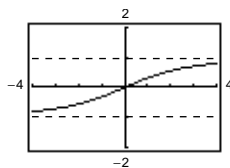
 $y = 0$  horizontal asymptote

 Minimum:  $(-1.10, -9.05)$ 

 Maximum:  $(1.10, 9.05)$ 

 Points of inflection:  $(-1.84, -7.86), (1.84, 7.86)$ 

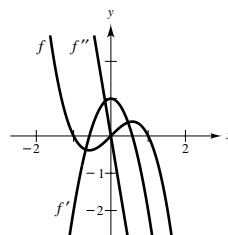
49.  $y = \frac{x}{\sqrt{x^2 + 7}}$


 $(0, 0)$  point of inflection

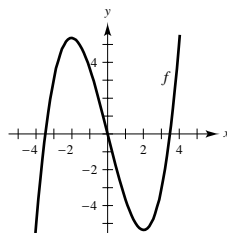
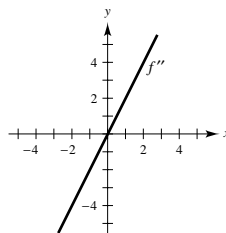
 $y = \pm 1$  horizontal asymptotes

 51.  $f$  is cubic.

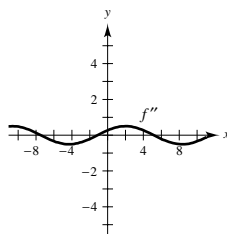
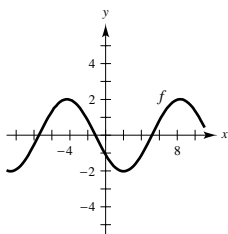
 $f'$  is quadratic.

 $f''$  is linear.


53.


 (any vertical translate of  $f$  will do)


55.



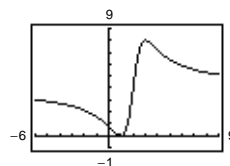
(any vertical translate of  $f$  will do)

57. Since the slope is negative, the function is decreasing on  $(2, 8)$ , and hence  $f(3) > f(5)$ .

$$59. f(x) = \frac{4(x-1)^2}{x^2 - 4x + 5}$$

Vertical asymptote: none

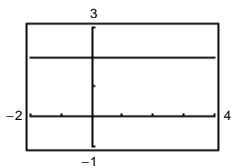
Horizontal asymptote:  $y = 4$



The graph crosses the horizontal asymptote  $y = 4$ . If a function has a vertical asymptote at  $x = c$ , the graph would not cross it since  $f(c)$  is undefined.

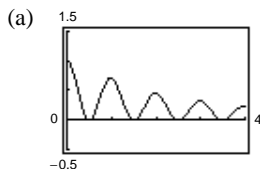
$$61. h(x) = \frac{6-2x}{3-x} = \frac{2(3-x)}{3-x} = \begin{cases} 2, & \text{if } x \neq 3 \\ \text{Undefined,} & \text{if } x = 3 \end{cases}$$

The rational function is not reduced to lowest terms.



hole at  $(3, 2)$

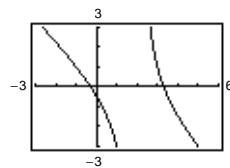
$$65. f(x) = \frac{\cos^2 \pi x}{\sqrt{x^2 + 1}}, \quad (0, 4)$$



On  $(0, 4)$  there seem to be 7 critical numbers:

0.5, 1.0, 1.5, 2.0, 2.5, 3.0, 3.5

$$63. f(x) = \frac{x^2 - 3x - 1}{x - 2} = -x + 1 + \frac{3}{x - 2}$$



The graph appears to approach the slant asymptote  $y = -x + 1$ .

$$(b) f'(x) = \frac{-\cos \pi x (x \cos \pi x + 2\pi(x^2 + 1)\sin \pi x)}{(x^2 + 1)^{3/2}} = 0$$

$$\text{Critical numbers} \approx \frac{1}{2}, 0.97, \frac{3}{2}, 1.98, \frac{5}{2}, 2.98, \frac{7}{2}$$

The critical numbers where maxima occur appear to be integers in part (a), but approximating them using  $f'$  shows that they are not integers.

67. Vertical asymptote:  $x = 5$

Horizontal asymptote:  $y = 0$

$$y = \frac{1}{x - 5}$$

71.  $f(x) = \frac{ax}{(x - b)^2}$

(a) The graph has a vertical asymptote at  $x = b$ . If  $a > 0$ , the graph approaches  $\infty$  as  $x \rightarrow b$ . If  $a < 0$ , the graph approaches  $-\infty$  as  $x \rightarrow b$ . The graph approaches its vertical asymptote faster as  $|a| \rightarrow 0$ .

73.  $f(x) = \frac{3x^n}{x^4 + 1}$

- (a) For  $n$  even,  $f$  is symmetric about the  $y$ -axis. For  $n$  odd,  $f$  is symmetric about the origin.
- (b) The  $x$ -axis will be the horizontal asymptote if the degree of the numerator is less than 4. That is,  $n = 0, 1, 2, 3$ .
- (c)  $n = 4$  gives  $y = 3$  as the horizontal asymptote.

69. Vertical asymptote:  $x = 5$

Slant asymptote:  $y = 3x + 2$

$$y = 3x + 2 + \frac{1}{x - 5} = \frac{3x^2 - 13x - 9}{x - 5}$$

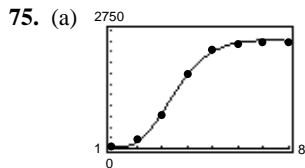
(b) As  $b$  varies, the position of the vertical asymptote changes:  $x = b$ . Also, the coordinates of the minimum ( $a > 0$ ) or maximum ( $a < 0$ ) are changed.

(d) There is a slant asymptote  $y = 3x$  if  $n = 5$ :

$$\frac{3x^5}{x^4 + 1} = 3x - \frac{3x}{x^4 + 1}$$

(e)

$n$	0	1	2	3	4	5
$M$	1	2	3	2	1	0
$N$	2	3	4	5	2	3



- (b) When  $t = 10$ ,  $N(10) \approx 2434$  bacteria.
- (c)  $N$  is a maximum when  $t \approx 7.2$  (seventh day).
- (d)  $N''(t) = 0$  for  $t \approx 3.2$
- (e)  $\lim_{t \rightarrow \infty} N(t) = \frac{13,250}{7} \approx 1892.86$

### Section 3.7 Optimization Problems

1. (a)

First Number, $x$	Second Number	Product, $P$
10	$110 - 10$	$10(110 - 10) = 1000$
20	$110 - 20$	$20(110 - 20) = 1800$
30	$110 - 30$	$30(110 - 30) = 2400$
40	$110 - 40$	$40(110 - 40) = 2800$
50	$110 - 50$	$50(110 - 50) = 3000$
60	$110 - 60$	$60(110 - 60) = 3000$

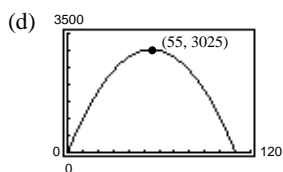
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## 1. —CONTINUED—

First Number, $x$	Second Number	Product, $P$
10	$110 - 10$	$10(110 - 10) = 1000$
20	$110 - 20$	$20(110 - 20) = 1800$
30	$110 - 30$	$30(110 - 30) = 2400$
40	$110 - 40$	$40(110 - 40) = 2800$
50	$110 - 50$	$50(110 - 50) = 3000$
60	$110 - 60$	$60(110 - 60) = 3000$
70	$110 - 70$	$70(110 - 70) = 2800$
80	$110 - 80$	$80(110 - 80) = 2400$
90	$110 - 90$	$90(110 - 90) = 1800$
100	$110 - 100$	$100(110 - 100) = 1000$

The maximum is attained near  $x = 50$  and  $60$ .

(c)  $P = x(110 - x) = 110x - x^2$



The solution appears to be  $x = 55$ .

(e)  $\frac{dP}{dx} = 110 - 2x = 0$  when  $x = 55$ .

$$\frac{d^2P}{dx^2} = -2 < 0$$

$P$  is a maximum when  $x = 110 - x = 55$ .

The two numbers are 55 and 55.

3. Let  $x$  and  $y$  be two positive numbers such that  $xy = 192$ .

$$S = x + y = x + \frac{192}{x}$$

$$\frac{dS}{dx} = 1 - \frac{192}{x^2} = 0 \text{ when } x = \sqrt{192}.$$

$$\frac{d^2S}{dx^2} = \frac{384}{x^3} > 0 \text{ when } x = \sqrt{192}.$$

$S$  is a minimum when  $x = y = \sqrt{192}$ .

7. Let  $x$  be the length and  $y$  the width of the rectangle.

$$2x + 2y = 100$$

$$y = 50 - x$$

$$A = xy = x(50 - x)$$

$$\frac{dA}{dx} = 50 - 2x = 0 \text{ when } x = 25.$$

$$\frac{d^2A}{dx^2} = -2 < 0 \text{ when } x = 25.$$

$A$  is maximum when  $x = y = 25$  meters.

5. Let  $x$  be a positive number.

$$S = x + \frac{1}{x}$$

$$\frac{dS}{dx} = 1 - \frac{1}{x^2} = 0 \text{ when } x = 1.$$

$$\frac{d^2S}{dx^2} = \frac{2}{x^3} > 0 \text{ when } x = 1.$$

The sum is a minimum when  $x = 1$  and  $1/x = 1$ .

9. Let  $x$  be the length and  $y$  the width of the rectangle.

$$xy = 64$$

$$y = \frac{64}{x}$$

$$P = 2x + 2y = 2x + 2\left(\frac{64}{x}\right) = 2x + \frac{128}{x}$$

$$\frac{dP}{dx} = 2 - \frac{128}{x^2} = 0 \text{ when } x = 8.$$

$$\frac{d^2P}{dx^2} = \frac{256}{x^3} > 0 \text{ when } x = 8.$$

$P$  is minimum when  $x = y = 8$  feet.

$$11. d = \sqrt{(x-4)^2 + (\sqrt{x}-0)^2}$$

$$= \sqrt{x^2 - 7x + 16}$$

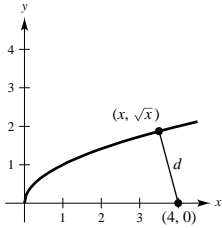
Since  $d$  is smallest when the expression inside the radical is smallest, you need only find the critical numbers of

$$f(x) = x^2 - 7x + 16.$$

$$f'(x) = 2x - 7 = 0$$

$$x = \frac{7}{2}$$

By the First Derivative Test, the point nearest to  $(4, 0)$  is  $(\frac{7}{2}, \sqrt{7/2})$ .



$$15. \frac{dQ}{dx} = kx(Q_0 - x) = kQ_0x - kx^2$$

$$\frac{d^2Q}{dx^2} = kQ_0 - 2kx$$

$$= k(Q_0 - 2x) = 0 \text{ when } x = \frac{Q_0}{2}.$$

$$\frac{d^3Q}{dx^3} = -2k < 0 \text{ when } x = \frac{Q_0}{2}.$$

$dQ/dx$  is maximum when  $x = Q_0/2$ .

