43. $r = \frac{v_0^2}{32}(\sin 2\theta)$

$v_0 = 2200$ ft/sec

θ changes from 10° to 11°

$dr = \frac{(2200)^2}{16}(\cos 2\theta)\,d\theta$

$\theta = 10\left(\frac{\pi}{180}\right)$

$d\theta = (11 - 10)\frac{\pi}{180}$

$\Delta r \approx dr$

$= \frac{(2200)^2}{16}\cos\left(\frac{20\pi}{180}\right)\left(\frac{\pi}{180}\right) \approx 4961$ feet

≈ 4961 feet

45. Let $f(x) = \sqrt{x}$, $x = 100$, $dx = -0.6$.

$f(x + \Delta x) \approx f(x) + f'(x)\,dx$

$= \sqrt{x} + \frac{1}{2\sqrt{x}}\,dx$

$f(x + \Delta x) = \sqrt{99.4}$

$\approx \sqrt{100} + \frac{1}{2\sqrt{100}}(-0.6) = 9.97$

Using a calculator: $\sqrt{99.4} \approx 9.96995$

47. Let $f(x) = \sqrt[4]{x}$, $x = 625$, $dx = -1$.

$f(x + \Delta x) \approx f(x) + f'(x)\,dx = \sqrt[4]{x} + \frac{1}{4\sqrt[4]{x^3}}\,dx$

$f(x + \Delta x) = \sqrt[4]{624} \approx \sqrt[4]{625} + \frac{1}{4\left(\sqrt[4]{625}\right)^3}(-1)$

$= 5 - \frac{1}{500} = 4.998$

Using a calculator, $\sqrt[4]{624} \approx 4.9980$.

49. Let $f(x) = \sqrt{x}$, $x = 4$, $dx = 0.02$, $f'(x) = 1/\left(2\sqrt{x}\right)$.

Then

$f(4.02) \approx f(4) + f'(4)\,dx$

$\sqrt{4.02} \approx \sqrt{4} + \frac{1}{2\sqrt{4}}(0.02) = 2 + \frac{1}{4}(0.02).$

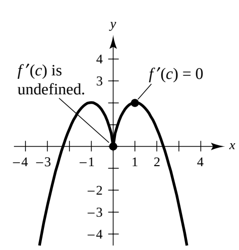
51. In general, when $\Delta x \to 0$, dy approaches Δy.

53. True

55. True

Review Exercises for Chapter 3

1. A number c in the domain of f is a critical number if $f'(c) = 0$ or f' is undefined at c.



3. $g(x) = 2x + 5\cos x$, $[0, 2\pi]$

$g'(x) = 2 - 5\sin x$

$= 0$ when $\sin x = \frac{2}{5}$.

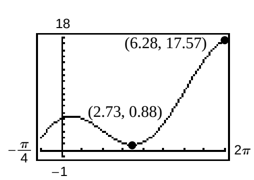
Critical numbers: $x \approx 0.41$, $x \approx 2.73$

Left endpoint: $(0, 5)$

Critical number: $(0.41, 5.41)$

Critical number: $(2.73, 0.88)$ Minimum

Right endpoint: $(2\pi, 17.57)$ Maximum



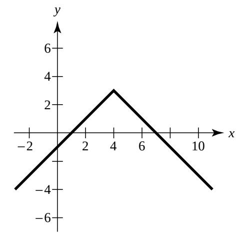
5. Yes. $f(-3) = f(2) = 0$. f is continuous on $[-3, 2]$, differentiable on $(-3, 2)$.

$$f'(x) = (x + 3)(3x - 1) = 0 \text{ for } x = \tfrac{1}{3}.$$

$$c = \tfrac{1}{3} \text{ satisfies } f'(c) - 0.$$

7. $f(x) = 3 - |x - 4|$

(a)



$f(1) = f(7) = 0$

(b) f is not differentiable at $x = 4$.

9.

$$f(x) = x^{2/3},\ 1 \le x \le 8$$

$$f'(x) = \frac{2}{3}x^{-1/3}$$

$$\frac{f(b) - f(a)}{b - a} = \frac{4 - 1}{8 - 1} = \frac{3}{7}$$

$$f'(c) = \frac{2}{3}c^{-1/3} = \frac{3}{7}$$

$$c = \left(\frac{14}{9}\right)^3 = \frac{2744}{729} \approx 3.764$$

11.

$$f(x) = x - \cos x,\ -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

$$f'(x) = 1 + \sin x$$

$$\frac{f(b) - f(a)}{b - a} = \frac{(\pi/2) - (-\pi/2)}{(\pi/2) - (-\pi/2)} = 1$$

$$f'(c) = 1 + \sin c = 1$$

$$c = 0$$

13.

