

CHAPTER 4

Integration

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CHAPTER 4

Integration

Section 4.1 Antiderivatives and Indefinite Integration

Solutions to Even-Numbered Exercises

$$2. \frac{d}{dx}\left(x^4 + \frac{1}{x} + C\right) = 4x^3 - \frac{1}{x^2}$$

$$4. \frac{d}{dx}\left(\frac{2(x^2 + 3)}{3\sqrt{x}} + C\right) = \frac{d}{dx}\left(\frac{2}{3}x^{3/2} + 2x^{-1/2} + C\right) \\ = x^{1/2} - x^{-3/2} = \frac{x^2 - 1}{x^{3/2}}$$

$$6. \frac{dr}{d\theta} = \pi$$

$$r = \pi\theta + C$$

$$\text{Check: } \frac{d}{d\theta}[\pi\theta + C] = \pi$$

$$8. \frac{dy}{dx} = 2x^{-3}$$

$$y = \frac{2x^{-2}}{-2} + C = \frac{-1}{x^2} + c$$

$$\text{Check: } \frac{d}{dx}\left[\frac{-1}{x^2} + C\right] = 2x^{-3}$$

Given

Rewrite

Integrate

Simplify

$$10. \int \frac{1}{x^2} dx \quad \int x^{-2} dx \quad \frac{x^{-1}}{-1} + C \quad -\frac{1}{x} + C$$

$$12. \int x(x^2 + 3) dx \quad \int (x^3 + 3x) dx \quad \frac{x^4}{4} + 3\left(\frac{x^2}{2}\right) + C \quad \frac{1}{4}x^4 + \frac{3}{2}x^2 + C$$

$$14. \int \frac{1}{(3x^2)} dx \quad \frac{1}{9} \int x^{-2} dx \quad \frac{1}{9}\left(\frac{x^{-1}}{-1}\right) + C \quad \frac{-1}{9x} + C$$

$$16. \int (5 - x) dx = 5x - \frac{x^2}{2} + C$$

$$\text{Check: } \frac{d}{dx}\left[5x - \frac{x^2}{2} + C\right] = 5 - x$$

$$18. \int (4x^3 + 6x^2 - 1) dx = x^4 + 2x^3 - x + C$$

$$\text{Check: } \frac{d}{dx}[x^4 + 2x^3 - x + C] = 4x^3 + 6x^2 - 1$$

$$20. \int (x^3 - 4x + 2) dx = \frac{x^4}{4} - 2x^2 + 2x + C$$

$$\text{Check: } \frac{d}{dx}\left[\frac{x^4}{4} - 2x^2 + 2x + C\right] = x^3 - 4x + 2$$

$$22. \int \left(\sqrt{x} + \frac{1}{2\sqrt{x}}\right) dx = \int \left(x^{1/2} + \frac{1}{2}x^{-1/2}\right) dx = \frac{x^{3/2}}{3/2} + \frac{1}{2}\left(\frac{x^{1/2}}{1/2}\right) + C = \frac{2}{3}x^{3/2} + x^{1/2} + C$$

$$\text{Check: } \frac{d}{dx}\left(\frac{2}{3}x^{3/2} + x^{1/2} + C\right) = x^{1/2} + \frac{1}{2}x^{-1/2} = \sqrt{x} + \frac{1}{2\sqrt{x}}$$

$$24. \int (\sqrt[4]{x^3} + 1) dx = \int (x^{3/4} + 1) dx = \frac{4}{7}x^{7/4} + x + C$$

$$\text{Check: } \frac{d}{dx} \left(\frac{4}{7}x^{7/4} + x + C \right) = x^{3/4} + 1 = \sqrt[4]{x^3} + 1$$

$$26. \int \frac{1}{x^4} dx = \int x^{-4} dx = \frac{x^{-3}}{-3} + C = -\frac{1}{3x^3} + C$$

$$\text{Check: } \frac{d}{dx} \left(-\frac{1}{3x^3} + C \right) = \frac{1}{x^4}$$

$$28. \int \frac{x^2 + 2x - 3}{x^4} dx = \int (x^{-2} + 2x^{-3} - 3x^{-4}) dx$$

$$= \frac{x^{-1}}{-1} + \frac{2x^{-2}}{-2} - \frac{3x^{-3}}{-3} + C$$

$$= -\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3} + C$$

$$\text{Check: } \frac{d}{dx} \left[-\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3} + C \right] = x^{-2} + 2x^{-3} - 3x^{-4}$$

$$= \frac{x^2 + 2x - 3}{x^4}$$

$$30. \int (2t^2 - 1)^2 dt = \int (4t^4 - 4t^2 + 1) dt$$

$$= \frac{4}{5}t^5 - \frac{4}{3}t^3 + t + C$$

$$\text{Check: } \frac{d}{dt} \left(\frac{4}{5}t^5 - \frac{4}{3}t^3 + t + C \right) = 4t^4 - 4t^2 + 1$$

$$= (2t^2 - 1)^2$$

$$32. \int (1 + 3t)t^2 dt = \int (t^2 + 3t^3) dt = \frac{1}{3}t^3 + \frac{3}{4}t^4 + C$$

$$\text{Check: } \frac{d}{dt} \left(\frac{1}{3}t^3 + \frac{3}{4}t^4 + C \right) = t^2 + 3t^3 = (1 + 3t)t^2$$

$$34. \int 3 dt = 3t + C$$

$$\text{Check: } \frac{d}{dt} (3t + C) = 3$$

$$36. \int (t^2 - \sin t) dt = \frac{1}{3}t^3 + \cos t + C$$

$$\text{Check: } \frac{d}{dt} \left(\frac{1}{3}t^3 + \cos t + C \right) = t^2 - \sin t$$

$$38. \int (\theta^2 + \sec^2 \theta) d\theta = \frac{1}{3}\theta^3 + \tan \theta + C$$

$$\text{Check: } \frac{d}{d\theta} \left(\frac{1}{3}\theta^3 + \tan \theta + C \right) = \theta^2 + \sec^2 \theta$$

$$40. \int \sec y (\tan y - \sec y) dy = \int (\sec y \tan y - \sec^2 y) dy$$

$$= \sec y - \tan y + C$$

$$\text{Check: } \frac{d}{dy} (\sec y - \tan y + C) = \sec y \tan y - \sec^2 y$$

$$= \sec y (\tan y - \sec y)$$

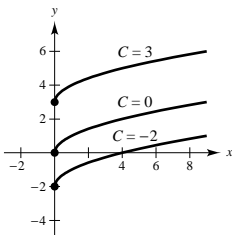
$$42. \int \frac{\cos x}{1 - \cos^2 x} dx = \int \frac{\cos x}{\sin^2 x} dx = \int \left(\frac{1}{\sin x} \right) \left(\frac{\cos x}{\sin x} \right) dx$$

$$= \int \csc x \cot x dx = -\csc x + C$$

$$\text{Check: } \frac{d}{dx} [-\csc x + C] = \csc x \cot x + \frac{1}{\sin x} \cdot \frac{\cos x}{\sin x}$$