19. (a)  $A = 4(\text{area of side}) + 2(\text{area of Top})$

$$(a) A = 4(3)(11) + 2(3)(3) = 150 \text{ square inches}$$

$$(b) A = 4(5)(5) + 2(5)(5) = 150 \text{ square inches}$$

$$(c) A = 4(3.25)(6) + 2(6)(6) = 150 \text{ square inches}$$

$$(c) S = 4xy + 2x^2 = 150 \Rightarrow y = \frac{150 - 2x^2}{4x}$$

$$V = x^2y = x^2\left(\frac{150 - 2x^2}{4x}\right) = \frac{75}{2}x - \frac{1}{2}x^3$$

$$V' = \frac{75}{2} - \frac{3}{2}x^2 = 0 \Rightarrow x = \pm 5$$

By the First Derivative Test,  $x = 5$  yields the maximum volume. Dimensions:  $5 \times 5 \times 5$ . (A cube!)

$$13. d = \sqrt{(x-2)^2 + [x^2 - (1/2)]^2}$$

$$= \sqrt{x^4 - 4x + (17/4)}$$

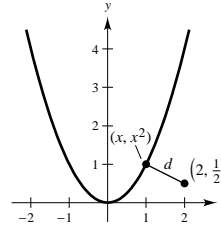
Since  $d$  is smallest when the expression inside the radical is smallest, you need only find the critical numbers of

$$f(x) = x^4 - 4x + \frac{17}{4}.$$

$$f'(x) = 4x^3 - 4 = 0$$

$$x = 1$$

By the First Derivative Test, the point nearest to  $(2, \frac{1}{2})$  is  $(1, 1)$ .



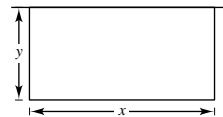
17.  $xy = 180,000$  (see figure)

$S = x + 2y = \left(x + \frac{360,000}{x}\right)$  where  $S$  is the length of fence needed.

$$\frac{dS}{dx} = 1 - \frac{360,000}{x^2} = 0 \text{ when } x = 600.$$

$$\frac{d^2S}{dx^2} = \frac{720,000}{x^3} > 0 \text{ when } x = 600.$$

$S$  is a minimum when  $x = 600$  meters and  $y = 300$  meters.

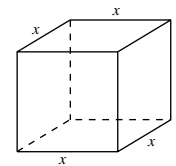


(b)  $V = (\text{length})(\text{width})(\text{height})$

$$(a) V = (3)(3)(11) = 99 \text{ cubic inches}$$

$$(b) V = (5)(5)(5) = 125 \text{ cubic inches}$$

$$(c) V = (6)(6)(3.25) = 117 \text{ cubic inches}$$



21. (a)  $V = x(s - 2x)^2, 0 < x < \frac{s}{2}$

$$\frac{dV}{dx} = 2x(s - 2x)(-2) + (s - 2x)^2$$

$$= (s - 2x)(s - 6x) = 0 \text{ when } x = \frac{s}{2}, \frac{s}{6} \text{ (} s/2 \text{ is not in the domain).}$$

$$\frac{d^2V}{dx^2} = 24x - 8s$$

$$\frac{d^2V}{dx^2} < 0 \text{ when } x = \frac{s}{6}.$$

$$V = \frac{2s^3}{27} \text{ is maximum when } x = \frac{s}{6}.$$

(b) If the length is doubled,  $V = \frac{2}{27}(2s)^3 = 8\left(\frac{2}{27}s^3\right)$ . Volume is increased by a factor of 8.

23.  $16 = 2y + x + \pi\left(\frac{x}{2}\right)$

$$32 = 4y + 2x + \pi x$$

$$y = \frac{32 - 2x - \pi x}{4}$$

$$A = xy + \frac{\pi}{2}\left(\frac{x}{2}\right)^2 = \left(\frac{32 - 2x - \pi x}{4}\right)x + \frac{\pi x^2}{8}$$

$$= 8x - \frac{1}{2}x^2 - \frac{\pi}{4}x^2 + \frac{\pi}{8}x^2$$

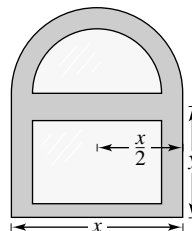
$$\frac{dA}{dx} = 8 - x - \frac{\pi}{2}x + \frac{\pi}{4}x = 8 - x\left(1 + \frac{\pi}{4}\right)$$

$$= 0 \text{ when } x = \frac{8}{1 + (\pi/4)} = \frac{32}{4 + \pi}.$$

$$\frac{d^2A}{dx^2} = -\left(1 + \frac{\pi}{4}\right) < 0 \text{ when } x = \frac{32}{4 + \pi}$$

$$y = \frac{32 - 2[32/(4 + \pi)] - \pi[32/(4 + \pi)]}{4} = \frac{16}{4 + \pi}$$

The area is maximum when  $y = \frac{16}{4 + \pi}$  feet and  $x = \frac{32}{4 + \pi}$  feet.



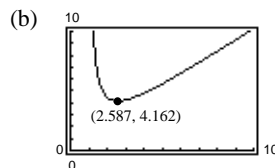
25. (a)  $\frac{y - 2}{0 - 1} = \frac{0 - 2}{x - 1}$

$$y = 2 + \frac{2}{x - 1}$$

$$L = \sqrt{x^2 + y^2} = \sqrt{x^2 + \left(2 + \frac{2}{x - 1}\right)^2}$$

$$= \sqrt{x^2 + 4 + \frac{8}{x - 1} + \frac{4}{(x - 1)^2}}, \quad x > 1$$

—CONTINUED—



$L$  is minimum when  $x \approx 2.587$  and  $L \approx 4.162$ .

## 25. —CONTINUED—

$$(c) \text{ Area} = A(x) = \frac{1}{2}xy = \frac{1}{2}x\left(2 + \frac{2}{x-1}\right) = x + \frac{x}{x-1}$$

$$A'(x) = 1 + \frac{(x-1) - x}{(x-1)^2} = 1 - \frac{1}{(x-1)^2} = 0$$

$$(x-1)^2 = 1$$

$$x-1 = \pm 1$$

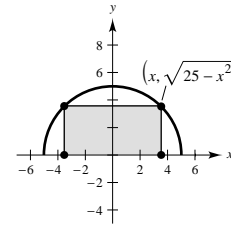
$$x = 0, 2 \text{ (select } x = 2\text{)}$$

Then  $y = 4$  and  $A = 4$ .

Vertices:  $(0, 0)$ ,  $(2, 0)$ ,  $(0, 4)$

$$27. A = 2xy = 2x\sqrt{25-x^2} \text{ (see figure)}$$

$$\begin{aligned} \frac{dA}{dx} &= 2x\left(\frac{1}{2}\right)\left(\frac{-2x}{\sqrt{25-x^2}}\right) + 2\sqrt{25-x^2} \\ &= 2\left(\frac{25-2x^2}{\sqrt{25-x^2}}\right) = 0 \text{ when } x = y = \frac{5\sqrt{2}}{2} \approx 3.54. \end{aligned}$$



By the First Derivative Test, the inscribed rectangle of maximum area has vertices

$$\left(\pm \frac{5\sqrt{2}}{2}, 0\right), \left(\pm \frac{5\sqrt{2}}{2}, \frac{5\sqrt{2}}{2}\right).$$

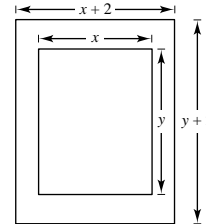
$$\text{Width: } \frac{5\sqrt{2}}{2}; \text{ Length: } 5\sqrt{2}$$

$$29. xy = 30 \Rightarrow y = \frac{30}{x}$$

$$A = (x+2)\left(\frac{30}{x} + 2\right) \text{ (see figure)}$$

$$\frac{dA}{dx} = (x+2)\left(\frac{-30}{x^2}\right) + \left(\frac{30}{x} + 2\right) = \frac{2(x^2-30)}{x^2} = 0 \text{ when } x = \sqrt{30}.$$

$$y = \frac{30}{\sqrt{30}} = \sqrt{30}$$



By the First Derivative Test, the dimensions  $(x+2)$  by  $(y+2)$  are  $(2 + \sqrt{30})$  by  $(2 + \sqrt{30})$  (approximately 7.477 by 7.477). These dimensions yield a minimum area.

$$31. V = \pi r^2 h = 22 \text{ cubic inches or } h = \frac{22}{\pi r^2}$$

(a)

Radius, $r$	Height	Surface Area
0.2	$\frac{22}{\pi(0.2)^2}$	$2\pi(0.2)\left[0.2 + \frac{22}{\pi(0.2)^2}\right] \approx 220.3$
0.4	$\frac{22}{\pi(0.4)^2}$	$2\pi(0.4)\left[0.4 + \frac{22}{\pi(0.4)^2}\right] \approx 111.0$
0.6	$\frac{22}{\pi(0.6)^2}$	$2\pi(0.6)\left[0.6 + \frac{22}{\pi(0.6)^2}\right] \approx 75.6$
0.8	$\frac{22}{\pi(0.8)^2}$	$2\pi(0.8)\left[0.8 + \frac{22}{\pi(0.8)^2}\right] \approx 59.0$

—CONTINUED—



## 31. —CONTINUED—

(b)

Radius, $r$	Height	Surface Area
0.2	$\frac{22}{\pi(0.2)^2}$	$2\pi(0.2)\left[0.2 + \frac{22}{\pi(0.2)^2}\right] \approx 220.3$
0.4	$\frac{22}{\pi(0.4)^2}$	$2\pi(0.4)\left[0.4 + \frac{22}{\pi(0.4)^2}\right] \approx 111.0$
0.6	$\frac{22}{\pi(0.6)^2}$	$2\pi(0.6)\left[0.6 + \frac{22}{\pi(0.6)^2}\right] \approx 75.6$
0.8	$\frac{22}{\pi(0.8)^2}$	$2\pi(0.8)\left[0.8 + \frac{22}{\pi(0.8)^2}\right] \approx 59.0$
1.0	$\frac{22}{\pi(1.0)^2}$	$2\pi(1.0)\left[1.0 + \frac{22}{\pi(1.0)^2}\right] \approx 50.3$
1.2	$\frac{22}{\pi(1.2)^2}$	$2\pi(1.2)\left[1.2 + \frac{22}{\pi(1.2)^2}\right] \approx 45.7$
1.4	$\frac{22}{\pi(1.4)^2}$	$2\pi(1.4)\left[1.4 + \frac{22}{\pi(1.4)^2}\right] \approx 43.7$
1.6	$\frac{22}{\pi(1.6)^2}$	$2\pi(1.6)\left[1.6 + \frac{22}{\pi(1.6)^2}\right] \approx 43.6$
1.8	$\frac{22}{\pi(1.8)^2}$	$2\pi(1.8)\left[1.8 + \frac{22}{\pi(1.8)^2}\right] \approx 44.8$
2.0	$\frac{22}{\pi(2.0)^2}$	$2\pi(2.0)\left[2.0 + \frac{22}{\pi(2.0)^2}\right] \approx 47.1$

The minimum seems to be about 43.6 for  $r = 1.6$ .