$$f(x) = Ax^2 + Bx + C$$

$$f'(x) = 2Ax + B$$

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{A(x_2{}^2 - x_1{}^2) + B(x_2 - x_1)}{x_2 - x_1}$$

$$= A(x_1 + x_2) + B$$

$$f'(c) = 2Ac + B = A(x_1 + x_2) + B$$

$$2Ac = A(x_1 + x_2)$$

$$c = \frac{x_1 + x_2}{2} = \text{Midpoint of } [x_1, x_2]$$

15. $f(x) = (x - 1)^2(x - 3)$

$$f'(x) = (x - 1)^2(1) + (x - 3)(2)(x - 1)$$

$$= (x - 1)(3x - 7)$$

Critical numbers: $x = 1$ and $x = \frac{7}{3}$

Interval:	$-\infty < x < 1$	$1 < x < \frac{7}{3}$	$\frac{7}{3} < x < \infty$
Sign of $f'(x)$:	$f'(x) > 0$	$f'(x) < 0$	$f'(x) > 0$
Conclusion:	Increasing	Decreasing	Increasing

17. $h(x) = \sqrt{x}(x - 3) = x^{3/2} - 3x^{1/2}$

Domain: $(0, \infty)$

$$h'(x) = \frac{3}{2}x^{1/2} - \frac{3}{2}x^{-1/2}$$

$$= \frac{3}{2}x^{-1/2}(x - 1) = \frac{3(x - 1)}{2\sqrt{x}}$$

Critical number: $x = 1$

Interval:	$0 < x < 1$	$1 < x < \infty$
Sign of $h'(x)$:	$h'(x) < 0$	$h'(x) > 0$
Conclusion:	Decreasing	Increasing

19. $h(t) = \frac{1}{4}t^4 - 8t$

$h'(t) = t^3 - 8 = 0$ when $t = 2$.

Relative minimum: $(2, -12)$

Test Interval:	$-\infty < t < 2$	$2 < t < \infty$
Sign of $h'(t)$:	$h'(t) < 0$	$h'(t) > 0$
Conclusion:	Decreasing	Increasing

21. $y = \frac{1}{3}\cos(12t) - \frac{1}{4}\sin(12t)$

$v = y' = -4\sin(12t) - 3\cos(12t)$

(a) When $t = \frac{\pi}{8}$, $y = \frac{1}{4}$ inch and $v = y' = 4$ inches/second.

(b) $y' = -4\sin(12t) - 3\cos(12t) = 0$ when $\frac{\sin(12t)}{\cos(12t)} = -\frac{3}{4} \Longrightarrow \tan(12t) = -\frac{3}{4}$.

Therefore, $\sin(12t) = -\frac{3}{5}$ and $\cos(12t) = \frac{4}{5}$. The maximum displacement is

$$y = \left(\frac{1}{3}\right)\left(\frac{4}{5}\right) - \frac{1}{4}\left(-\frac{3}{5}\right) = \frac{5}{12} \text{ inch.}$$

(c) Period: $\frac{2\pi}{12} = \frac{\pi}{6}$

Frequency: $\frac{1}{\pi/6} = \frac{6}{\pi}$

23. $f(x) = x + \cos x,\ 0 \le x \le 2\pi$

$f'(x) = 1 - \sin x$

$f''(x) = -\cos x = 0$ when $x = \frac{\pi}{2}, \frac{3\pi}{2}$.

Points of inflection: $\left(\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{3\pi}{2}, \frac{3\pi}{2}\right)$

Test Interval:	$0 < x < \frac{\pi}{2}$	$\frac{\pi}{2} < x < \frac{3\pi}{2}$	$\frac{3\pi}{2} < x < 2\pi$
Sign of $f''(x)$:	$f''(x) < 0$	$f''(x) > 0$	$f''(x) < 0$
Conclusion:	Concave downward	Concave upward	Concave downward

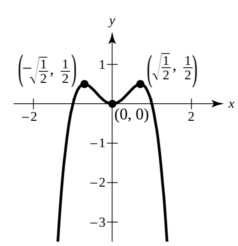
25. $g(x) = 2x^2(1 - x^2)$

$g'(x) = -4x(2x^2 - 1)$ Critical numbers: $x = 0, \pm\frac{1}{\sqrt{2}}$

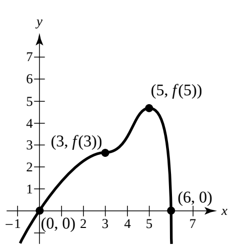
$g''(x) = 4 - 24x^2$

$g''(0) = 4 > 0$ Relative minimum at $(0, 0)$

$g''\left(\pm\frac{1}{\sqrt{2}}\right) = -8 < 0$ Relative maximums at $\left(\pm\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$



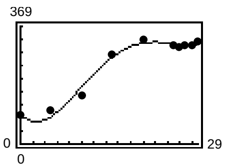
27.