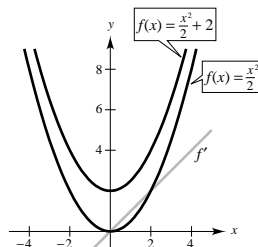
$$= \frac{\cos x}{1 - \cos^2 x}$$

$$44. f(x) = \sqrt{x}$$



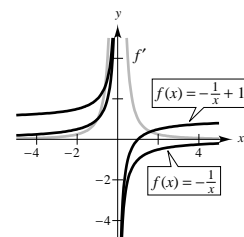
$$46. f'(x) = x$$

$$f(x) = \frac{x^2}{2} + C$$



$$48. f'(x) = \frac{1}{x^2}$$

$$f(x) = -\frac{1}{x} + C$$



$$50. \frac{dy}{dx} = 2(x-1) = 2x-2, (3, 2)$$

$$y = \int 2(x-1) dx = x^2 - 2x + C$$

$$2 = (3)^2 - 2(3) + C \Rightarrow C = -1$$

$$y = x^2 - 2x - 1$$

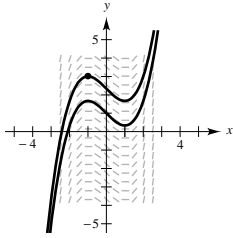
$$52. \frac{dy}{dx} = -\frac{1}{x^2} = -x^{-2}, (1, 3)$$

$$y = \int -x^{-2} dx = \frac{1}{x} + C$$

$$3 = \frac{1}{1} + C \Rightarrow C = 2$$

$$y = \frac{1}{x} + 2, x > 0$$

54. (a)



$$(b) \frac{dy}{dx} = x^2 - 1, (-1, 3)$$

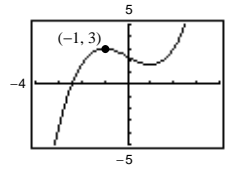
$$y = \frac{x^3}{3} - x + C$$

$$3 = \frac{(-1)^3}{3} - (-1) + C$$

$$3 = -\frac{1}{3} + 1 + C$$

$$C = \frac{7}{3}$$

$$y = \frac{x^3}{3} - x + \frac{7}{3}$$



$$56. g'(x) = 6x^2, g(0) = -1$$

$$g(x) = \int 6x^2 dx = 2x^3 + C$$

$$g(0) = -1 = 2(0)^3 + C \Rightarrow C = -1$$

$$g(x) = 2x^3 - 1$$

$$58. f'(s) = 6s - 8s^3, f(2) = 3$$

$$f(s) = \int (6s - 8s^3) ds = 3s^2 - 2s^4 + C$$

$$f(2) = 3 = 3(2)^2 - 2(2)^4 + C = 12 - 32 + C \Rightarrow C = 23$$

$$f(s) = 3s^2 - 2s^4 + 23$$

$$60. f''(x) = x^2$$

$$f'(0) = 6$$

$$f(0) = 3$$

$$f'(x) = \int x^2 dx = \frac{1}{3}x^3 + C_1$$

$$f'(0) = 0 + C_1 = 6 \Rightarrow C_1 = 6$$

$$f'(x) = \frac{1}{3}x^3 + 6$$

$$f(x) = \int \left(\frac{1}{3}x^3 + 6 \right) dx = \frac{1}{12}x^4 + 6x + C_2$$

$$f(0) = 0 + 0 + C_2 = 3 \Rightarrow C_2 = 3$$

$$f(x) = \frac{1}{12}x^4 + 6x + 3$$

$$62. f''(x) = \sin x$$

$$f'(0) = 1$$

$$f(0) = 6$$

$$f'(x) = \int \sin x dx = -\cos x + C_1$$

$$f'(0) = -1 + C_1 = 1 \Rightarrow C_1 = 2$$

$$f'(x) = -\cos x + 2$$

$$f(x) = \int (-\cos x + 2) dx = -\sin x + 2x + C_2$$

$$f(0) = 0 + 0 + C_2 = 6 \Rightarrow C_2 = 6$$

$$f(x) = -\sin x + 2x + 6$$

$$64. \frac{dP}{dt} = k\sqrt{t}, \quad 0 \leq t \leq 10$$

$$P(t) = \int kt^{1/2} dt = \frac{2}{3}kt^{3/2} + C$$

$$P(0) = 0 + C = 500 \Rightarrow C = 500$$

$$P(1) = \frac{2}{3}k + 500 = 600 \Rightarrow k = 150$$

$$P(t) = \frac{2}{3}(150)t^{3/2} + 500 = 100t^{3/2} + 500$$

$$P(7) = 100(7)^{3/2} + 500 \approx 2352 \text{ bacteria}$$

$$68. f''(t) = a(t) = -32 \text{ ft/sec}^2$$

$$f'(0) = v_0$$

$$f(0) = s_0$$

$$f'(t) = v(t) = \int -32 dt = -32t + C_1$$

$$f'(0) = 0 + C_1 = v_0 \Rightarrow C_1 = v_0$$

$$f'(t) = -32t + v_0$$

$$f(t) = s(t) = \int (-32t + v_0) dt = -16t^2 + v_0t + C_2$$

$$f(0) = 0 + 0 + C_2 = s_0 \Rightarrow C_2 = s_0$$

$$f(t) = -16t^2 + v_0t + s_0$$

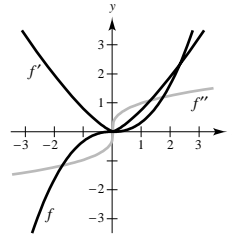
72. From Exercise 71, $f(t) = -4.9t^2 + 1600$. (Using the canyon floor as position 0.)

$$f(t) = 0 = -4.9t^2 + 1600$$

$$4.9t^2 = 1600$$

$$t^2 = \frac{1600}{4.9} \Rightarrow t \approx \sqrt{326.53} \approx 18.1 \text{ sec}$$

66. Since f'' is negative on $(-\infty, 0)$, f' is decreasing on $(-\infty, 0)$. Since f'' is positive on $(0, \infty)$, f' is increasing on $(0, \infty)$. f' has a relative minimum at $(0, 0)$. Since f' is positive on $(-\infty, \infty)$, f is increasing on $(-\infty, \infty)$.