33. Let  $x$  be the sides of the square ends and  $y$  the length of the package.

$$P = 4x + y = 108 \Rightarrow y = 108 - 4x$$

$$V = x^2y = x^2(108 - 4x) = 108x^2 - 4x^3$$

$$\frac{dV}{dx} = 216x - 12x^2$$

$$= 12x(18 - x) = 0 \text{ when } x = 18.$$

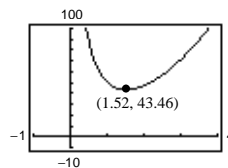
$$\frac{d^2V}{dx^2} = 216 - 24x = -216 < 0 \text{ when } x = 18.$$

The volume is maximum when  $x = 18$  inches and  $y = 108 - 4(18) = 36$  inches.

$$(c) S = 2\pi r^2 + 2\pi rh$$

$$= 2\pi r(r + h) = 2\pi r\left[r + \frac{22}{\pi r^2}\right] = 2\pi r^2 + \frac{44}{r}$$

(d)



The minimum seems to be 43.46 for  $r \approx 1.52$ .

$$(e) \frac{dS}{dr} = 4\pi r - \frac{44}{r^2} = 0 \text{ when } r = \sqrt[3]{11/\pi} \approx 1.52 \text{ in.}$$

$$h = \frac{22}{\pi r^2} \approx 3.04 \text{ in.}$$

**Note:** Notice that

$$h = \frac{22}{\pi r^2} = \frac{22}{\pi(11/\pi)^{2/3}} = 2\left(\frac{11^{1/3}}{\pi^{1/3}}\right) = 2r.$$

35.  $V = \frac{1}{3}\pi x^2 h = \frac{1}{3}\pi x^2(r + \sqrt{r^2 - x^2})$  (see figure)

$$\frac{dV}{dx} = \frac{1}{3}\pi \left[ \frac{-x^3}{\sqrt{r^2 - x^2}} + 2x(r + \sqrt{r^2 - x^2}) \right] = \frac{\pi x}{3\sqrt{r^2 - x^2}} (2r^2 + 2r\sqrt{r^2 - x^2} - 3x^2) = 0$$

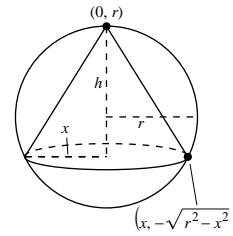
$$2r^2 + 2r\sqrt{r^2 - x^2} - 3x^2 = 0$$

$$2r\sqrt{r^2 - x^2} = 3x^2 - 2r^2$$

$$4r^2(r^2 - x^2) = 9x^4 - 12x^2r^2 + 4r^4$$

$$0 = 9x^4 - 8x^2r^2 = x^2(9x^2 - 8r^2)$$

$$x = 0, \frac{2\sqrt{2}r}{3}$$



By the First Derivative Test, the volume is a maximum when

$$x = \frac{2\sqrt{2}r}{3} \text{ and } h = r + \sqrt{r^2 - x^2} = \frac{4r}{3}.$$

Thus, the maximum volume is

$$V = \frac{1}{3}\pi \left( \frac{8r^2}{9} \right) \left( \frac{4r}{3} \right) = \frac{32\pi r^3}{81} \text{ cubic units.}$$

37. No, there is no minimum area. If the sides are  $x$  and  $y$ , then  $2x + 2y = 20 \Rightarrow y = 10 - x$ . The area is  $A(x) = x(10 - x) = 10x - x^2$ . This can be made arbitrarily small by selecting  $x \approx 0$ .

39.  $V = 12 = \frac{4}{3}\pi r^3 + \pi r^2 h$

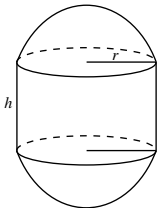
$$h = \frac{12 - (4/3)\pi r^3}{\pi r^2} = \frac{12}{\pi r^2} - \frac{4}{3}r$$

$$S = 4\pi r^2 + 2\pi r h = 4\pi r^2 + 2\pi r \left( \frac{12}{\pi r^2} - \frac{4}{3}r \right) \\ = 4\pi r^2 + \frac{24}{r} - \frac{8}{3}\pi r^2 = \frac{4}{3}\pi r^2 + \frac{24}{r}$$

$$\frac{dS}{dr} = \frac{8}{3}\pi r - \frac{24}{r^2} = 0 \text{ when } r = \sqrt[3]{9/\pi} \approx 1.42 \text{ cm.}$$

$$\frac{d^2S}{dr^2} = \frac{8}{3}\pi + \frac{48}{r^3} > 0 \text{ when } r = \sqrt[3]{9/\pi} \text{ cm.}$$

The surface area is minimum when  $r = \sqrt[3]{9/\pi}$  cm and  $h = 0$ . The resulting solid is a sphere of radius  $r \approx 1.42$  cm.



41. Let  $x$  be the length of a side of the square and  $y$  the length of a side of the triangle.

$$4x + 3y = 10$$

$$A = x^2 + \frac{1}{2}y \left( \frac{\sqrt{3}}{2}y \right) \\ = \frac{(10 - 3y)^2}{16} + \frac{\sqrt{3}}{4}y^2$$

$$\frac{dA}{dy} = \frac{1}{8}(10 - 3y)(-3) + \frac{\sqrt{3}}{2}y = 0$$

$$-30 + 9y + 4\sqrt{3}y = 0$$

$$y = \frac{30}{9 + 4\sqrt{3}}$$

$$\frac{d^2A}{dy^2} = \frac{9 + 4\sqrt{3}}{8} > 0$$

$A$  is minimum when

$$y = \frac{30}{9 + 4\sqrt{3}} \text{ and } x = \frac{10\sqrt{3}}{9 + 4\sqrt{3}}.$$

43. Let  $S$  be the strength and  $k$  the constant of proportionality. Given  $h^2 + w^2 = 24^2$ ,  $h^2 = 24^2 - w^2$ ,

$$S = kwh^2$$

$$S = kw(576 - w^2) = k(576w - w^3)$$

$$\frac{dS}{dw} = k(576 - 3w^2) = 0 \text{ when } w = 8\sqrt{3}, h = 8\sqrt{6}.$$

$$\frac{d^2S}{dw^2} = -6kw < 0 \text{ when } w = 8\sqrt{3}.$$

These values yield a maximum.

47.  $\sin \alpha = \frac{h}{s} \Rightarrow s = \frac{h}{\sin \alpha}, 0 < \alpha < \frac{\pi}{2}$

$$\tan \alpha = \frac{h}{2} \Rightarrow h = 2 \tan \alpha \Rightarrow s = \frac{2 \tan \alpha}{\sin \alpha} = 2 \sec \alpha$$

$$I = \frac{k \sin \alpha}{s^2} = \frac{k \sin \alpha}{4 \sec^2 \alpha} = \frac{k}{4} \sin \alpha \cos^2 \alpha$$

$$\frac{dI}{d\alpha} = \frac{k}{4} [\sin \alpha (-2 \sin \alpha \cos \alpha) + \cos^2 \alpha (\cos \alpha)]$$

$$= \frac{k}{4} \cos \alpha [\cos^2 \alpha - 2 \sin^2 \alpha]$$

$$= \frac{k}{4} \cos \alpha [1 - 3 \sin^2 \alpha]$$

$$= 0 \text{ when } \alpha = \frac{\pi}{2}, \frac{3\pi}{2}, \text{ or when } \sin \alpha = \pm \frac{1}{\sqrt{3}}.$$

Since  $\alpha$  is acute, we have

$$\sin \alpha = \frac{1}{\sqrt{3}} \Rightarrow h = 2 \tan \alpha = 2 \left( \frac{1}{\sqrt{2}} \right) = \sqrt{2} \text{ feet.}$$

Since  $(d^2I)/(d\alpha^2) = (k/4) \sin \alpha (9 \sin^2 \alpha - 7) < 0$  when  $\sin \alpha = 1/\sqrt{3}$ , this yields a maximum.

49.  $S = \sqrt{x^2 + 4}, L = \sqrt{1 + (3 - x)^2}$

$$\text{Time} = T = \frac{\sqrt{x^2 + 4}}{2} + \frac{\sqrt{x^2 - 6x + 10}}{4}$$

$$\frac{dT}{dx} = \frac{x}{2\sqrt{x^2 + 4}} + \frac{x - 3}{4\sqrt{x^2 - 6x + 10}} = 0$$

$$\frac{x^2}{x^2 + 4} = \frac{9 - 6x + x^2}{4(x^2 - 6x + 10)}$$

$$x^4 - 6x^3 + 9x^2 + 8x - 12 = 0$$

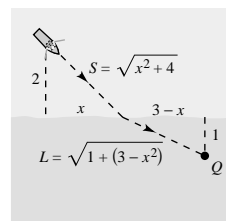
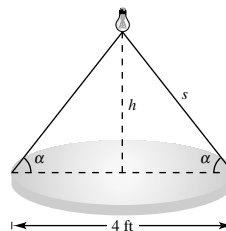
You need to find the roots of this equation in the interval  $[0, 3]$ . By using a computer or graphics calculator, you can determine that this equation has only one root in this interval ( $x = 1$ ). Testing at this value and at the endpoints, you see that  $x = 1$  yields the minimum time. Thus, the man should row to a point 1 mile from the nearest point on the coast.

45.  $R = \frac{v_0^2}{g} \sin 2\theta$

$$\frac{dR}{d\theta} = \frac{2v_0^2}{g} \cos 2\theta = 0 \text{ when } \theta = \frac{\pi}{4}, \frac{3\pi}{4}.$$

$$\frac{d^2R}{d\theta^2} = -\frac{4v_0^2}{g} \sin 2\theta < 0 \text{ when } \theta = \frac{\pi}{4}.$$

By the Second Derivative Test,  $R$  is maximum when  $\theta = \pi/4$ .



$$51. T = \frac{\sqrt{x^2 + 4}}{v_1} + \frac{\sqrt{x^2 - 6x + 10}}{v_2}$$

$$\frac{dT}{dx} = \frac{x}{v_1\sqrt{x^2 + 4}} + \frac{x - 3}{v_2\sqrt{x^2 - 6x + 10}} = 0$$

Since

$$\frac{x}{\sqrt{x^2 + 4}} = \sin \theta_1 \text{ and } \frac{x - 3}{\sqrt{x^2 - 6x + 10}} = -\sin \theta_2$$

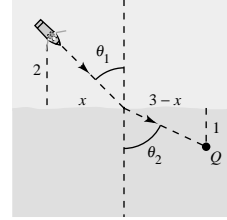
we have

$$\frac{\sin \theta_1}{v_1} - \frac{\sin \theta_2}{v_2} = 0 \Rightarrow \frac{\sin \theta_1}{v_1} = \frac{\sin \theta_2}{v_2}.$$

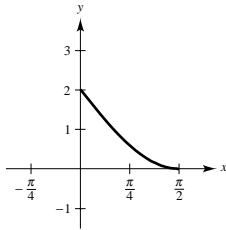
Since

$$\frac{d^2T}{dx^2} = \frac{4}{v_1(x^2 + 4)^{3/2}} + \frac{1}{v_2(x^2 - 6x + 10)^{3/2}} > 0$$

this condition yields a minimum time.