29. The first derivative is positive and the second derivative is negative. The graph is increasing and is concave down.

31. (a) $D = 0.0034t^4 - 0.2352t^3 + 4.9423t^2 - 20.8641t + 94.4025$

(b)



(c) Maximum at (21.9, 319.5) ($\approx$1992)

Minimum at (2.6, 69.6) ($\approx$1972)

(d) Outlays increasing at greatest rate at the point of inflection (9.8, 173.7) ($\approx$1979)

33. $\lim_{x\to\infty} \frac{2x^2}{3x^2+5} = \lim_{x\to\infty} \frac{2}{3+5/x^2} = \frac{2}{3}$

35. $\lim_{x\to\infty} \frac{5\cos x}{x} = 0$, since $|5\cos x| \le 5$.

37. $h(x) = \frac{2x+3}{x-4}$

Discontinuity: $x = 4$

$$\lim_{x\to\infty} \frac{2x+3}{x-4} = \lim_{x\to\infty} \frac{2+(3/x)}{1-(4/x)} = 2$$

Vertical asymptote: $x = 4$

Horizontal asymptote: $y = 2$

39. $f(x) = \frac{3}{x} - 2$

Discontinuity: $x = 0$

$$\lim_{x\to\infty} \left(\frac{3}{x} - 2\right) = -2$$

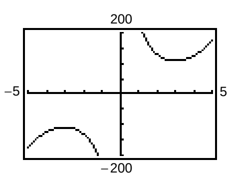
Vertical asymptote: $x = 0$

Horizontal asymptote: $y = -2$

41. $f(x) = x^3 + \frac{243}{x}$

Relative minimum: (3, 108)

Relative maximum: $(-3, -108)$

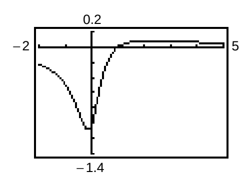


Vertical asymptote: $x = 0$

43. $f(x) = \frac{x-1}{1+3x^2}$

Relative minimum: $(-0.155, -1.077)$

Relative maximum: (2.155, 0.077)



Horizontal asymptote: $y = 0$

45. $f(x) = 4x - x^2 = x(4 - x)$

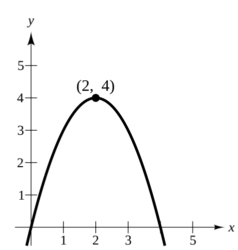
Domain: $(-\infty, \infty)$; Range: $(-\infty, 4)$

$f'(x) = 4 - 2x = 0$ when $x = 2$.

$f''(x) = -2$

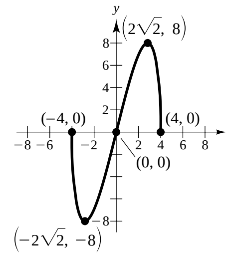
Therefore, (2, 4) is a relative maximum.

Intercepts: (0, 0), (4, 0)



47. $f(x) = x\sqrt{16 - x^2}$, Domain: $[-4, 4]$, Range: $[-8, 8]$

Domain: $[-4, 4]$; Range: $[-8, 8]$



$$f'(x) = \frac{16 - 2x^2}{\sqrt{16 - x^2}} = 0 \text{ when } x = \pm 2\sqrt{2} \text{ and undefined when } x = \pm 4.$$

$$f''(x) = \frac{2x(x^2 - 24)}{(16 - x^2)^{3/2}}$$

$f'(-2\sqrt{2}) > 0$

Therefore, $(-2\sqrt{2}, -8)$ is a relative minimum.

$f''(2\sqrt{2}) < 0$

Therefore, $(2\sqrt{2}, 8)$ is a relative maximum.

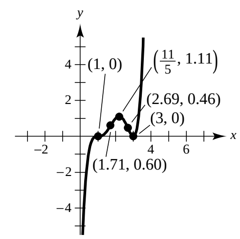
Point of inflection: $(0, 0)$

Intercepts: $(-4, 0)$, $(0, 0)$, $(4, 0)$

Symmetry with respect to origin

49. $f(x) = (x - 1)^3(x - 3)^2$

Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$



$$f'(x) = (x - 1)^2(x - 3)(5x - 11) = 0 \text{ when } x = 1, \frac{11}{5}, 3.$$

$$f''(x) = 4(x - 1)(5x^2 - 22x + 23) = 0 \text{ when } x = 1, \frac{11 \pm \sqrt{6}}{5}.$$

$f''(3) > 0$

Therefore, $(3, 0)$ is a relative minimum.

$f''\left(\frac{11}{5}\right) < 0$

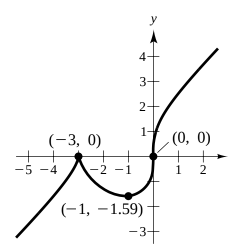
Therefore, $\left(\frac{11}{5}, \frac{3456}{3125}\right)$ is a relative maximum.