$$70. v_0 = 16 \text{ ft/sec}$$

$$s_0 = 64 \text{ ft}$$

$$(a) \quad s(t) = -16t^2 + 16t + 64 = 0$$

$$-16(t^2 - t - 4) = 0$$

$$t = \frac{1 \pm \sqrt{17}}{2}$$

Choosing the positive value,

$$t = \frac{1 + \sqrt{17}}{2} \approx 2.562 \text{ seconds.}$$

$$(b) \quad v(t) = s'(t) = -32t + 16$$

$$v\left(\frac{1 + \sqrt{17}}{2}\right) = -32\left(\frac{1 + \sqrt{17}}{2}\right) + 16$$

$$= -16\sqrt{17} \approx -65.970 \text{ ft/sec}$$

74. From Exercise 71, $f(t) = -4.9t^2 + v_0t + 2$. If

$$f(t) = 200 = -4.9t^2 + v_0t + 2,$$

then

$$v(t) = -9.8t + v_0 = 0$$

for this t value. Hence, $t = v_0/9.8$ and we solve

$$-4.9\left(\frac{v_0}{9.8}\right)^2 + v_0\left(\frac{v_0}{9.8}\right) + 2 = 200$$

$$\frac{-4.9 v_0^2}{(9.8)^2} + \frac{v_0^2}{9.8} = 198$$

$$-4.9 v_0^2 + 9.8 v_0^2 = (9.8)^2 198$$

$$4.9 v_0^2 = (9.8)^2 198$$

$$v_0^2 = 3880.8 \Rightarrow v_0 \approx 62.3 \text{ m/sec.}$$

$$76. \int v \, dv = -GM \int \frac{1}{y^2} \, dy$$

$$\frac{1}{2}v^2 = \frac{GM}{y} + C$$

When $y = R$, $v = v_0$.

$$\frac{1}{2}v_0^2 = \frac{GM}{R} + C$$

$$C = \frac{1}{2}v_0^2 - \frac{GM}{R}$$

$$\frac{1}{2}v^2 = \frac{GM}{y} + \frac{1}{2}v_0^2 - \frac{GM}{R}$$

$$v^2 = \frac{2GM}{y} + v_0^2 - \frac{2GM}{R}$$

$$v^2 = v_0^2 + 2GM\left(\frac{1}{y} - \frac{1}{R}\right)$$

$$80. (a) a(t) = \cos t$$

$$v(t) = \int a(t) \, dt = \int \cos t \, dt = \sin t + C_1 = \sin t \quad (\text{since } v_0 = 0)$$

$$f(t) = \int v(t) \, dt = \int \sin t \, dt = -\cos t + C_2$$

$$f(0) = 3 = -\cos(0) + C_2 = -1 + C_2 \Rightarrow C_2 = 4$$

$$f(t) = -\cos t + 4$$

$$(b) v(t) = 0 = \sin t \text{ for } t = k\pi, k = 0, 1, 2, \dots$$

$$82. \quad v(0) = 45 \text{ mph} = 66 \text{ ft/sec}$$

$$30 \text{ mph} = 44 \text{ ft/sec}$$

$$15 \text{ mph} = 22 \text{ ft/sec}$$

$$a(t) = -a$$

$$v(t) = -at + 66$$

$$s(t) = -\frac{a}{2}t^2 + 66t \quad (\text{Let } s(0) = 0.)$$

$$v(t) = 0 \text{ after car moves 132 ft.}$$

$$-at + 66 = 0 \text{ when } t = \frac{66}{a}.$$

$$s\left(\frac{66}{a}\right) = -\frac{a}{2}\left(\frac{66}{a}\right)^2 + 66\left(\frac{66}{a}\right)$$

$$= 132 \text{ when } a = \frac{33}{2} = 16.5.$$

$$a(t) = -16.5$$

$$v(t) = -16.5t + 66$$

$$s(t) = -8.25t^2 + 66t$$

$$78. x(t) = (t-1)(t-3)^2 \quad 0 \leq t \leq 5$$

$$= t^3 - 7t^2 + 15t - 9$$

$$(a) v(t) = x'(t) = 3t^2 - 14t + 15 = (3t-5)(t-3)$$

$$a(t) = v'(t) = 6t - 14$$

$$(b) v(t) > 0 \text{ when } 0 < t < \frac{5}{3} \text{ and } 3 < t < 5.$$

$$(c) a(t) = 6t - 14 = 0 \text{ when } t = \frac{7}{3}.$$

$$v\left(\frac{7}{3}\right) = \left(3\left(\frac{7}{3}\right) - 5\right)\left(\frac{7}{3} - 3\right) = 2\left(-\frac{2}{3}\right) = -\frac{4}{3}$$

$$(a) -16.5t + 66 = 44$$

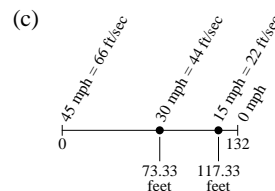
$$t = \frac{22}{16.5} \approx 1.333$$

$$s\left(\frac{22}{16.5}\right) \approx 73.33 \text{ ft}$$

$$(b) -16.5t + 66 = 22$$

$$t = \frac{44}{16.5} \approx 2.667$$

$$s\left(\frac{44}{16.5}\right) \approx 117.33 \text{ ft}$$



It takes 1.333 seconds to reduce the speed from 45 mph to 30 mph, 1.333 seconds to reduce the speed from 30 mph to 15 mph, and 1.333 seconds to reduce the speed from 15 mph to 0 mph. Each time, less distance is needed to reach the next speed reduction.

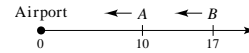
84. No, car 2 will be ahead of car 1. If $v_1(t)$ and $v_2(t)$ are the respective velocities, then $\int_0^{30} |v_2(t)| dt > \int_0^{30} |v_1(t)| dt$.