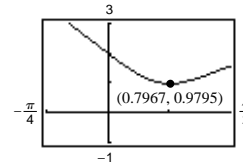


$$53. f(x) = 2 - 2 \sin x$$



- (a) Distance from origin to y-intercept is 2.  
Distance from origin to x-intercept is  $\pi/2 \approx 1.57$ .

$$(b) d = \sqrt{x^2 + y^2} = \sqrt{x^2 + (2 - 2 \sin x)^2}$$



Minimum distance = 0.9795 at  $x = 0.7967$ .

$$(c) \text{ Let } f(x) = d^2(x) = x^2 + (2 - 2 \sin x)^2.$$

$$f'(x) = 2x + 2(2 - 2 \sin x)(-2 \cos x)$$

Setting  $f'(x) = 0$ , you obtain  $x \approx 0.7967$ , which corresponds to  $d = 0.9795$ .

$$55. F \cos \theta = k(W - F \sin \theta)$$

$$F = \frac{kW}{\cos \theta + k \sin \theta}$$

$$\frac{dF}{d\theta} = \frac{-kW(k \cos \theta - \sin \theta)}{(\cos \theta + k \sin \theta)^2} = 0$$

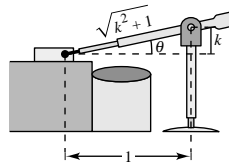
$$k \cos \theta = \sin \theta \Rightarrow k = \tan \theta \Rightarrow \theta = \arctan k$$

Since

$$\cos \theta + k \sin \theta = \frac{1}{\sqrt{k^2 + 1}} + \frac{k^2}{\sqrt{k^2 + 1}} = \sqrt{k^2 + 1},$$

the minimum force is

$$F = \frac{kW}{\cos \theta + k \sin \theta} = \frac{kW}{\sqrt{k^2 + 1}}.$$



57. (a)

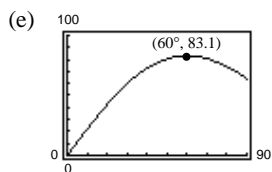
Base 1	Base 2	Altitude	Area
8	$8 + 16 \cos 10^\circ$	$8 \sin 10^\circ$	$\approx 22.1$
8	$8 + 16 \cos 20^\circ$	$8 \sin 20^\circ$	$\approx 42.5$
8	$8 + 16 \cos 30^\circ$	$8 \sin 30^\circ$	$\approx 59.7$
8	$8 + 16 \cos 40^\circ$	$8 \sin 40^\circ$	$\approx 72.7$
8	$8 + 16 \cos 50^\circ$	$8 \sin 50^\circ$	$\approx 80.5$
8	$8 + 16 \cos 60^\circ$	$8 \sin 60^\circ$	$\approx 83.1$

(b)

Base 1	Base 2	Altitude	Area
8	$8 + 16 \cos 10^\circ$	$8 \sin 10^\circ$	$\approx 22.1$
8	$8 + 16 \cos 20^\circ$	$8 \sin 20^\circ$	$\approx 42.5$
8	$8 + 16 \cos 30^\circ$	$8 \sin 30^\circ$	$\approx 59.7$
8	$8 + 16 \cos 40^\circ$	$8 \sin 40^\circ$	$\approx 72.7$
8	$8 + 16 \cos 50^\circ$	$8 \sin 50^\circ$	$\approx 80.5$
8	$8 + 16 \cos 60^\circ$	$8 \sin 60^\circ$	$\approx 83.1$
8	$8 + 16 \cos 70^\circ$	$8 \sin 70^\circ$	$\approx 80.7$
8	$8 + 16 \cos 80^\circ$	$8 \sin 80^\circ$	$\approx 74.0$
8	$8 + 16 \cos 90^\circ$	$8 \sin 90^\circ$	$\approx 64.0$

The maximum cross-sectional area is approximately 83.1 square feet.

(c)  $A = (a + b)\frac{h}{2}$   
 $= [8 + (8 + 16 \cos \theta)]\frac{8 \sin \theta}{2}$   
 $= 64(1 + \cos \theta)\sin \theta, 0^\circ < \theta < 90^\circ$



(d)  $\frac{dA}{d\theta} = 64(1 + \cos \theta)\cos \theta + (-64 \sin \theta)\sin \theta$   
 $= 64(\cos \theta + \cos^2 \theta - \sin^2 \theta)$   
 $= 64(2 \cos^2 \theta + \cos \theta - 1)$   
 $= 64(2 \cos \theta - 1)(\cos \theta + 1)$   
 $= 0$  when  $\theta = 60^\circ, 180^\circ, 300^\circ$ .

The maximum occurs when  $\theta = 60^\circ$ .

59.  $C = 100\left(\frac{200}{x^2} + \frac{x}{x + 30}\right), 1 \leq x$

$$C' = 100\left(-\frac{400}{x^3} + \frac{30}{(x + 30)^2}\right)$$

Approximation:  $x \approx 40.45$  units, or 4045 units

61.  $S_1 = (4m - 1)^2 + (5m - 6)^2 + (10m - 3)^2$

$$\frac{dS_1}{dm} = 2(4m - 1)(4) + 2(5m - 6)(5) + 2(10m - 3)(10) = 282m - 128 = 0 \text{ when } m = \frac{64}{141}$$

Line:  $y = \frac{64}{141}x$

$$S = \left|4\left(\frac{64}{141}\right) - 1\right| + \left|5\left(\frac{64}{141}\right) - 6\right| + \left|10\left(\frac{64}{141}\right) - 3\right|$$

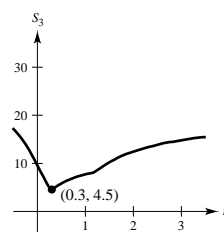
$$= \left|\frac{256}{141} - 1\right| + \left|\frac{320}{141} - 6\right| + \left|\frac{640}{141} - 3\right| = \frac{858}{141} \approx 6.1 \text{ mi}$$

63.  $S_3 = \frac{|4m - 1|}{\sqrt{m^2 + 1}} + \frac{|5m - 6|}{\sqrt{m^2 + 1}} + \frac{|10m - 3|}{\sqrt{m^2 + 1}}$

Using a graphing utility, you can see that the minimum occurs when  $x \approx 0.3$ .

Line:  $y \approx 0.3x$

$$S_3 = \frac{|4(0.3) - 1| + |5(0.3) - 6| + |10(0.3) - 3|}{\sqrt{(0.3)^2 + 1}} \approx 4.5 \text{ mi.}$$



## Section 3.8 Newton's Method

1.  $f(x) = x^2 - 3$

$f'(x) = 2x$

$x_1 = 1.7$

$n$	$x_n$	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.7000	-0.1100	3.4000	-0.0324	1.7324
2	1.7324	0.0012	3.4648	0.0003	1.7321

3.  $f(x) = \sin x$

$f'(x) = \cos x$

$x_1 = 3$

$n$	$x_n$	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	3.0000	0.1411	-0.9900	-0.1425	3.1425
2	3.1425	-0.0009	-1.0000	0.0009	3.1416

5.  $f(x) = x^3 + x - 1$

$f'(x) = 3x^2 + 1$

Approximation of the zero of  $f$  is 0.682.

$n$	$x_n$	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	0.5000	-0.3750	1.7500	-0.2143	0.7143
2	0.7143	0.0788	2.5307	0.0311	0.6832
3	0.6832	0.0021	2.4003	0.0009	0.6823

7.  $f(x) = 3\sqrt{x-1} - x$

$f'(x) = \frac{3}{2\sqrt{x-1}} - 1$

Approximation of the zero of  $f$  is 1.146.

Similarly, the other zero is approximately 7.854.

$n$	$x_n$	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.2000	0.1416	2.3541	0.0602	1.1398
2	1.1398	-0.0181	3.0118	-0.0060	1.1458
3	1.1458	-0.0003	2.9284	-0.0001	1.1459

9.  $f(x) = x^3 + 3$

$f'(x) = 3x^2$

Approximation of the zero of  $f$  is -1.442.

$n$	$x_n$	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	-1.5000	-0.3750	6.7500	-0.0556	-1.4444
2	-1.4444	-0.0134	6.2589	-0.0021	-1.4423
3	-1.4423	-0.0003	6.2407	-0.0001	-1.4422

11.  $f(x) = x^3 - 3.9x^2 + 4.79x - 1.881$

$f'(x) = 3x^2 - 7.8x + 4.79$

$n$	$x_n$	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	0.5000	-0.3360	1.6400	-0.2049	0.7049
2	0.7049	-0.0921	0.7824	-0.1177	0.8226
3	0.8226	-0.0231	0.4037	-0.0573	0.8799
4	0.8799	-0.0045	0.2495	-0.0181	0.8980
5	0.8980	-0.0004	0.2048	-0.0020	0.9000
6	0.9000	0.0000	0.2000	0.0000	0.9000

Approximation of the zero of  $f$  is 0.900.

—CONTINUED—

## 11. —CONTINUED—

$n$	$x_n$	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.1	0.0000	-0.1600	-0.0000	1.1000

Approximation of the zero of  $f$  is 1.100.

$n$	$x_n$	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.9	0.0000	0.8000	0.0000	1.9000

Approximation of the zero of  $f$  is 1.900.

13.  $f(x) = x + \sin(x + 1)$

$f'(x) = 1 + \cos(x + 1)$

Approximation of the zero of  $f$  is -0.489.

$n$	$x_n$	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	-0.5000	-0.0206	1.8776	-0.0110	-0.4890
2	-0.4890	0.0000	1.8723	0.0000	-0.4890

15.  $h(x) = f(x) - g(x) = 2x + 1 - \sqrt{x + 4}$

$h'(x) = 2 - \frac{1}{2\sqrt{x + 4}}$

Point of intersection of the graphs of  $f$  and  $g$  occurs when  $x \approx 0.569$ .

$n$	$x_n$	$h(x_n)$	$h'(x_n)$	$\frac{h(x_n)}{h'(x_n)}$	$x_n - \frac{h(x_n)}{h'(x_n)}$
1	0.6000	0.0552	1.7669	0.0313	0.5687
2	0.5687	-0.0001	1.7661	0.0000	0.5687

17.  $h(x) = f(x) - g(x) = x - \tan x$

$h'(x) = 1 - \sec^2 x$

Point of intersection of the graphs of  $f$  and  $g$  occurs when  $x \approx 4.493$ .

$n$	$x_n$	$h(x_n)$	$h'(x_n)$	$\frac{h(x_n)}{h'(x_n)}$	$x_n - \frac{h(x_n)}{h'(x_n)}$
1	4.5000	-0.1373	-21.5048	0.0064	4.4936
2	4.4936	-0.0039	-20.2271	0.0002	4.4934

19.  $f(x) = x^2 - a = 0$

$f'(x) = 2x$

$x_{i+1} = x_i - \frac{x_i^2 - a}{2x_i}$

$= \frac{2x_i^2 - x_i^2 + a}{2x_i} = \frac{x_i^2 + a}{2x_i} = \frac{x_i}{2} + \frac{a}{2x_i}$

21.  $x_{i+1} = \frac{x_i^2 + 7}{2x_i}$

$i$	1	2	3	4	5
$x_i$	2.0000	2.7500	2.6477	2.6458	2.6458

$\sqrt{7} \approx 2.646$

23.  $x_{i+1} = \frac{3x_i^4 + 6}{4x_i^3}$

$i$	1	2	3	4
$x_i$	1.5000	1.5694	1.5651	1.5651

$\sqrt[4]{6} \approx 1.565$

25.  $f(x) = 1 + \cos x$

$f'(x) = -\sin x$

Approximation of the zero: 3.141

$n$	$x_n$	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	3.0000	0.0100	-0.1411	-0.0709	3.0709
2	3.0709	0.0025	-0.0706	-0.0354	3.1063
3	3.1063	0.0006	-0.0353	-0.0176	3.1239
4	3.1239	0.0002	-0.0177	-0.0088	3.1327
5	3.1327	0.0000	-0.0089	-0.0044	3.1371
6	3.1371	0.0000	-0.0045	-0.0022	3.1393
7	3.1393	0.0000	-0.0023	-0.0011	3.1404
8	3.1404	0.0000	-0.0012	-0.0006	3.1410

27.  $y = 2x^3 - 6x^2 + 6x - 1 = f(x)$

$y' = 6x^2 - 12x + 6 = f'(x)$

$x_1 = 1$

$f'(x) = 0$ ; therefore, the method fails.