Points of inflection: $(1, 0)$, $\left(\frac{11 - \sqrt{6}}{5}, 0.60\right)$, $\left(\frac{11 + \sqrt{6}}{5}, 0.46\right)$

Intercepts: $(0, -9)$, $(1, 0)$, $(3, 0)$

51. $f(x) = x^{1/3}(x + 3)^{2/3}$

Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$



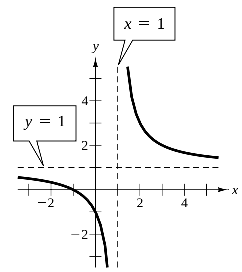
$$f'(x) = \frac{x + 1}{(x + 3)^{1/3}x^{2/3}} = 0 \text{ when } x = -1 \text{ and undefined when } x = -3, 0.$$

$$f''(x) = \frac{-2}{x^{5/3}(x + 3)^{4/3}} \text{ is undefined when } x = 0, -3.$$

By the First Derivative Test $(-3, 0)$ is a relative maximum and $(-1, -\sqrt[3]{4})$ is a relative minimum. $(0, 0)$ is a point of inflection.

Intercepts: $(-3, 0)$, $(0, 0)$

53. $f(x) = \dfrac{x+1}{x-1}$



Domain: $(-\infty, 1), (1, \infty)$; Range: $(-\infty, 1), (1, \infty)$

$f'(x) = \dfrac{-2}{(x-1)^2} < 0$ if $x \neq 1$.

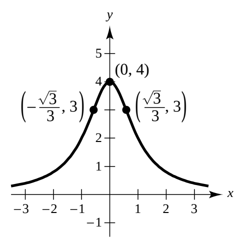
$f''(x) = \dfrac{4}{(x-1)^3}$

Horizontal asymptote: $y = 1$

Vertical asymptote: $x = 1$

Intercepts: $(-1, 0), (0, -1)$

55. $f(x) = \dfrac{4}{1+x^2}$



Domain: $(-\infty, \infty)$; Range: $(0, 4]$

$f'(x) = \dfrac{-8x}{(1+x^2)^2} = 0$ when $x = 0$.

$f''(x) = \dfrac{-8(1-3x^2)}{(1+x^2)^3} = 0$ when $x = \pm\dfrac{\sqrt{3}}{3}$.

$f''(0) < 0$

Therefore, $(0, 4)$ is a relative maximum.

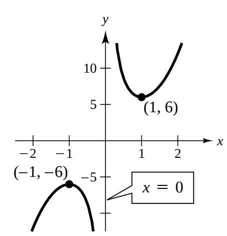
Points of inflection: $\left(\pm\sqrt{3}/3, 3\right)$

Intercept: $(0, 4)$

Symmetric to the y-axis

Horizontal asymptote: $y = 0$

57. $f(x) = x^3 + x + \dfrac{4}{x}$



Domain: $(-\infty, 0), (0, \infty)$; Range: $(-\infty, -6], [6, \infty)$

$f'(x) = 3x^2 + 1 - \dfrac{4}{x^2} = \dfrac{3x^4 + x^2 - 4}{x^2} = 0$ when $x = \pm 1$.

$f''(x) = 6x + \dfrac{8}{x^3} = \dfrac{6x^4 + 8}{x^3} \neq 0$

$f''(-1) < 0$

Therefore, $(-1, -6)$ is a relative maximum.

$f''(1) > 0$

Therefore, $(1, 6)$ is a relative minimum.

Vertical asymptote: $x = 0$

Symmetric with respect to origin

59. $f(x) = |x^2 - 9|$

Domain: $(-\infty, \infty)$; Range: $[0, \infty)$

$$f'(x) = \frac{2x(x^2 - 9)}{|x^2 - 9|} = 0 \text{ when } x = 0 \text{ and is undefined when } x = \pm 3.$$

$$f''(x) = \frac{2(x^2 - 9)}{|x^2 - 9|} \text{ is undefined at } x = \pm 3.$$

$$f''(0) < 0$$

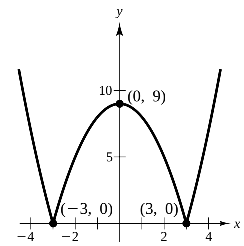
Therefore, $(0, 9)$ is a relative maximum.