86. (a) $v = 0.6139t^3 - 5.525t^2 + 0.0492t + 65.9881$ (b) $s(t) = \int v(t)dt = \frac{0.6139t^4}{4} - \frac{5.525t^3}{3} + \frac{0.0492t^2}{2} + 65.9881t$
 (Note: Assume $s(0) = 0$ is initial position)
 $s(6) \approx 196.1$ feet

88. Let the aircrafts be located 10 and 17 miles away from the airport, as indicated in the figure.

$$v_A(t) = k_A t - 150 \quad v_B = k_B t - 250$$

$$s_A(t) = \frac{1}{2}k_A t^2 - 150t + 10 \quad s_B = \frac{1}{2}k_B t^2 - 250t + 17$$



(a) When aircraft A lands at time t_A you have

$$v_A(t_A) = k_A t_A - 150 = -100 \Rightarrow k_A = \frac{50}{t_A}$$

$$s_A(t_A) = \frac{1}{2}k_A t_A^2 - 150t_A + 10 = 0$$

$$\frac{1}{2}\left(\frac{50}{t_A}\right)t_A^2 - 150t_A = -10$$

$$125t_A = 10$$

$$t_A = \frac{10}{125}$$

$$k_A = \frac{50}{t_A} = 50\left(\frac{125}{10}\right) = 625 \Rightarrow s_A(t) = S_1(t) = \frac{625}{2}t^2 - 150t + 10$$

Similarly, when aircraft B lands at time t_B you have

$$v_B(t_B) = k_B t_B - 250 = -115 \Rightarrow k_B = \frac{135}{t_B}$$

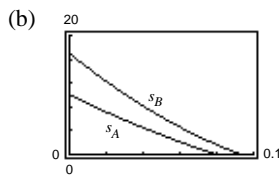
$$s_B(t_B) = \frac{1}{2}k_B t_B^2 - 250t_B + 17 = 0$$

$$\frac{1}{2}\left(\frac{135}{t_B}\right)t_B^2 - 250t_B = -17$$

$$\frac{365}{2}t_B = 17$$

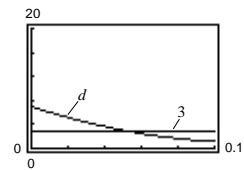
$$t_B = \frac{34}{365}$$

$$k_B = \frac{135}{t_B} = 135\left(\frac{365}{34}\right) = \frac{49,275}{34} \Rightarrow s_B(t) = S_2(t) = \frac{49,275}{68}t^2 - 250t + 17$$



(c) $d = s_B(t) - s_A(t)$

Yes, $d < 3$ for $t > 0.0505$.



90. True

92. True

94. False. f has an infinite number of antiderivatives, each differing by a constant.

$$\begin{aligned} 96. \frac{d}{dx} [s(x)]^2 + [c(x)]^2 &= 2s(x)s'(x) + 2c(x)c'(x) \\ &= 2s(x)c(x) - 2c(x)s(x) \\ &= 0 \end{aligned}$$

Thus, $[s(x)]^2 + [c(x)]^2 = k$ for some constant k . Since, $s(0) = 0$ and $c(0) = 1$, $k = 1$.

Therefore,

$$[s(x)]^2 + [c(x)]^2 = 1.$$

[Note that $s(x) = \sin x$ and $c(x) = \cos x$ satisfy these properties.]

Section 4.2 Area

$$2. \sum_{k=3}^6 k(k-2) = 3(1) + 4(2) + 5(3) + 6(4) = 50$$

$$4. \sum_{j=3}^5 \frac{1}{j} = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{47}{60}$$

$$6. \sum_{i=1}^4 [(i-1)^2 + (i+1)^2] = (0+8) + (1+27) + (4+64) + (9+125) = 238$$

$$8. \sum_{i=1}^{15} \frac{5}{1+i}$$

$$10. \sum_{j=1}^4 \left[1 - \left(\frac{j}{4} \right)^2 \right]$$

$$12. \frac{2}{n} \sum_{i=1}^n \left[1 - \left(\frac{2i}{n} - 1 \right)^2 \right]$$

$$14. \frac{1}{n} \sum_{i=0}^{n-1} \sqrt{1 - \left(\frac{i}{n} \right)^2}$$

$$\begin{aligned} 16. \sum_{i=1}^{15} (2i-3) &= 2 \sum_{i=1}^{15} i - 3(15) \\ &= 2 \left[\frac{15(16)}{2} \right] - 45 = 195 \end{aligned}$$

$$\begin{aligned} 18. \sum_{i=1}^{10} (i^2 - 1) &= \sum_{i=1}^{10} i^2 - \sum_{i=1}^{10} 1 \\ &= \left[\frac{10(11)(21)}{6} \right] - 10 = 375 \end{aligned}$$

$$\begin{aligned} 20. \sum_{i=1}^{10} i(i^2 + 1) &= \sum_{i=1}^{10} i^3 + \sum_{i=1}^{10} i \\ &= \frac{10^2(11)^2}{4} + \left[\frac{10(11)}{2} \right] = 3080 \end{aligned}$$

$$\begin{aligned} 22. \text{sum seq}(x \square 3 - 2x, x, 1, 15, 1) &= 14,160 \quad (TI-82) \\ \sum_{i=1}^{15} (i^3 - 2i) &= \frac{(15)^2(15+1)^2}{4} - 2 \frac{15(15+1)}{2} \\ &= \frac{(15)^2(16)^2}{4} - 15(16) = 14,160 \end{aligned}$$

$$\begin{aligned} 24. S &= [5 + 5 + 4 + 2](1) = 16 \\ s &= [4 + 4 + 2 + 0](1) = 10 \end{aligned}$$

$$\begin{aligned} 26. S &= \left[5 + 2 + 1 + \frac{2}{3} + \frac{1}{2} \right] = \frac{55}{6} \\ s &= \left[2 + 1 + \frac{2}{3} + \frac{1}{2} + \frac{1}{3} \right] = \frac{9}{2} = 4.5 \end{aligned}$$

$$\begin{aligned} 28. S(8) &= \left(\sqrt{\frac{1}{4}} + 2 \right) \frac{1}{4} + \left(\sqrt{\frac{1}{2}} + 2 \right) \frac{1}{4} + \left(\sqrt{\frac{3}{4}} + 2 \right) \frac{1}{4} + (\sqrt{1} + 2) \frac{1}{4} \\ &\quad + \left(\sqrt{\frac{5}{4}} + 2 \right) \frac{1}{4} + \left(\sqrt{\frac{3}{2}} + 2 \right) \frac{1}{4} + \left(\sqrt{\frac{7}{4}} + 2 \right) \frac{1}{4} + (\sqrt{2} + 2) \frac{1}{4} \\ &= \frac{1}{4} \left(16 + \frac{1}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} + 1 + \frac{\sqrt{5}}{2} + \frac{\sqrt{6}}{2} + \frac{\sqrt{7}}{2} + \sqrt{2} \right) \approx 6.038 \end{aligned}$$

$$s(8) = (0+2) \frac{1}{4} + \left(\sqrt{\frac{1}{4}} + 2 \right) \frac{1}{4} + \left(\sqrt{\frac{1}{2}} + 2 \right) \frac{1}{4} + \cdots + \left(\sqrt{\frac{7}{4}} + 2 \right) \frac{1}{4} \approx 5.685$$