$n$	$x_n$	$f(x_n)$	$f'(x_n)$
1	1	1	0

29.  $y = -x^3 + 6x^2 - 10x + 6 = f(x)$

$y' = -3x^2 + 12x - 10 = f'(x)$

$x_1 = 2$

$x_2 = 1$

$x_3 = 2$

$x_4 = 1$  and so on.

Fails to converge

31. Answers will vary. See page 222.

Newton's Method uses tangent lines to approximate  $c$  such that  $f(c) = 0$ .

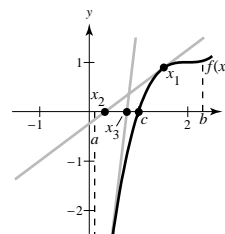
First, estimate an initial  $x_1$  close to  $c$  (see graph).

Then determine  $x_2$  by  $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$ .

Calculate a third estimate by  $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$ .

Continue this process until  $|x_n - x_{n+1}|$  is within the desired accuracy.

Let  $x_{n+1}$  be the final approximation of  $c$ .



33. Let  $g(x) = f(x) - x = \cos x - x$

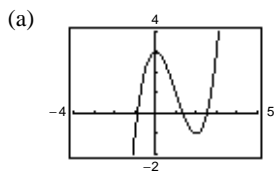
$g'(x) = -\sin x - 1$ .

The fixed point is approximately 0.74.

$n$	$x_n$	$g(x_n)$	$g'(x_n)$	$\frac{g(x_n)}{g'(x_n)}$	$x_n - \frac{g(x_n)}{g'(x_n)}$
1	1.0000	-0.4597	-1.8415	0.2496	0.7504
2	0.7504	-0.0190	-1.6819	0.0113	0.7391
3	0.7391	0.0000	-1.6736	0.0000	0.7391



35.  $f(x) = x^3 - 3x^2 + 3$ ,  $f'(x) = 3x^2 - 6x$



(c)  $x_1 = \frac{1}{4}$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 2.405$$

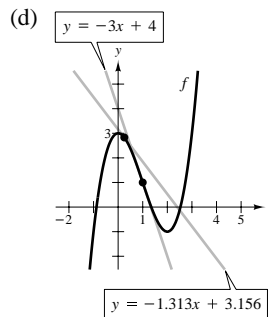
Continuing, the zero is 2.532.

- (e) If the initial guess  $x_1$  is not "close to" the desired zero of the function, the x-intercept of the tangent line may approximate another zero of the function.

(b)  $x_1 = 1$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 1.333$$

Continuing, the zero is 1.347.



The x-intercepts correspond to the values resulting from the first iteration of Newton's Method.

37.  $f(x) = \frac{1}{x} - a = 0$

$$f'(x) = -\frac{1}{x^2}$$

$$x_{n+1} = x_n - \frac{(1/x_n) - a}{-1/x_n^2} = x_n + x_n^2 \left( \frac{1}{x_n} - a \right) = x_n + x_n - x_n^2 a = 2x_n - x_n^2 a = x_n(2 - ax_n)$$

39.  $f(x) = x \cos x$ ,  $(0, \pi)$

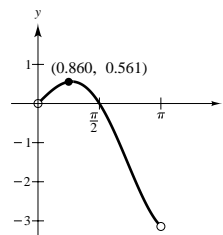
$$f'(x) = -x \sin x + \cos x = 0$$

Letting  $F(x) = f'(x)$ , we can use Newton's Method as follows.

$$[F'(x) = -2 \sin x + x \cos x]$$

$n$	$x_n$	$F(x_n)$	$F'(x_n)$	$\frac{F(x_n)}{F'(x_n)}$	$x_n - \frac{F(x_n)}{F'(x_n)}$
1	0.9000	-0.0834	-2.1261	0.0392	0.8608
2	0.8608	-0.0010	-2.0778	0.0005	0.8603

Approximation to the critical number: 0.860



41.  $y = f(x) = 4 - x^2, (1, 0)$

$$d = \sqrt{(x-1)^2 + (y-0)^2} = \sqrt{(x-1)^2 + (4-x^2)^2} = \sqrt{x^4 - 7x^2 - 2x + 17}$$

$d$  is minimized when  $D = x^4 - 7x^2 - 2x + 17$  is a minimum.

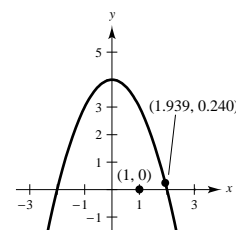
$$g(x) = D' = 4x^3 - 14x - 2$$

$$g'(x) = 12x^2 - 14$$

$n$	$x_n$	$g(x_n)$	$g'(x_n)$	$\frac{g(x_n)}{g'(x_n)}$	$x_n - \frac{g(x_n)}{g'(x_n)}$
1	2.0000	2.0000	34.0000	0.0588	1.9412
2	1.9412	0.0830	31.2191	0.0027	1.9385
3	1.9385	-0.0012	31.0934	0.0000	1.9385

$$x \approx 1.939$$

Point closest to  $(1, 0)$  is  $\approx (1.939, 0.240)$ .



43. Minimize:  $T = \frac{\text{Distance rowed}}{\text{Rate rowed}} + \frac{\text{Distance walked}}{\text{Rate walked}}$

$$T = \frac{\sqrt{x^2 + 4}}{3} + \frac{\sqrt{x^2 - 6x + 10}}{4}$$

$$T' = \frac{x}{3\sqrt{x^2 + 4}} + \frac{x - 3}{4\sqrt{x^2 - 6x + 10}} = 0$$

$$4x\sqrt{x^2 - 6x + 10} = -3(x - 3)\sqrt{x^2 + 4}$$

$$16x^2(x^2 - 6x + 10) = 9(x - 3)^2(x^2 + 4)$$

$$7x^4 - 42x^3 + 43x^2 + 216x - 324 = 0$$

Let  $f(x) = 7x^4 - 42x^3 + 43x^2 + 216x - 324$  and  $f'(x) = 28x^3 - 126x^2 + 86x + 216$ . Since  $f(1) = -100$  and  $f(2) = 56$ , the solution is in the interval  $(1, 2)$ .

$n$	$x_n$	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.7000	19.5887	135.6240	0.1444	1.5556
2	1.5556	-1.0480	150.2780	-0.0070	1.5626
3	1.5626	0.0014	49.5591	0.0000	1.5626

Approximation:  $x \approx 1.563$  miles

45.  $2,500,000 = -76x^3 + 4830x^2 - 320,000$

$$76x^3 - 4830x^2 + 2,820,000 = 0$$

Let  $f(x) = 76x^3 - 4830x^2 + 2,820,000$

$$f'(x) = 228x^2 - 9660x.$$

From the graph, choose  $x_1 = 40$ .

$n$	$x_n$	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	40.0000	-44000.0000	-21600.0000	2.0370	37.9630
2	37.9630	17157.6209	-38131.4039	-0.4500	38.4130
3	38.4130	780.0914	-34642.2263	-0.0225	38.4355
4	38.4355	2.6308	-34465.3435	-0.0001	38.4356

The zero occurs when  $x \approx 38.4356$  which corresponds to \$384,356.

47. False. Let  $f(x) = (x^2 - 1)/(x - 1)$ .  $x = 1$  is a discontinuity. It is not a zero of  $f(x)$ . This statement would be true if  $f(x) = p(x)/q(x)$  is given in **reduced** form.

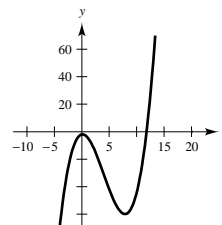
49. True

51.  $f(x) = \frac{1}{4}x^3 - 3x^2 + \frac{3}{4}x - 2$

$$f'(x) = \frac{3}{4}x^2 - 6x + \frac{3}{4}$$

Let  $x_1 = 12$ .

$n$	$x_n$	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	12.0000	7.0000	36.7500	0.1905	11.8095
2	11.8095	0.2151	34.4912	0.0062	11.8033
3	11.8033	0.0015	34.4186	0.0000	11.8033



Approximation:  $x \approx 11.803$

## Section 3.9 Differentials

1.  $f(x) = x^2$

$$f'(x) = 2x$$

Tangent line at  $(2, 4)$ :  $y - f(2) = f'(2)(x - 2)$

$$y - 4 = 4(x - 2)$$

$$y = 4x - 4$$

$x$	1.9	1.99	2	2.01	2.1
$f(x) = x^2$	3.6100	3.9601	4	4.0401	4.4100
$T(x) = 4x - 4$	3.6000	3.9600	4	4.0400	4.4000

3.  $f(x) = x^5$

$$f'(x) = 5x^4$$

Tangent line at  $(2, 32)$ :  $y - f(2) = f'(2)(x - 2)$

$$y - 32 = 80(x - 2)$$

$$y = 80x - 128$$

$x$	1.9	1.99	2	2.01	2.1
$f(x) = x^5$	24.7610	31.2080	32	32.8080	40.8410
$T(x) = 80x - 128$	24.0000	31.2000	32	32.8000	40.0000

5.  $f(x) = \sin x$

$$f'(x) = \cos x$$

Tangent line at  $(2, \sin 2)$ :

$$y - f(2) = f'(2)(x - 2)$$

$$y - \sin 2 = (\cos 2)(x - 2)$$

$$y = (\cos 2)(x - 2) + \sin 2$$

$x$	1.9	1.99	2	2.01	2.1
$f(x) = \sin x$	0.9463	0.9134	0.9093	0.9051	0.8632
$T(x) = (\cos 2)(x - 2) + \sin 2$	0.9509	0.9135	0.9093	0.9051	0.8677

7.  $y = f(x) = \frac{1}{2}x^3, f'(x) = \frac{3}{2}x^2, x = 2, \Delta x = dx = 0.1$

$$\Delta y = f(x + \Delta x) - f(x)$$

$$= f(2.1) - f(2)$$

$$= 0.6305$$

$$dy = f'(x)dx$$

$$= f'(2)(0.1)$$

$$= 6(0.1) = 0.6$$

$$9. y = f(x) = x^4 + 1, f'(x) = 4x^3, x = -1, \Delta x = dx = 0.01$$

$$\begin{aligned} \Delta y &= f(x + \Delta x) - f(x) \\ &= f(-0.99) - f(-1) \\ &= [(-0.99)^4 + 1] - [(-1)^4 + 1] \approx -0.0394 \end{aligned}$$

$$\begin{aligned} dy &= f'(x) dx \\ &= f'(-1)(0.01) \\ &= (-4)(0.01) = -0.04 \end{aligned}$$

$$11. y = 3x^2 - 4$$

$$dy = 6x dx$$

$$13. y = \frac{x + 1}{2x - 1}$$

$$dy = \frac{-3}{(2x - 1)^2} dx$$

$$15. y = x\sqrt{1 - x^2}$$

$$dy = \left( x \frac{-x}{\sqrt{1 - x^2}} + \sqrt{1 - x^2} \right) dx = \frac{1 - 2x^2}{\sqrt{1 - x^2}} dx$$

$$17. y = 2x - \cot^2 x$$

$$\begin{aligned} dy &= (2 + 2 \cot x \csc^2 x) dx \\ &= (2 + 2 \cot x + 2 \cot^3 x) dx \end{aligned}$$

$$19. y = \frac{1}{3} \cos\left(\frac{6\pi x - 1}{2}\right)$$

$$dy = -\pi \sin\left(\frac{6\pi x - 1}{2}\right) dx$$

$$\begin{aligned} 21. (a) f(1.9) &= f(2 - 0.1) \approx f(2) + f'(2)(-0.1) \\ &\approx 1 + (1)(-0.1) = 0.9 \end{aligned}$$