Relative minima: $(\pm 3, 0)$

Points of inflection: $(\pm 3, 0)$

Intercepts: $(\pm 3, 0)$, $(0, 9)$

Symmetric to the y-axis



61. $f(x) = x + \cos x$

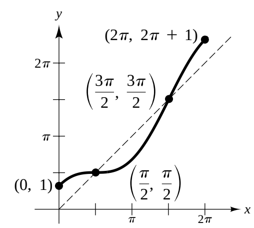
Domain: $[0, 2\pi]$; Range: $[1, 1 + 2\pi]$

$f'(x) = 1 - \sin x \geq 0$, f is increasing.

$$f''(x) = -\cos x = 0 \text{ when } x = \frac{\pi}{2}, \frac{3\pi}{2}.$$

Points of inflection: $\left(\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{3\pi}{2}, \frac{3\pi}{2}\right)$

Intercept: $(0, 1)$



63. $x^2 + 4y^2 - 2x - 16y + 13 = 0$

(a) $(x^2 - 2x + 1) + 4(y^2 - 4y + 4) = -13 + 1 + 16$

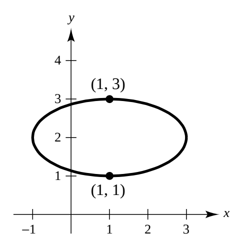
$$(x - 1)^2 + 4(y - 2)^2 = 4$$

$$\frac{(x - 1)^2}{4} + \frac{(y - 2)^2}{1} = 1$$

The graph is an ellipse:

Maximum: $(1, 3)$

Minimum: $(1, 1)$



(b) $x^2 + 4y^2 - 2x - 16y + 13 = 0$

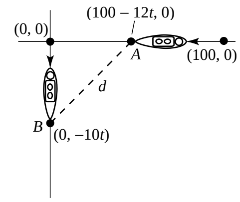
$$2x + 8y\frac{dy}{dx} - 2 - 16\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(8y - 16) = 2 - 2x$$

$$\frac{dy}{dx} = \frac{2 - 2x}{8y - 16} = \frac{1 - x}{4y - 8}$$

The critical numbers are $x = 1$ and $y = 2$. These correspond to the points $(1, 1)$, $(1, 3)$, $(2, -1)$, and $(2, 3)$. Hence, the maximum is $(1, 3)$ and the minimum is $(1, 1)$.

65. Let $t = 0$ at noon.



$$L = d^2 = (100 - 12t)^2 + (-10t)^2 = 10{,}000 - 2400t + 244t^2$$

$$\frac{dL}{dt} = -2400 + 488t = 0 \text{ when } t = \frac{300}{61} \approx 4.92 \text{ hr.}$$

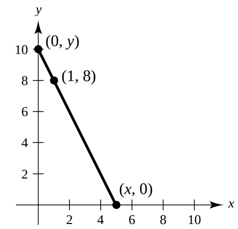
Ship A at $(40.98, 0)$; Ship B at $(0, -49.18)$

$$d^2 = 10{,}000 - 2400t + 244t^2$$

$$\approx 4098.36 \text{ when } t \approx 4.92 \approx 4{:}55 \text{ P.M.}.$$

$$d \approx 64 \text{ km}$$

67. We have points $(0, y)$, $(x, 0)$, and $(1, 8)$. Thus,



$$m = \frac{y - 8}{0 - 1} = \frac{0 - 8}{x - 1} \text{ or } y = \frac{8x}{x - 1}.$$

Let $f(x) = L^2 = x^2 + \left(\frac{8x}{x - 1}\right)^2$.

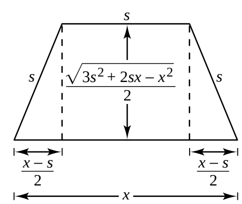
$$f'(x) = 2x + 128\left(\frac{x}{x - 1}\right)\left[\frac{(x - 1) - x}{(x - 1)^2}\right] = 0$$

$$x - \frac{64x}{(x - 1)^3} = 0$$

$$x[(x - 1)^3 - 64] = 0 \text{ when } x = 0, 5 \text{ (minimum)}.$$

Vertices of triangle: $(0, 0)$, $(5, 0)$, $(0, 10)$

69. $A = $ (Average of bases)(Height)



$$= \left(\frac{x + s}{2}\right)\frac{\sqrt{3s^2 + 2sx - x^2}}{2} \text{ (see figure)}$$

$$\frac{dA}{dx} = \frac{1}{4}\left[\frac{(s - x)(s + x)}{\sqrt{3s^2 + 2sx - x^2}} + \sqrt{3s^2 + 2sx - x^2}\right]$$

$$= \frac{2(2s - x)(s + x)}{4\sqrt{3s^2 + 2sx - x^2}} = 0 \text{ when } x = 2s.$$

A is a maximum when $x = 2s$.

71. You can form a right triangle with vertices $(0, 0)$, $(x, 0)$ and $(0, y)$. Assume that the hypotenuse of length L passes through $(4, 6)$.

$$m = \frac{y - 6}{0 - 4} = \frac{6 - 0}{4 - x} \text{ or } y = \frac{6x}{x - 4}$$

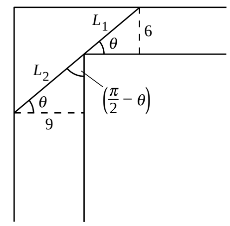
Let $f(x) = L^2 = x^2 + y^2 = x^2 + \left(\frac{6x}{x - 4}\right)^2$.