$$\begin{aligned}
30. S(5) &= 1\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{1}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{2}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{3}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{4}{5}\right)^2}\left(\frac{1}{5}\right) \\
&= \frac{1}{5}\left[1 + \frac{\sqrt{24}}{5} + \frac{\sqrt{21}}{5} + \frac{\sqrt{16}}{5} + \frac{\sqrt{9}}{5}\right] \approx 0.859 \\
s(5) &= \sqrt{1 - \left(\frac{1}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{2}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{3}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{4}{5}\right)^2}\left(\frac{1}{5}\right) + 0 \approx 0.659
\end{aligned}$$

$$32. \lim_{n \rightarrow \infty} \left[\frac{\left(\frac{64}{n^3}\right)n(n+1)(2n+1)}{6} \right] = \frac{64}{6} \lim_{n \rightarrow \infty} \left[\frac{2n^3 + 3n^2 + n}{n^3} \right] = \frac{64}{6}(2) = \frac{64}{3}$$

$$34. \lim_{n \rightarrow \infty} \left[\frac{\left(\frac{1}{n^2}\right)n(n+1)}{2} \right] = \frac{1}{2} \lim_{n \rightarrow \infty} \left[\frac{n^2 + n}{n^2} \right] = \frac{1}{2}(1) = \frac{1}{2}$$

$$36. \sum_{j=1}^n \frac{4j+3}{n^2} = \frac{1}{n^2} \sum_{j=1}^n (4j+3) = \frac{1}{n^2} \left[\frac{4n(n+1)}{2} + 3n \right] = \frac{2n+5}{n} = S(n)$$

$$S(10) = \frac{25}{10} = 2.5$$

$$S(100) = 2.05$$

$$S(1000) = 2.005$$

$$S(10,000) = 2.0005$$

$$\begin{aligned}
38. \sum_{i=1}^n \frac{4i^2(i-1)}{n^4} &= \frac{4}{n^4} \sum_{i=1}^n (i^3 - i^2) = \frac{4}{n^4} \left[\frac{n^2(n+1)^2}{4} - \frac{n(n+1)(2n+1)}{6} \right] \\
&= \frac{4}{n^3} \left[\frac{n^3 + 2n^2 + n}{4} - \frac{2n^2 + 3n + 1}{6} \right] \\
&= \frac{1}{3n^3} [3n^3 + 6n^2 + 3n - 4n^2 - 6n - 2] \\
&= \frac{1}{3n^3} [3n^3 + 2n^2 - 3n - 2] = S(n)
\end{aligned}$$

$$S(10) = 1.056$$

$$S(100) = 1.006566$$

$$S(1000) = 1.00066567$$

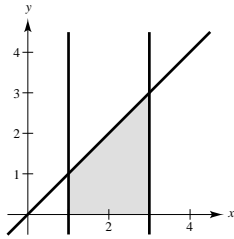
$$S(10,000) = 1.000066657$$

$$40. \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{2i}{n}\right)\left(\frac{2}{n}\right) = \lim_{n \rightarrow \infty} \frac{4}{n^2} \sum_{i=1}^n i = \lim_{n \rightarrow \infty} \frac{4}{n^2} \left(\frac{n(n+1)}{2}\right) = \lim_{n \rightarrow \infty} \frac{4}{2} \left(1 + \frac{1}{n}\right) = 2$$

$$\begin{aligned}
42. \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right) &= \lim_{n \rightarrow \infty} \frac{2}{n^3} \sum_{i=1}^n (n + 2i)^2 \\
&= \lim_{n \rightarrow \infty} \frac{2}{n^3} \left[\sum_{i=1}^n n^2 + 4n \sum_{i=1}^n i + 4 \sum_{i=1}^n i^2 \right] \\
&= \lim_{n \rightarrow \infty} \frac{2}{n^3} \left[n^3 + (4n) \left(\frac{n(n+1)}{2}\right) + \frac{4(n)(n+1)(2n+1)}{6} \right] \\
&= 2 \lim_{n \rightarrow \infty} \left[1 + 2 + \frac{2}{n} + \frac{4}{3} + \frac{2}{n} + \frac{2}{3n^2} \right] \\
&= 2 \left(1 + 2 + \frac{4}{3} \right) = \frac{26}{3}
\end{aligned}$$

$$\begin{aligned}
 44. \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(1 + \frac{2i}{n}\right)^3 \left(\frac{2}{n}\right) &= 2 \lim_{n \rightarrow \infty} \frac{1}{n^4} \sum_{i=1}^n (n + 2i)^3 \\
 &= 2 \lim_{n \rightarrow \infty} \frac{1}{n^4} \sum_{i=1}^n (n^3 + 6n^2i + 12ni^2 + 8i^3) \\
 &= 2 \lim_{n \rightarrow \infty} \frac{1}{n^4} \left[n^4 + 6n^2 \left(\frac{n(n+1)}{2}\right) + 12n \left(\frac{n(n+1)(2n+1)}{6}\right) + 8 \left(\frac{n^2(n+1)^2}{4}\right) \right] \\
 &= 2 \lim_{n \rightarrow \infty} \left(1 + 3 + \frac{3}{n} + 4 + \frac{6}{n} + \frac{2}{n^2} + 2 + \frac{4}{n} + \frac{2}{n^2}\right) \\
 &= 2 \lim_{n \rightarrow \infty} \left(10 + \frac{13}{n} + \frac{4}{n^2}\right) = 20
 \end{aligned}$$

46. (a)



(b) $\Delta x = \frac{3-1}{n} = \frac{2}{n}$

Endpoints:

$$1 < 1 + \frac{2}{n} < 1 + \frac{4}{n} < \dots < 1 + \frac{2n}{n} = 3$$

$$1 < 1 + 1\left(\frac{2}{n}\right) < 1 + 2\left(\frac{2}{n}\right) < \dots < 1 + (n-1)\left(\frac{2}{n}\right) < 1 + n\left(\frac{2}{n}\right)$$

 (c) Since $y = x$ is increasing, $f(m_i) = f(x_{i-1})$ on $[x_{i-1}, x_i]$.

$$\begin{aligned}
 s(n) &= \sum_{i=1}^n f(x_{i-1}) \Delta x \\
 &= \sum_{i=1}^n f\left[1 + (i-1)\left(\frac{2}{n}\right)\right] \left(\frac{2}{n}\right) = \sum_{i=1}^n \left[1 + (i-1)\left(\frac{2}{n}\right)\right] \left(\frac{2}{n}\right)
 \end{aligned}$$

 (d) $f(M_i) = f(x_i)$ on $[x_{i-1}, x_i]$

$$S(n) = \sum_{i=1}^n f(x_i) \Delta x = \sum_{i=1}^n f\left[1 + i\left(\frac{2}{n}\right)\right] \left(\frac{2}{n}\right) = \sum_{i=1}^n \left[1 + i\left(\frac{2}{n}\right)\right] \left(\frac{2}{n}\right)$$