$$\begin{aligned} (b) f(2.04) &= f(2 + 0.04) \approx f(2) + f'(2)(0.04) \\ &\approx 1 + (1)(0.04) = 1.04 \end{aligned}$$

$$\begin{aligned} 23. (a) f(1.9) &= f(2 - 0.1) \approx f(2) + f'(2)(-0.1) \\ &\approx 1 + \left(-\frac{1}{2}\right)(-0.1) = 1.05 \end{aligned}$$

$$\begin{aligned} (b) f(2.04) &= f(2 + 0.04) \approx f(2) + f'(2)(0.04) \\ &\approx 1 + \left(-\frac{1}{2}\right)(0.04) = 0.98 \end{aligned}$$

$$\begin{aligned} 25. (a) g(2.93) &= g(3 - 0.07) \approx g(3) + g'(3)(-0.07) \\ &\approx 8 + \left(-\frac{1}{2}\right)(-0.07) = 8.035 \end{aligned}$$

$$\begin{aligned} (b) g(3.1) &= g(3 + 0.1) \approx g(3) + g'(3)(0.1) \\ &\approx 8 + \left(-\frac{1}{2}\right)(0.1) = 7.95 \end{aligned}$$

$$\begin{aligned} 27. (a) g(2.93) &= g(3 - 0.07) \approx g(3) + g'(3)(-0.07) \\ &\approx 8 + 0(-0.07) = 8 \end{aligned}$$

$$\begin{aligned} (b) g(3.1) &= g(3 + 0.1) \approx g(3) + g'(3)(0.1) \\ &\approx 8 + 0(0.1) = 8 \end{aligned}$$

$$29. A = x^2$$

$$x = 12$$

$$\Delta x = dx = \pm \frac{1}{64}$$

$$dA = 2x dx$$

$$\begin{aligned} \Delta A &\approx dA = 2(12)\left(\pm \frac{1}{64}\right) \\ &= \pm \frac{3}{8} \text{ square inches} \end{aligned}$$

$$31. A = \pi r^2$$

$$r = 14$$

$$\Delta r = dr = \pm \frac{1}{4}$$

$$\begin{aligned} \Delta A &\approx dA = 2\pi r dr = \pi(28)\left(\pm \frac{1}{4}\right) \\ &= \pm 7\pi \text{ square inches} \end{aligned}$$

33. (a)
- $x = 15$
- centimeter

$$\Delta x = dx = \pm 0.05 \text{ centimeters}$$

$$A = x^2$$

$$dA = 2x dx = 2(15)(\pm 0.05)$$

$$= \pm 1.5 \text{ square centimeters}$$

Percentage error:

$$\frac{dA}{A} = \frac{\pm 1.5}{(15)^2} = 0.006666 \dots = \frac{2}{3}\%$$

$$(b) \frac{dA}{A} = \frac{2x dx}{x^2} = \frac{2 dx}{x} \leq 0.025$$

$$\frac{dx}{x} \leq \frac{0.025}{2} = 0.0125 = 1.25\%$$

- 37.
- $V = \pi r^2 h = 40\pi r^2$
- ,
- $r = 5$
- cm,
- $h = 40$
- cm,
- $dr = 0.2$
- cm

$$\Delta V \approx dV = 80\pi r dr = 80\pi(5)(0.2) = 80\pi \text{ cm}^3$$

39. (a)
- $T = 2\pi\sqrt{L/g}$

$$dT = \frac{\pi}{g\sqrt{L/g}} dL$$

Relative error:

$$\frac{dT}{T} = \frac{(\pi dL)/(g\sqrt{L/g})}{2\pi\sqrt{L/g}}$$

$$= \frac{dL}{2L}$$

$$= \frac{1}{2} (\text{relative error in } L)$$

$$= \frac{1}{2}(0.005) = 0.0025$$

$$\text{Percentage error: } \frac{dT}{T}(100) = 0.25\% = \frac{1}{4}\%$$

- 41.
- $\theta = 26^\circ 45' = 26.75^\circ$

$$d\theta = \pm 15' = \pm 0.25^\circ$$

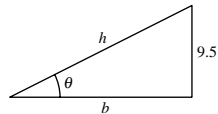
$$(a) h = 9.5 \csc \theta$$

$$dh = -9.5 \csc \theta \cot \theta d\theta$$

$$\frac{dh}{h} = -\cot \theta d\theta$$

$$\left| \frac{dh}{h} \right| = (\cot 26.75^\circ)(0.25^\circ)$$

Converting to radians,  $(\cot 0.4669)(0.0044)$   
 $\approx 0.0087 = 0.87\%$  (in radians).



- 35.
- $r = 6$
- inches

$$\Delta r = dr = \pm 0.02 \text{ inches}$$

$$(a) V = \frac{4}{3}\pi r^3$$

$$dV = 4\pi r^2 dr = 4\pi(6)^2(\pm 0.02) = \pm 2.88\pi \text{ cubic inches}$$

$$(b) S = 4\pi r^2$$

$$dS = 8\pi r dr = 8\pi(6)(\pm 0.02) = \pm 0.96\pi \text{ square inches}$$

$$(c) \text{Relative error: } \frac{dV}{V} = \frac{4\pi r^2 dr}{(4/3)\pi r^3} = \frac{3dr}{r}$$

$$= \frac{3}{6}(0.02) = 0.01 = 1\%$$

$$\text{Relative error: } \frac{dS}{S} = \frac{8\pi r dr}{4\pi r^2} = \frac{2dr}{r}$$

$$= \frac{2(0.02)}{6} = 0.006666 \dots = \frac{2}{3}\%$$

$$(b) (0.0025)(3600)(24) = 216 \text{ seconds}$$

$$= 3.6 \text{ minutes}$$

$$(b) \left| \frac{dh}{h} \right| = \cot \theta d\theta \leq 0.02$$

$$\frac{d\theta}{\theta} \leq \frac{0.02}{\theta(\cot \theta)} = \frac{0.02 \tan \theta}{\theta}$$

$$\frac{d\theta}{\theta} \leq \frac{0.02 \tan 26.75^\circ}{26.75^\circ} \approx \frac{0.02 \tan 0.4669}{0.4669}$$

$$\approx 0.0216 = 2.16\% \text{ (in radians)}$$

$$43. r = \frac{v_0^2}{32} (\sin 2\theta)$$

$$v_0 = 2200 \text{ ft/sec}$$

$\theta$  changes from  $10^\circ$  to  $11^\circ$

$$dr = \frac{(2200)^2}{16} (\cos 2\theta) d\theta$$

$$\theta = 10 \left( \frac{\pi}{180} \right)$$

$$d\theta = (11 - 10) \frac{\pi}{180}$$

$$\Delta r \approx dr$$

$$= \frac{(2200)^2}{16} \cos \left( \frac{20\pi}{180} \right) \left( \frac{\pi}{180} \right) \approx 4961 \text{ feet}$$

$$\approx 4961 \text{ feet}$$

$$47. \text{ Let } f(x) = \sqrt[4]{x}, x = 625, dx = -1.$$

$$f(x + \Delta x) \approx f(x) + f'(x) dx = \sqrt[4]{x} + \frac{1}{4^4 \sqrt{x^3}} dx$$

$$f(x + \Delta x) = \sqrt[4]{624} \approx \sqrt[4]{625} + \frac{1}{4(\sqrt[4]{625})^3} (-1)$$

$$= 5 - \frac{1}{500} = 4.998$$

Using a calculator,  $\sqrt[4]{624} \approx 4.9980$ .

51. In general, when  $\Delta x \rightarrow 0$ ,  $dy$  approaches  $\Delta y$ .

53. True

$$45. \text{ Let } f(x) = \sqrt{x}, x = 100, dx = -0.6.$$

$$f(x + \Delta x) \approx f(x) + f'(x) dx$$

$$= \sqrt{x} + \frac{1}{2\sqrt{x}} dx$$

$$f(x + \Delta x) = \sqrt{99.4}$$

$$\approx \sqrt{100} + \frac{1}{2\sqrt{100}} (-0.6) = 9.97$$

Using a calculator:  $\sqrt{99.4} \approx 9.96995$

$$49. \text{ Let } f(x) = \sqrt{x}, x = 4, dx = 0.02, f'(x) = 1/(2\sqrt{x}).$$

Then

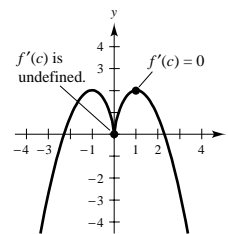
$$f(4.02) \approx f(4) + f'(4) dx$$

$$\sqrt{4.02} \approx \sqrt{4} + \frac{1}{2\sqrt{4}} (0.02) = 2 + \frac{1}{4} (0.02).$$

55. True

## Review Exercises for Chapter 3

1. A number  $c$  in the domain of  $f$  is a critical number if  $f'(c) = 0$  or  $f'$  is undefined at  $c$ .



$$3. g(x) = 2x + 5 \cos x, [0, 2\pi]$$

$$g'(x) = 2 - 5 \sin x$$

$$= 0 \text{ when } \sin x = \frac{2}{5}.$$

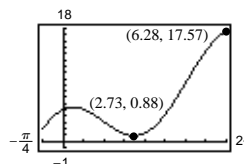
Critical numbers:  $x \approx 0.41, x \approx 2.73$

Left endpoint:  $(0, 5)$

Critical number:  $(0.41, 5.41)$

Critical number:  $(2.73, 0.88)$  Minimum

Right endpoint:  $(2\pi, 17.57)$  Maximum



5. Yes.  $f(-3) = f(2) = 0$ .  $f$  is continuous on  $[-3, 2]$ , differentiable on  $(-3, 2)$ .

$$f'(x) = (x + 3)(3x - 1) = 0 \text{ for } x = \frac{1}{3}.$$

$$c = \frac{1}{3} \text{ satisfies } f'(c) = 0.$$

9.  $f(x) = x^{2/3}, 1 \leq x \leq 8$

$$f'(x) = \frac{2}{3}x^{-1/3}$$

$$\frac{f(b) - f(a)}{b - a} = \frac{4 - 1}{8 - 1} = \frac{3}{7}$$

$$f'(c) = \frac{2}{3}c^{-1/3} = \frac{3}{7}$$

$$c = \left(\frac{14}{9}\right)^3 = \frac{2744}{729} \approx 3.764$$

13.  $f(x) = Ax^2 + Bx + C$

$$f'(x) = 2Ax + B$$

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{A(x_2^2 - x_1^2) + B(x_2 - x_1)}{x_2 - x_1}$$

$$= A(x_1 + x_2) + B$$

$$f'(c) = 2Ac + B = A(x_1 + x_2) + B$$

$$2Ac = A(x_1 + x_2)$$

$$c = \frac{x_1 + x_2}{2} = \text{Midpoint of } [x_1, x_2]$$

15.  $f(x) = (x - 1)^2(x - 3)$

$$f'(x) = (x - 1)^2(1) + (x - 3)(2)(x - 1)$$

$$= (x - 1)(3x - 7)$$

$$\text{Critical numbers: } x = 1 \text{ and } x = \frac{7}{3}$$

17.  $h(x) = \sqrt{x}(x - 3) = x^{3/2} - 3x^{1/2}$

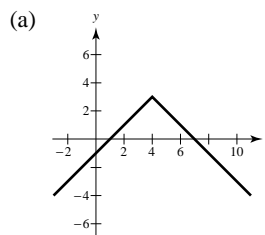
$$\text{Domain: } (0, \infty)$$

$$h'(x) = \frac{3}{2}x^{1/2} - \frac{3}{2}x^{-1/2}$$

$$= \frac{3}{2}x^{-1/2}(x - 1) = \frac{3(x - 1)}{2\sqrt{x}}$$

$$\text{Critical number: } x = 1$$

7.  $f(x) = 3 - |x - 4|$



$$f(1) = f(7) = 0$$

- (b)  $f$  is not differentiable at  $x = 4$ .