$$f'(x) = 2x + 72\left(\frac{x}{x - 4}\right)\left[\frac{-4}{(x - 4)^2}\right] = 0$$

$$x[(x - 4)^3 - 144] = 0 \text{ when } x = 0 \text{ or } x = 4 + \sqrt[3]{144}.$$

$$L \approx 14.05 \text{ feet}$$

73. $\csc\theta = \dfrac{L_1}{6}$ or $L_1 = 6\csc\theta$ (see figure)



$$\csc\left(\frac{\pi}{2} - \theta\right) = \frac{L_2}{9} \text{ or } L_2 = 9\csc\left(\frac{\pi}{2} - \theta\right)$$

$$L = L_1 + L_2 = 6\csc\theta + 9\csc\left(\frac{\pi}{2} - \theta\right) = 6\csc\theta + 9\sec\theta$$

$$\frac{dL}{d\theta} = -6\csc\theta\cot\theta + 9\sec\theta\tan\theta = 0$$

$$\tan^3\theta = \frac{2}{3} \Longrightarrow \tan\theta = \frac{\sqrt[3]{2}}{\sqrt[3]{3}}$$

$$\sec\theta = \sqrt{1 + \tan^2\theta} = \sqrt{1 + \left(\frac{2}{3}\right)^{2/3}} = \frac{\sqrt{3^{2/3} + 2^{2/3}}}{3^{1/3}}$$

$$\csc\theta = \frac{\sec\theta}{\tan\theta} = \frac{\sqrt{3^{2/3} + 2^{2/3}}}{2^{1/3}}$$

$$L = 6\frac{(3^{2/3} + 2^{2/3})^{1/2}}{2^{1/3}} + 9\frac{(3^{2/3} + 2^{2/3})^{1/2}}{3^{1/3}} = 3(3^{2/3} + 2^{2/3})^{3/2} \text{ ft} \approx 21.07 \text{ ft}$$ (Compare to Exercise 72 using $a = 9$ and $b = 6$.)

75. Total cost = (Cost per hour)(Number of hours)

$$T = \left(\frac{v^2}{600} + 5\right)\left(\frac{110}{v}\right) = \frac{11v}{60} + \frac{550}{v}$$

$$\frac{dT}{dv} = \frac{11}{60} - \frac{550}{v^2} = \frac{11v^2 - 33{,}000}{60v^2}$$

$= 0$ when $v = \sqrt{3000} = 10\sqrt{30} \approx 54.8$ mph.

$\dfrac{d^2T}{dv^2} = \dfrac{1100}{v^3} > 0$ when $v = 10\sqrt{30}$ so this value yields a minimum.

77. $f(x) = x^3 - 3x - 1$

From the graph you can see that $f(x)$ has three real zeros.

$f'(x) = 3x^2 - 3$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	−1.5000	0.1250	3.7500	0.0333	−1.5333
2	−1.5333	−0.0049	4.0530	−0.0012	−1.5321

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	−0.5000	0.3750	−2.2500	−0.1667	−0.3333
2	−0.3333	−0.0371	−2.6667	0.0139	−0.3472
3	−0.3472	−0.0003	−2.6384	0.0001	−0.3473

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	−1.9000	0.1590	7.8300	0.0203	1.8797
2	1.8797	0.0024	7.5998	0.0003	1.8794

The three real zeros of $f(x)$ are $x \approx -1.532$, $x \approx -0.347$, and $x \approx 1.879$.

79. Find the zeros of $f(x) = x^4 - x - 3$.

$$f'(x) = 4x^3 - 1$$

From the graph you can see that $f(x)$ has two real zeros.

f changes sign in $[-2, -1]$.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	-1.2000	0.2736	-7.9120	-0.0346	-1.1654
2	-1.1654	0.0100	-7.3312	-0.0014	-1.1640

On the interval $[-2, -1]$: $x \approx -1.164$.

f changes sign in $[1, 2]$.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.5000	0.5625	12.5000	0.0450	1.4550
2	1.4550	0.0268	11.3211	0.0024	1.4526
3	1.4526	-0.0003	11.2602	0.0000	1.4526

On the interval $[1, 2]$; $x \approx 1.453$.

81. $y = x(1 - \cos x) = x - x\cos x$

$$\frac{dy}{dx} = 1 + x\sin x - \cos x$$

$$dy = (1 + x\sin x - \cos x)\,dx$$

83. $S = 4\pi r^2$. $dr = \Delta r = \pm 0.025$

$$dS = 8\pi r\,dr = 8\pi(9)(\pm 0.025)$$

$$= \pm 1.8\pi \text{ square cm}$$

$$\frac{dS}{S}(100) = \frac{8\pi r\,dr}{4\pi r^2}(100) = \frac{2\,dr}{r}(100)$$

$$= \frac{2(\pm 0.025)}{9}(100) \approx \pm 0.56\%$$

$$V = \frac{4}{3}\pi r^3$$

$$dV = 4\pi r^2\,dr = 4\pi(9)^2(\pm 0.025)$$

$$= \pm 8.1\pi \text{ cubic cm}$$

$$\frac{dV}{V}(100) = \frac{4\pi r^2\,dr}{(4/3)\pi r^3}(100) = \frac{3\,dr}{r}(100)$$