(e)

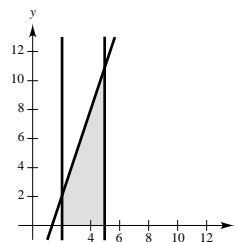
x	5	10	50	100
$s(n)$	3.6	3.8	3.96	3.98
$S(n)$	4.4	4.2	4.04	4.02

$$\begin{aligned}
 (f) \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[1 + (i-1)\left(\frac{2}{n}\right)\right] \left(\frac{2}{n}\right) &= \lim_{n \rightarrow \infty} \left(\frac{2}{n}\right) \left[n + \frac{2(n(n+1)}{2} - n)\right] \\
 &= \lim_{n \rightarrow \infty} \left[2 + \frac{2n+2}{n} - \frac{4}{n}\right] = \lim_{n \rightarrow \infty} \left[4 - \frac{2}{n}\right] = 4 \\
 \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[1 + i\left(\frac{2}{n}\right)\right] \left(\frac{2}{n}\right) &= \lim_{n \rightarrow \infty} \frac{2}{n} \left[n + \left(\frac{2}{n}\right) \frac{n(n+1)}{2}\right] \\
 &= \lim_{n \rightarrow \infty} \left[2 + \frac{2(n+1)}{n}\right] = \lim_{n \rightarrow \infty} \left[4 + \frac{2}{n}\right] = 4
 \end{aligned}$$

48. $y = 3x - 4$ on $[2, 5]$. (Note: $\Delta x = \frac{5-2}{n} = \frac{3}{n}$)

$$\begin{aligned} S(n) &= \sum_{i=1}^n f\left(2 + \frac{3i}{n}\right)\left(\frac{3}{n}\right) \\ &= \sum_{i=1}^n \left[3\left(2 + \frac{3i}{n}\right) - 4\right]\left(\frac{3}{n}\right) = 18 + 3\left(\frac{3}{n}\right)^2 \sum_{i=1}^n i - 12 \\ &= 6 + \frac{27}{n^2} \left(\frac{(n+1)n}{2}\right) = 6 + \frac{27}{2} \left(1 + \frac{1}{n}\right) \end{aligned}$$

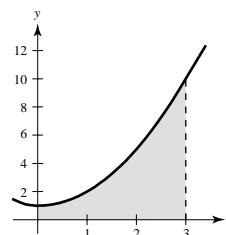
$$\text{Area} = \lim_{n \rightarrow \infty} S(n) = 6 + \frac{27}{2} = \frac{39}{2}$$



50. $y = x^2 + 1$ on $[0, 3]$. (Note: $\Delta x = \frac{3-0}{n} = \frac{3}{n}$)

$$\begin{aligned} S(n) &= \sum_{i=1}^n f\left(\frac{3i}{n}\right)\left(\frac{3}{n}\right) = \sum_{i=1}^n \left[\left(\frac{3i}{n}\right)^2 + 1\right]\left(\frac{3}{n}\right) \\ &= \frac{27}{n^3} \sum_{i=1}^n i^2 + \frac{3}{n} \sum_{i=1}^n 1 \\ &= \frac{27}{n^3} \frac{n(n+1)(2n+1)}{6} + \frac{3}{n}(n) = \frac{9}{2} \frac{2n^2 + 3n + 1}{n^2} + 3 \end{aligned}$$

$$\text{Area} = \lim_{n \rightarrow \infty} S(n) = \frac{9}{2}(2) + 3 = 12$$

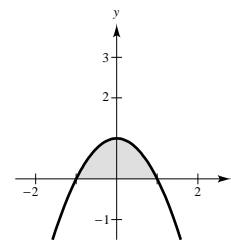


52. $y = 1 - x^2$ on $[-1, 1]$. Find area of region over the interval $[0, 1]$. (Note: $\Delta x = \frac{1}{n}$)

$$\begin{aligned} s(n) &= \sum_{i=1}^n f\left(\frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^n \left[1 - \left(\frac{i}{n}\right)^2\right]\left(\frac{1}{n}\right) \\ &= 1 - \frac{1}{n^3} \sum_{i=1}^n i^2 = 1 - \frac{n(n+1)(2n+1)}{6n^3} = 1 - \frac{1}{6} \left(2 + \frac{3}{n} + \frac{1}{n^2}\right) \end{aligned}$$

$$\frac{1}{2} \text{Area} = \lim_{n \rightarrow \infty} s(n) = 1 - \frac{1}{3} = \frac{2}{3}$$

$$\text{Area} = \frac{4}{3}$$

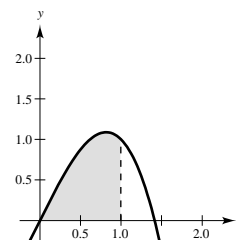


54. $y = 2x - x^3$ on $[0, 1]$. (Note: $\Delta x = \frac{1-0}{n} = \frac{1}{n}$)

Since y both increases and decreases on $[0, 1]$, $T(n)$ is neither an upper nor lower sum.

$$\begin{aligned} T(n) &= \sum_{i=1}^n f\left(\frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^n \left[2\left(\frac{i}{n}\right) - \left(\frac{i}{n}\right)^3\right]\left(\frac{1}{n}\right) \\ &= \frac{2}{n^2} \sum_{i=1}^n i - \frac{1}{n^4} \sum_{i=1}^n i^3 = \frac{n(n+1)}{n^2} - \frac{1}{n^4} \left[\frac{n^2(n+1)^2}{4}\right] \\ &= 1 + \frac{1}{n} - \frac{1}{4} - \frac{2}{4n} - \frac{1}{4n^2} \end{aligned}$$

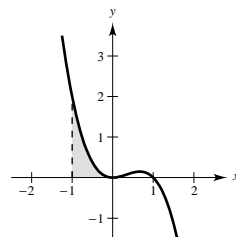
$$\text{Area} = \lim_{n \rightarrow \infty} T(n) = 1 - \frac{1}{4} = \frac{3}{4}$$



56. $y = x^2 - x^3$ on $[-1, 0]$. (Note: $\Delta x = \frac{0 - (-1)}{n} = \frac{1}{n}$)

$$\begin{aligned} s(n) &= \sum_{i=1}^n f\left(-1 + \frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^n \left[\left(-1 + \frac{i}{n}\right)^2 - \left(-1 + \frac{i}{n}\right)^3 \right] \left(\frac{1}{n}\right) \\ &= \sum_{i=1}^n \left[\left(2 - \frac{5i}{n} + \frac{4i^2}{n^2} - \frac{i^3}{n^3}\right) - \left(-1 + \frac{3i}{n} - \frac{3i^2}{n^2} + \frac{i^3}{n^3}\right) \right] \left(\frac{1}{n}\right) \\ &= 2 - \frac{5}{2} - \frac{5}{2n} + \frac{4}{3} + \frac{2}{n} + \frac{2}{3n^3} - \frac{1}{4} - \frac{1}{2n} - \frac{1}{4n^2} \end{aligned}$$