11.  $f(x) = x - \cos x, -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

$$f'(x) = 1 + \sin x$$

$$\frac{f(b) - f(a)}{b - a} = \frac{(\pi/2) - (-\pi/2)}{(\pi/2) - (-\pi/2)} = 1$$

$$f'(c) = 1 + \sin c = 1$$

$$c = 0$$

Interval:	$-\infty < x < 1$	$1 < x < \frac{7}{3}$	$\frac{7}{3} < x < \infty$
Sign of $f'(x)$ :	$f'(x) > 0$	$f'(x) < 0$	$f'(x) > 0$
Conclusion:	Increasing	Decreasing	Increasing

Interval:	$0 < x < 1$	$1 < x < \infty$
Sign of $h'(x)$ :	$h'(x) < 0$	$h'(x) > 0$
Conclusion:	Decreasing	Increasing

19.  $h(t) = \frac{1}{4}t^4 - 8t$

$h'(t) = t^3 - 8 = 0$  when  $t = 2$ .

 Relative minimum:  $(2, -12)$ 

Test Interval:	$-\infty < t < 2$	$2 < t < \infty$
Sign of $h'(t)$ :	$h'(t) < 0$	$h'(t) > 0$
Conclusion:	Decreasing	Increasing

21.  $y = \frac{1}{3} \cos(12t) - \frac{1}{4} \sin(12t)$

$v = y' = -4 \sin(12t) - 3 \cos(12t)$

 (a) When  $t = \frac{\pi}{8}$ ,  $y = \frac{1}{4}$  inch and  $v = y' = 4$  inches/second.

 (b)  $y' = -4 \sin(12t) - 3 \cos(12t) = 0$  when  $\frac{\sin(12t)}{\cos(12t)} = -\frac{3}{4} \Rightarrow \tan(12t) = -\frac{3}{4}$ .

 Therefore,  $\sin(12t) = -\frac{3}{5}$  and  $\cos(12t) = \frac{4}{5}$ . The maximum displacement is

$$y = \left(\frac{1}{3}\right)\left(\frac{4}{5}\right) - \frac{1}{4}\left(-\frac{3}{5}\right) = \frac{5}{12} \text{ inch.}$$

 (c) Period:  $\frac{2\pi}{12} = \frac{\pi}{6}$ 

Frequency:  $\frac{1}{\pi/6} = \frac{6}{\pi}$

23.  $f(x) = x + \cos x$ ,  $0 \leq x \leq 2\pi$

$f'(x) = 1 - \sin x$

$f''(x) = -\cos x = 0$  when  $x = \frac{\pi}{2}, \frac{3\pi}{2}$ .

 Points of inflection:  $\left(\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{3\pi}{2}, \frac{3\pi}{2}\right)$ 

Test Interval:	$0 < x < \frac{\pi}{2}$	$\frac{\pi}{2} < x < \frac{3\pi}{2}$	$\frac{3\pi}{2} < x < 2\pi$
Sign of $f''(x)$ :	$f''(x) < 0$	$f''(x) > 0$	$f''(x) < 0$
Conclusion:	Concave downward	Concave upward	Concave downward

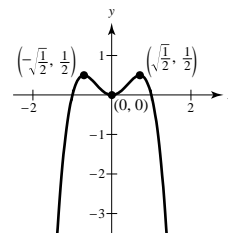
25.  $g(x) = 2x^2(1 - x^2)$

$g'(x) = -4x(2x^2 - 1)$  Critical numbers:  $x = 0, \pm \frac{1}{\sqrt{2}}$

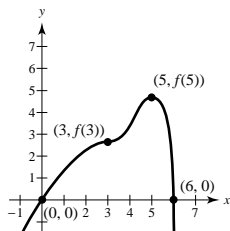
$g''(x) = 4 - 24x^2$

 $g''(0) = 4 > 0$  Relative minimum at  $(0, 0)$ 

$g''\left(\pm \frac{1}{\sqrt{2}}\right) = -8 < 0$  Relative maximums at  $\left(\pm \frac{1}{\sqrt{2}}, \frac{1}{2}\right)$



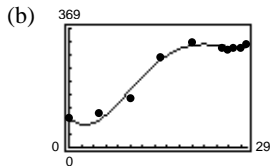
27.



29. The first derivative is positive and the second derivative is negative. The graph is increasing and is concave down.



31. (a)  $D = 0.0034t^4 - 0.2352t^3 + 4.9423t^2 - 20.8641t + 94.4025$



(c) Maximum at (21.9, 319.5) ( $\approx 1992$ )

Minimum at (2.6, 69.6) ( $\approx 1972$ )

(d) Outlays increasing at greatest rate at the point of inflection (9.8, 173.7) ( $\approx 1979$ )

33.  $\lim_{x \rightarrow \infty} \frac{2x^2}{3x^2 + 5} = \lim_{x \rightarrow \infty} \frac{2}{3 + 5/x^2} = \frac{2}{3}$

35.  $\lim_{x \rightarrow \infty} \frac{5 \cos x}{x} = 0$ , since  $|5 \cos x| \leq 5$ .

37.  $h(x) = \frac{2x + 3}{x - 4}$

Discontinuity:  $x = 4$

$$\lim_{x \rightarrow \infty} \frac{2x + 3}{x - 4} = \lim_{x \rightarrow \infty} \frac{2 + (3/x)}{1 - (4/x)} = 2$$

Vertical asymptote:  $x = 4$

Horizontal asymptote:  $y = 2$

39.  $f(x) = \frac{3}{x} - 2$

Discontinuity:  $x = 0$

$$\lim_{x \rightarrow \infty} \left( \frac{3}{x} - 2 \right) = -2$$

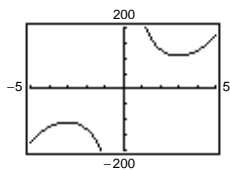
Vertical asymptote:  $x = 0$

Horizontal asymptote:  $y = -2$

41.  $f(x) = x^3 + \frac{243}{x}$

Relative minimum: (3, 108)

Relative maximum: (-3, -108)

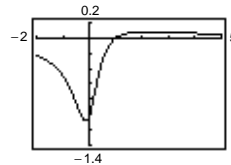


Vertical asymptote:  $x = 0$

43.  $f(x) = \frac{x - 1}{1 + 3x^2}$

Relative minimum: (-0.155, -1.077)

Relative maximum: (2.155, 0.077)



Horizontal asymptote:  $y = 0$

45.  $f(x) = 4x - x^2 = x(4 - x)$

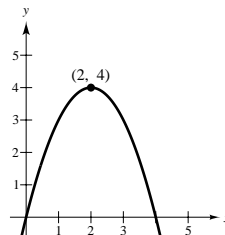
Domain:  $(-\infty, \infty)$ ; Range:  $(-\infty, 4)$

$$f'(x) = 4 - 2x = 0 \text{ when } x = 2.$$

$$f''(x) = -2$$

Therefore, (2, 4) is a relative maximum.

Intercepts: (0, 0), (4, 0)



47.  $f(x) = x\sqrt{16 - x^2}$ , Domain:  $[-4, 4]$ , Range:  $[-8, 8]$

Domain:  $[-4, 4]$ ; Range:  $[-8, 8]$

$$f'(x) = \frac{16 - 2x^2}{\sqrt{16 - x^2}} = 0 \text{ when } x = \pm 2\sqrt{2} \text{ and undefined when } x = \pm 4.$$

$$f''(x) = \frac{2x(x^2 - 24)}{(16 - x^2)^{3/2}}$$

$$f'(-2\sqrt{2}) > 0$$

Therefore,  $(-2\sqrt{2}, -8)$  is a relative minimum.

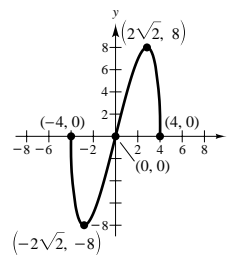
$$f'(2\sqrt{2}) < 0$$

Therefore,  $(2\sqrt{2}, 8)$  is a relative maximum.

Point of inflection:  $(0, 0)$

Intercepts:  $(-4, 0)$ ,  $(0, 0)$ ,  $(4, 0)$

Symmetry with respect to origin



49.  $f(x) = (x - 1)^3(x - 3)^2$

Domain:  $(-\infty, \infty)$ ; Range:  $(-\infty, \infty)$

$$f'(x) = (x - 1)^2(x - 3)(5x - 11) = 0 \text{ when } x = 1, \frac{11}{5}, 3.$$

$$f''(x) = 4(x - 1)(5x^2 - 22x + 23) = 0 \text{ when } x = 1, \frac{11 \pm \sqrt{6}}{5}.$$

$$f''(3) > 0$$

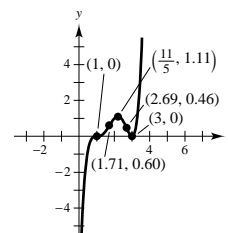
Therefore,  $(3, 0)$  is a relative minimum.

$$f''\left(\frac{11}{5}\right) < 0$$

Therefore,  $\left(\frac{11}{5}, \frac{3456}{3125}\right)$  is a relative maximum.

Points of inflection:  $(1, 0)$ ,  $\left(\frac{11 - \sqrt{6}}{5}, 0.60\right)$ ,  $\left(\frac{11 + \sqrt{6}}{5}, 0.46\right)$

Intercepts:  $(0, -9)$ ,  $(1, 0)$ ,  $(3, 0)$



51.  $f(x) = x^{1/3}(x + 3)^{2/3}$

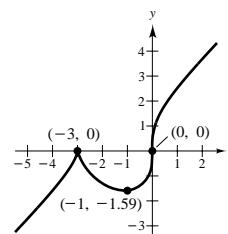
Domain:  $(-\infty, \infty)$ ; Range:  $(-\infty, \infty)$

$$f'(x) = \frac{x + 1}{(x + 3)^{1/3}x^{2/3}} = 0 \text{ when } x = -1 \text{ and undefined when } x = -3, 0.$$

$$f''(x) = \frac{-2}{x^{5/3}(x + 3)^{4/3}} \text{ is undefined when } x = 0, -3.$$

By the First Derivative Test  $(-3, 0)$  is a relative maximum and  $(-1, -\sqrt[3]{4})$  is a relative minimum.  $(0, 0)$  is a point of inflection.