$$= \frac{3(\pm 0.025)}{9}(100) \approx \pm 0.83\%$$

Problem Solving for Chapter 3

1. Assume $y_1 < d < y_2$. Let $g(x) = f(x) - d(x - a)$. g is continuous on $[a, b]$ and therefore has a minimum $(c, g(c))$ on $[a, b]$. The point c cannot be an endpoint of $[a, b]$ because

$$g'(a) = f'(a) - d = y_1 - d < 0$$

$$g'(b) = f'(b) - d = y_2 - d > 0$$

Hence, $a < c < b$ and $g'(c) = 0 \implies f'(c) = d$.

3. (a) For $a = -3, -2, -1, 0$, p has a relative maximum at $(0, 0)$.

For $a = 1, 2, 3$, p has a relative maximum at $(0, 0)$ and 2 relative minima.

(b) $p'(x) = 4ax^3 - 12x = 4x(ax^2 - 3) = 0 \implies x = 0, \pm\sqrt{\frac{3}{a}}$

$p''(x) = 12ax^2 - 12 = 12(ax^2 - 1)$

For $x = 0$, $p''(0) = -12 < 0 \implies p$ has a relative maximum at $(0, 0)$.

(c) If $a > 0$, $x = \pm\sqrt{\frac{3}{a}}$ are the remaining critical numbers.

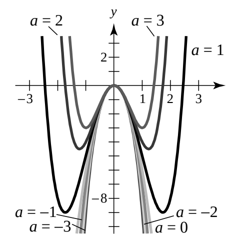
$p''\left(\pm\sqrt{\frac{3}{a}}\right) = 12\left(\frac{3}{a}\right) - 12 = 24 > 0 \implies p$ has relative minima for $a > 0$.

(d) $(0, 0)$ lies on $y = -3x^2$.

Let $x = \pm\sqrt{\frac{3}{a}}$. Then

$$p(x) = a\left(\frac{3}{a}\right)^2 - 6\left(\frac{3}{a}\right) = \frac{9}{a} - \frac{18}{a} = -\frac{9}{a}.$$

Thus, $y = -\frac{9}{a} = -3\left(\pm\sqrt{\frac{3}{a}}\right)^2 = -3x^2$ is satisfied by all the relative extrema of p.



5. $p(x) = x^4 + ax^2 + 1$

(a) $p'(x) = 4x^4 + 2ax = 2x(2x^2 + a)$

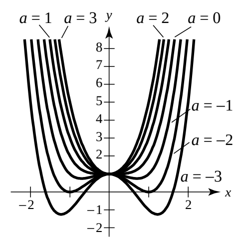
$p''(x) = 16x^2 + 2a$

For $a \geq 0$, there is one relative minimum at $(0, 0)$.

(b) For $a < \;\; 0$, there is a relative maximum at $(0, 1)$.

(c) For $a < 0$, there are two relative minima at $x = \pm\sqrt{-\frac{a}{2}}$.

(d) There are either 1 or 3 critical points. The above analysis shows that there cannot be exactly two relative extrema.



7. $f(x) = \frac{c}{x} + x^2$

$f'(x) = -\frac{c}{x^2} + 2x = 0 \implies \frac{c}{x^2} = 2x \implies x^3 = \frac{c}{2} \implies x = \sqrt[3]{\frac{c}{x}}$

$f''(x) = \frac{2c}{x^3} + 2$

If $c = 0$, $f(x) = x^2$ has a relative minimum, but no relative maximum.

If $c > 0$, $x = \sqrt[3]{\frac{c}{2}}$ is a relative minimum, because $f''\left(\sqrt[3]{\frac{c}{2}}\right) > 0$.

If $c < 0$, $x = \sqrt[3]{\frac{c}{2}}$ is a relative minimum too.

Answer: all c.

9. Set $\dfrac{f(b) - f(a) - f'(a)(b - a)}{(b - a)^2} = k$.

Define $F(x) = f(x) - f(a) - f'(a)(x - a) - k(x - a)^2$.

$F(a) = 0, F(b) = f(b) - f(a) - f'(a)(b - a) - k(b - a)^2 = 0$

F is continuous on $[a, b]$ and differentiable on (a, b).

There exists $c_1, a < c_1 < b$, satisfying $F'(c_1) = 0$.

$F'(x) = f'(x) - f'(a) - 2k(x - a)$ satisfies the hypothesis of Rolle's Theorem on $[a, c_1]$:

$F'(a) = 0, F'(c_1) = 0$.

There exists $c_2, a < c_2 < c_1$ satisfying $F''(c_2) = 0$.