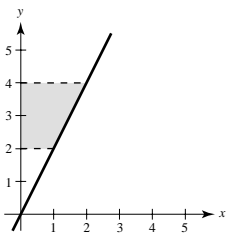
$$\text{Area} = \lim_{n \rightarrow \infty} s(n) = 2 - \frac{5}{2} + \frac{4}{3} - \frac{1}{4} = \frac{7}{12}$$



58. $g(y) = \frac{1}{2}y$, $2 \leq y \leq 4$. (Note: $\Delta y = \frac{4 - 2}{n} = \frac{2}{n}$)

$$\begin{aligned} S(n) &= \sum_{i=1}^n g\left(2 + \frac{2i}{n}\right)\left(\frac{2}{n}\right) \\ &= \sum_{i=1}^n \frac{1}{2} \left(2 + \frac{2i}{n}\right) \left(\frac{2}{n}\right) = \frac{2}{n} \sum_{i=1}^n \left(1 + \frac{i}{n}\right) \\ &= \frac{2}{n} \left[n + \frac{1}{n} \frac{n(n+1)}{2} \right] = 2 + \frac{n+1}{n} \end{aligned}$$

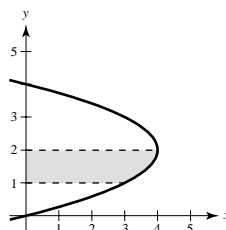
$$\text{Area} = \lim_{n \rightarrow \infty} S(n) = 2 + 1 = 3$$



60. $f(y) = 4y - y^2$, $1 \leq y \leq 2$. (Note: $\Delta y = \frac{2 - 1}{n} = \frac{1}{n}$)

$$\begin{aligned} S(n) &= \sum_{i=1}^n f\left(1 + \frac{i}{n}\right)\left(\frac{1}{n}\right) \\ &= \frac{1}{n} \sum_{i=1}^n \left[4\left(1 + \frac{i}{n}\right) - \left(1 + \frac{i}{n}\right)^2 \right] \\ &= \frac{1}{n} \sum_{i=1}^n \left(4 + \frac{4i}{n} - 1 - \frac{2i}{n} - \frac{i^2}{n^2} \right) \\ &= \frac{1}{n} \sum_{i=1}^n \left(3 + \frac{2i}{n} - \frac{i^2}{n^2} \right) \\ &= \frac{1}{n} \left[3n + \frac{2}{n} \frac{n(n+1)}{2} - \frac{1}{n^2} \frac{n(n+1)(2n+1)}{6} \right] \\ &= 3 + \frac{n+1}{n} - \frac{(n+1)(2n+1)}{6n} \end{aligned}$$

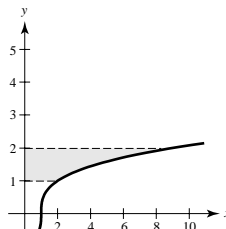
$$\text{Area} = \lim_{n \rightarrow \infty} S(n) = 3 + 1 - \frac{1}{3} = \frac{11}{3}$$



62. $h(y) = y^3 + 1$, $1 \leq y \leq 2$ (Note: $\Delta y = \frac{1}{n}$)

$$\begin{aligned} S(n) &= \sum_{i=1}^n h\left(1 + \frac{i}{n}\right)\left(\frac{1}{n}\right) \\ &= \sum_{i=1}^n \left[\left(1 + \frac{i}{n}\right)^3 + 1 \right] \frac{1}{n} \\ &= \frac{1}{n} \sum_{i=1}^n \left(2 + \frac{3i}{n} + \frac{3i^2}{n^2} + \frac{i^3}{n^3} \right) \\ &= \frac{1}{n} \left[2n + \frac{3}{n} \frac{n(n+1)}{2} + \frac{3}{n^2} \frac{n(n+1)(2n+1)}{6} + \frac{1}{n^3} \frac{n(n+1)(2n+1)(3n+1)}{24} \right] \\ &= 2 + \frac{(n+1)^2}{n^2} + \frac{1}{2} \frac{(n+1)(2n+1)}{n^2} + \frac{3(n+1)}{2n} \end{aligned}$$

$$\text{Area} = \lim_{n \rightarrow \infty} S(n) = 2 + \frac{1}{4} + 1 + \frac{3}{2} = \frac{19}{4}$$



64. $f(x) = x^2 + 4x, 0 \leq x \leq 4, n = 4$

Let $c_i = \frac{x_i + x_{i-1}}{2}$.

$$\Delta x = 1, c_1 = \frac{1}{2}, c_2 = \frac{3}{2}, c_3 = \frac{5}{2}, c_4 = \frac{7}{2}$$

$$\begin{aligned} \text{Area} &\approx \sum_{i=1}^n f(c_i) \Delta x = \sum_{i=1}^4 [c_i^2 + 4c_i](1) \\ &= \left[\left(\frac{1}{4} + 2 \right) + \left(\frac{9}{4} + 6 \right) + \left(\frac{25}{4} + 10 \right) + \left(\frac{49}{4} + 14 \right) \right] \\ &= 53 \end{aligned}$$

66. $f(x) = \sin x, 0 \leq x \leq \frac{\pi}{2}, n = 4$

Let $c_i = \frac{x_i + x_{i-1}}{2}$.

$$\Delta x = \frac{\pi}{8}, c_1 = \frac{\pi}{16}, c_2 = \frac{3\pi}{16}, c_3 = \frac{5\pi}{16}, c_4 = \frac{7\pi}{16}$$

$$\begin{aligned} \text{Area} &\approx \sum_{i=1}^n f(c_i) \Delta x = \sum_{i=1}^4 (\sin c_i) \left(\frac{\pi}{8} \right) \\ &= \frac{\pi}{8} \left(\sin \frac{\pi}{16} + \sin \frac{3\pi}{16} + \sin \frac{5\pi}{16} + \sin \frac{7\pi}{16} \right) \approx 1.006 \end{aligned}$$

68. $f(x) = \frac{8}{x^2 + 1}$ on $[2, 6]$.

n	4	8	12	16	20
Approximate area	2.3397	2.3755	2.3824	2.3848	2.3860

70. $f(x) = \cos \sqrt{x}$ on $[0, 2]$.

n	4	8	12	16	20
Approximate area	1.1041	1.1053	1.1055	1.1056	1.1056

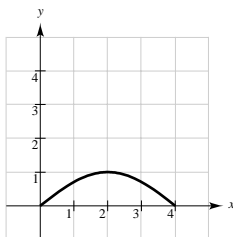
72. See the Definition of Area. Page 259.