Intercepts:  $(-3, 0)$ ,  $(0, 0)$

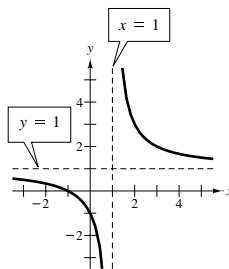


53.  $f(x) = \frac{x+1}{x-1}$

Domain:  $(-\infty, 1), (1, \infty)$ ; Range:  $(-\infty, 1), (1, \infty)$ 

$$f'(x) = \frac{-2}{(x-1)^2} < 0 \text{ if } x \neq 1.$$

$$f''(x) = \frac{4}{(x-1)^3}$$

Horizontal asymptote:  $y = 1$ Vertical asymptote:  $x = 1$ Intercepts:  $(-1, 0), (0, -1)$ 

55.  $f(x) = \frac{4}{1+x^2}$

Domain:  $(-\infty, \infty)$ ; Range:  $(0, 4]$ 

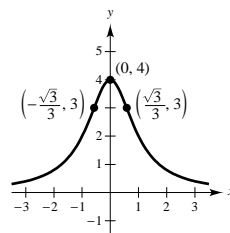
$$f'(x) = \frac{-8x}{(1+x^2)^2} = 0 \text{ when } x = 0.$$

$$f''(x) = \frac{-8(1-3x^2)}{(1+x^2)^3} = 0 \text{ when } x = \pm \frac{\sqrt{3}}{3}.$$

$$f''(0) < 0$$

Therefore,  $(0, 4)$  is a relative maximum.Points of inflection:  $(\pm\sqrt{3}/3, 3)$ Intercept:  $(0, 4)$ 

Symmetric to the y-axis

Horizontal asymptote:  $y = 0$ 

57.  $f(x) = x^3 + x + \frac{4}{x}$

Domain:  $(-\infty, 0), (0, \infty)$ ; Range:  $(-\infty, -6], [6, \infty)$ 

$$f'(x) = 3x^2 + 1 - \frac{4}{x^2} = \frac{3x^4 + x^2 - 4}{x^2} = 0 \text{ when } x = \pm 1.$$

$$f''(x) = 6x + \frac{8}{x^3} = \frac{6x^4 + 8}{x^3} \neq 0$$

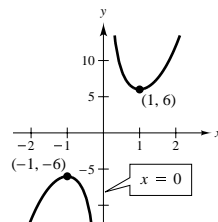
$$f''(-1) < 0$$

Therefore,  $(-1, -6)$  is a relative maximum.

$$f''(1) > 0$$

Therefore,  $(1, 6)$  is a relative minimum.Vertical asymptote:  $x = 0$ 

Symmetric with respect to origin



59.  $f(x) = |x^2 - 9|$

Domain:  $(-\infty, \infty)$ ; Range:  $[0, \infty)$ 

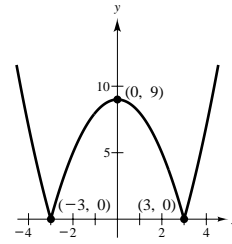
$$f'(x) = \frac{2x(x^2 - 9)}{|x^2 - 9|} = 0 \text{ when } x = 0 \text{ and is undefined when } x = \pm 3.$$

$$f''(x) = \frac{2(x^2 - 9)}{|x^2 - 9|} \text{ is undefined at } x = \pm 3.$$

$$f''(0) < 0$$

Therefore,  $(0, 9)$  is a relative maximum.Relative minima:  $(\pm 3, 0)$ Points of inflection:  $(\pm 3, 0)$ Intercepts:  $(\pm 3, 0), (0, 9)$ 

Symmetric to the y-axis



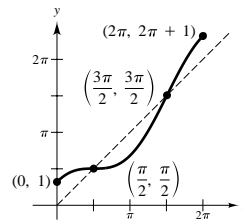
61.  $f(x) = x + \cos x$

Domain:  $[0, 2\pi]$ ; Range:  $[1, 1 + 2\pi]$ 

$$f'(x) = 1 - \sin x \geq 0, \text{ } f \text{ is increasing.}$$

$$f''(x) = -\cos x = 0 \text{ when } x = \frac{\pi}{2}, \frac{3\pi}{2}.$$

$$\text{Points of inflection: } \left(\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{3\pi}{2}, \frac{3\pi}{2}\right)$$

Intercept:  $(0, 1)$ 

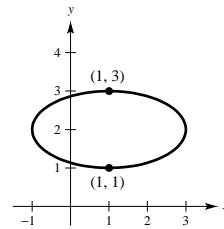
63.  $x^2 + 4y^2 - 2x - 16y + 13 = 0$

(a)  $(x^2 - 2x + 1) + 4(y^2 - 4y + 4) = -13 + 1 + 16$

$$(x - 1)^2 + 4(y - 2)^2 = 4$$

$$\frac{(x - 1)^2}{4} + \frac{(y - 2)^2}{1} = 1$$

The graph is an ellipse:

Maximum:  $(1, 3)$ Minimum:  $(1, 1)$ 

(b)  $x^2 + 4y^2 - 2x - 16y + 13 = 0$

$$2x + 8y \frac{dy}{dx} - 2 - 16 \frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(8y - 16) = 2 - 2x$$

$$\frac{dy}{dx} = \frac{2 - 2x}{8y - 16} = \frac{1 - x}{4y - 8}$$

The critical numbers are  $x = 1$  and  $y = 2$ . These correspond to the points  $(1, 1)$ ,  $(1, 3)$ ,  $(2, -1)$ , and  $(2, 3)$ . Hence, the maximum is  $(1, 3)$  and the minimum is  $(1, 1)$ .

65. Let  $t = 0$  at noon.

$$L = d^2 = (100 - 12t)^2 + (-10t)^2 = 10,000 - 2400t + 244t^2$$

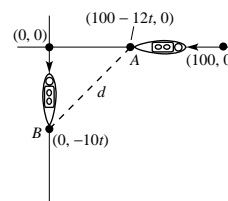
$$\frac{dL}{dt} = -2400 + 488t = 0 \text{ when } t = \frac{300}{61} \approx 4.92 \text{ hr.}$$

Ship A at (40.98, 0); Ship B at (0, -49.18)

$$d^2 = 10,000 - 2400t + 244t^2$$

$$\approx 4098.36 \text{ when } t \approx 4.92 \approx 4:55 \text{ P.M..}$$

$$d \approx 64 \text{ km}$$



67. We have points  $(0, y)$ ,  $(x, 0)$ , and  $(1, 8)$ . Thus,

$$m = \frac{y - 8}{0 - 1} = \frac{0 - 8}{x - 1} \text{ or } y = \frac{8x}{x - 1}.$$

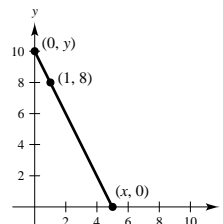
$$\text{Let } f(x) = L^2 = x^2 + \left(\frac{8x}{x - 1}\right)^2.$$

$$f'(x) = 2x + 128\left(\frac{x}{x - 1}\right)\left[\frac{(x - 1) - x}{(x - 1)^2}\right] = 0$$

$$x - \frac{64x}{(x - 1)^3} = 0$$

$$x[(x - 1)^3 - 64] = 0 \text{ when } x = 0, 5 \text{ (minimum).}$$

Vertices of triangle:  $(0, 0)$ ,  $(5, 0)$ ,  $(0, 10)$



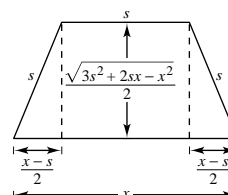
69.  $A = (\text{Average of bases})(\text{Height})$

$$= \left(\frac{x + s}{2}\right) \frac{\sqrt{3s^2 + 2sx - x^2}}{2} \text{ (see figure)}$$

$$\frac{dA}{dx} = \frac{1}{4} \left[ \frac{(s - x)(s + x)}{\sqrt{3s^2 + 2sx - x^2}} + \sqrt{3s^2 + 2sx - x^2} \right]$$

$$= \frac{2(2s - x)(s + x)}{4\sqrt{3s^2 + 2sx - x^2}} = 0 \text{ when } x = 2s.$$

$A$  is a maximum when  $x = 2s$ .



71. You can form a right triangle with vertices  $(0, 0)$ ,  $(x, 0)$  and  $(0, y)$ .

Assume that the hypotenuse of length  $L$  passes through  $(4, 6)$ .

$$m = \frac{y - 6}{0 - 4} = \frac{6 - 0}{4 - x} \text{ or } y = \frac{6x}{x - 4}$$

$$\text{Let } f(x) = L^2 = x^2 + y^2 = x^2 + \left(\frac{6x}{x - 4}\right)^2.$$

$$f'(x) = 2x + 72\left(\frac{x}{x - 4}\right)\left[\frac{-4}{(x - 4)^2}\right] = 0$$

$$x[(x - 4)^3 - 144] = 0 \text{ when } x = 0 \text{ or } x = 4 + \sqrt[3]{144}.$$

$$L \approx 14.05 \text{ feet}$$

73.  $\csc \theta = \frac{L_1}{6}$  or  $L_1 = 6 \csc \theta$  (see figure)

$$\csc\left(\frac{\pi}{2} - \theta\right) = \frac{L_2}{9} \text{ or } L_2 = 9 \csc\left(\frac{\pi}{2} - \theta\right)$$

$$L = L_1 + L_2 = 6 \csc \theta + 9 \csc\left(\frac{\pi}{2} - \theta\right) = 6 \csc \theta + 9 \sec \theta$$

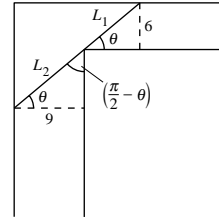
$$\frac{dL}{d\theta} = -6 \csc \theta \cot \theta + 9 \sec \theta \tan \theta = 0$$

$$\tan^3 \theta = \frac{2}{3} \Rightarrow \tan \theta = \frac{\sqrt[3]{2}}{\sqrt[3]{3}}$$

$$\sec \theta = \sqrt{1 + \tan^2 \theta} = \sqrt{1 + \left(\frac{2}{3}\right)^{2/3}} = \frac{\sqrt{3^{2/3} + 2^{2/3}}}{3^{1/3}}$$

$$\csc \theta = \frac{\sec \theta}{\tan \theta} = \frac{\sqrt{3^{2/3} + 2^{2/3}}}{2^{1/3}}$$

$$L = 6 \frac{(3^{2/3} + 2^{2/3})^{1/2}}{2^{1/3}} + 9 \frac{(3^{2/3} + 2^{2/3})^{1/2}}{3^{1/3}} = 3(3^{2/3} + 2^{2/3})^{3/2} \text{ ft} \approx 21.07 \text{ ft (Compare to Exercise 72 using } a = 9 \text{ and } b = 6.)$$



75. Total cost = (Cost per hour)(Number of hours)

$$T = \left(\frac{v^2}{600} + 5\right)\left(\frac{110}{v}\right) = \frac{11v}{60} + \frac{550}{v}$$

$$\frac{dT}{dv} = \frac{11}{60} - \frac{550}{v^2} = \frac{11v^2 - 33,000}{60v^2}$$

$$= 0 \text{ when } v = \sqrt{3000} = 10\sqrt{30} \approx 54.8 \text{ mph.}$$

$$\frac{d^2T}{dv^2} = \frac{1100}{v^3} > 0 \text{ when } v = 10\sqrt{30} \text{ so this value yields a minimum.}$$

77.  $f(x) = x^3 - 3x - 1$

From the graph you can see that  $f(x)$  has three real zeros.

$$f'(x) = 3x^2 - 3$$

$n$	$x_n$	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	-1.5000	0.1250	3.7500	0.0333	-1.5333
2	-1.5333	-0.0049	4.0530	-0.0012	-1.5321

$n$	$x_n$	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	-0.5000	0.3750	-2.2500	-0.1667	-0.3333
2	-0.3333	-0.0371	-2.6667	0.0139	-0.3472
3	-0.3472	-0.0003	-2.6384	0.0001	-0.3473

$n$	$x_n$	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	-1.9000	0.1590	7.8300	0.0203	1.8797
2	1.8797	0.0024	7.5998	0.0003	1.8794

The three real zeros of  $f(x)$  are  $x \approx -1.532$ ,  $x \approx -0.347$ , and  $x \approx 1.879$ .