Finally, $F''(x) = f''(x) - 2k$ and $F''(c_2) = 0$ implies that

$k = \dfrac{f''(c_2)}{2}$.

Thus, $k = \dfrac{f(b) - f(a) - f'(a)(b - a)}{(b - a)^2} = \dfrac{f''(c_2)}{2} \Longrightarrow f(b) = f(a) + f'(a)(b - a) + \dfrac{1}{2}f''(c_2)(b - a)^2$.

11. $E(\phi) = \dfrac{\tan \phi(1 - 0.1 \tan \phi)}{0.1 + \tan \phi} = \dfrac{10 \tan \phi - \tan^2 \phi}{1 + 10 \tan \phi}$

$E'(\phi) = \dfrac{(1 + 10 \tan \phi)(10 \sec^2 \phi - 2 \tan \phi \sec^2 \phi) - (10 \tan \phi - \tan^2 \phi)10 \sec^2 \phi}{(1 + 10 \tan \phi)} = 0$

$\Longrightarrow (1 + 10 \tan \phi)(10 \sec^2 \phi - 2 \tan \phi \sec^2 \phi) = (10 \tan \phi - \tan^2 \phi)10 \sec^2 \phi$

$\Longrightarrow 10 \sec^2 \phi - 2 \tan \phi \sec^2 \phi + 100 \tan \phi \sec^2 \phi - 20 \tan^2 \phi \sec^2 \phi$

$= 100 \tan \phi \sec^2 \phi - 10 \tan^2 \phi \sec^2 \phi$

$\Longrightarrow 10 - 2 \tan \phi = 10 \tan^2 \phi$

$\Longrightarrow 10 \tan^2 \phi + 2 \tan \phi - 10 = 0$

$\tan \phi = \dfrac{-2 \pm \sqrt{4 + 400}}{20} \approx 0.90499, -1.10499$

Using the positive value, $\phi \approx 0.7356$, or $42.14°$.

13. $v = -2400\pi \sin \theta$

$v' = -2400\pi \cos \theta = 0$

$\theta = \dfrac{\pi}{2} + 2n\pi, \dfrac{3\pi}{2} + 2n\pi$, n an integer

15. The line has equation $\frac{x}{3} + \frac{y}{4} = 1$ or $y = -\frac{4}{3}x + 4$.

Rectangle:

$$\text{Area} = A = xy = x\left(-\frac{4}{3}x + 4\right) = -\frac{4}{3}x^2 + 4x.$$

$$A'(x) = -\frac{8}{3}x + 4 = 0 \implies \frac{8}{3}x = 4 \implies x = \frac{3}{2}$$

Dimensions: $\frac{3}{2} \times 2$ Calculus was helpful.

Circle: The distance from the center (r, r) to the line $\frac{x}{3} + \frac{y}{4} - 1 = 0$ must be r:

$$r = \frac{\left|\frac{r}{3} + \frac{r}{4} - 1\right|}{\sqrt{\frac{1}{9} + \frac{1}{16}}} = \frac{12}{5}\left|\frac{7r - 12}{12}\right| = \frac{|7r - 12|}{5}$$

$$5r = |7r - 12| \implies r = 1 \text{ or } r = 6.$$

Clearly, $r = 1$.

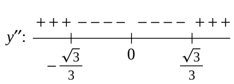
Semicircle: The center lies on the line $\frac{x}{3} + \frac{y}{4} = 1$ and satisfies $x = y = r$.

Thus $\frac{r}{3} + \frac{r}{4} = 1 \implies \frac{7}{12}r = 1 \implies r = \frac{12}{7}$. No calculus necessary.

17. $y = (1 + x^2)^{-1}$

$$y' = \frac{-2x}{(1 + x^2)^2}$$

$$y'' = \frac{2(3x^2 - 1)}{(x^2 + 1)^3} = 0 \implies x = \pm\frac{1}{\sqrt{3}} = \pm\frac{\sqrt{3}}{3}$$



The tangent line has greatest slope at $\left(-\frac{\sqrt{3}}{3}, \frac{3}{4}\right)$ and least slope at $\left(\frac{\sqrt{3}}{3}, \frac{3}{4}\right)$.

19. (a)

x	0.1	0.2	0.3	0.4	0.5	1.0
$\sin x$	0.09983	0.19867	0.29552	0.38942	0.47943	0.84147

$\sin x \le x$

(b) Let $f(x) = \sin x$. Then $f'(x) = \cos x$ and on $[0, x]$ you have by the Mean Value Theorem,

$$f'(c) = \frac{f(x) - f(0)}{x - 0}, \quad 0 < c < x$$

$$\cos(c) = \frac{\sin x}{x}$$

Hence, $\left|\frac{\sin x}{x}\right| = |\cos(c)| \le 1$

$$\implies |\sin x| \le |x|$$

$$\implies \sin x \le x$$