74. $f(x) = \sqrt[3]{x}, 0 \leq x \leq 8$

n	10	20	50	100	200
$s(n)$	10.998	11.519	11.816	11.910	11.956
$S(n)$	12.598	12.319	12.136	12.070	12.036
$M(n)$	12.040	12.016	12.005	12.002	12.001

(Note: exact answer is 12.)

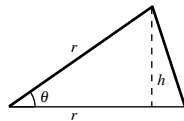
76.

a. $A \approx 3$ square units

78. True. (Theorem 4.3)

80. (a) $\theta = \frac{2\pi}{n}$

(b) $\sin \theta = \frac{h}{r}$



$$h = r \sin \theta$$

$$A = \frac{1}{2}bh = \frac{1}{2}r(r \sin \theta) = \frac{1}{2}r^2 \sin \theta$$

(c) $A_n = n \left(\frac{1}{2}r^2 \sin \frac{2\pi}{n} \right) = \frac{r^2 n}{2} \sin \frac{2\pi}{n} = \pi r^2 \left(\frac{\sin(2\pi/n)}{2\pi/n} \right)$

Let $x = 2\pi/n$. As $n \rightarrow \infty$, $x \rightarrow 0$.

$$\lim_{n \rightarrow \infty} A_n = \lim_{x \rightarrow 0} \pi r^2 \left(\frac{\sin x}{x} \right) = \pi r^2(1) = \pi r^2$$

82. (a) $\sum_{i=1}^n 2i = n(n+1)$

The formula is true for $n = 1$: $2 = 1(1+1) = 2$ Assume that the formula is true for $n = k$:

$$\sum_{i=1}^k 2i = k(k+1).$$

$$\begin{aligned} \text{Then we have } \sum_{i=1}^{k+1} 2i &= \sum_{i=1}^k 2i + 2(k+1) \\ &= k(k+1) + 2(k+1) \\ &= (k+1)(k+2) \end{aligned}$$

Which shows that the formula is true for $n = k+1$.

(b) $\sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}$

The formula is true for $n = 1$ because

$$1^3 = \frac{1^2(1+1)^2}{4} = \frac{4}{4} = 1$$

Assume that the formula is true for $n = k$:

$$\sum_{i=1}^k i^3 = \frac{k^2(k+1)^2}{4}$$

$$\begin{aligned} \text{Then we have } \sum_{i=1}^{k+1} i^3 &= \sum_{i=1}^k i^3 + (k+1)^3 \\ &= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\ &= \frac{(k+1)^2}{4} [k^2 + 4(k+1)] \\ &= \frac{(k+1)^2}{4} (k+2)^2 \end{aligned}$$

which shows that the formula is true for $n = k+1$.

Section 4.3 Riemann Sums and Definite Integrals

2. $f(x) = \sqrt[3]{x}$, $y = 0$, $x = 0$, $x = 1$, $c_i = \frac{i^3}{n^3}$

$$\Delta x_i = \frac{i^3}{n^3} - \frac{(i-1)^3}{n^3} = \frac{3i^2 - 3i + 1}{n^3}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{i=1}^n f(c_i) \Delta x_i &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \sqrt[3]{\frac{i^3}{n^3}} \left[\frac{3i^2 - 3i + 1}{n^3} \right] \\ &= \lim_{n \rightarrow \infty} \frac{1}{n^4} \sum_{i=1}^n (3i^3 - 3i^2 + i) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n^4} \left[3 \left(\frac{n^2(n+1)^2}{4} \right) - 3 \left(\frac{n(n+1)(2n+1)}{6} \right) + \frac{n(n+1)}{2} \right] \\ &= \lim_{n \rightarrow \infty} \frac{1}{n^4} \left[\frac{3n^4 + 6n^3 + 3n^2}{4} - \frac{2n^3 + 3n^2 + n}{2} + \frac{n^2 + n}{2} \right] \\ &= \lim_{n \rightarrow \infty} \frac{1}{n^4} \left[\frac{3n^4}{4} + \frac{n^3}{2} - \frac{n^2}{4} \right] = \lim_{n \rightarrow \infty} \left[\frac{3}{4} + \frac{1}{2n} - \frac{1}{4n^2} \right] = \frac{3}{4} \end{aligned}$$

4. $y = x$ on $[-2, 3]$. (Note: $\Delta x = \frac{3 - (-2)}{n} = \frac{5}{n}$, $\|\Delta\| \rightarrow 0$ as $n \rightarrow \infty$)

$$\begin{aligned}\sum_{i=1}^n f(c_i) \Delta x_i &= \sum_{i=1}^n f\left(-2 + \frac{5i}{n}\right) \left(\frac{5}{n}\right) = \sum_{i=1}^n \left(-2 + \frac{5i}{n}\right) \left(\frac{5}{n}\right) = -10 + \frac{25}{n^2} \sum_{i=1}^n i \\ &= -10 + \left(\frac{25}{n^2}\right) \frac{n(n+1)}{2} = -10 + \frac{25}{2} \left(1 + \frac{1}{n}\right) = \frac{5}{2} + \frac{25}{2n}\end{aligned}$$

$$\int_{-2}^3 x \, dx = \lim_{n \rightarrow \infty} \left(\frac{5}{2} + \frac{25}{2n}\right) = \frac{5}{2}$$

6. $y = 3x^2$ on $[1, 3]$. (Note: $\Delta x = \frac{3 - 1}{n} = \frac{2}{n}$, $\|\Delta\| \rightarrow 0$ as $n \rightarrow \infty$)

$$\begin{aligned}\sum_{i=1}^n f(c_i) \Delta x_i &= \sum_{i=1}^n f\left(1 + \frac{2i}{n}\right) \left(\frac{2}{n}\right) \\ &= \sum_{i=1}^n 3 \left(1 + \frac{2i}{n}\right)^2 \left(\frac{2}{n}\right) \\ &= \frac{6}{n} \sum_{i=1}^n \left(1 + \frac{4i}{n} + \frac{4i^2}{n^2}\right) \\ &= \frac{6}{n} \left[n + \frac{4}{n} \frac{n(n+1)}{2} + \frac{4}{n^2} \frac{n(n+1)(2n+1)}{6} \right] \\ &= 6 + 12 \frac{n+1}{n} + 4 \frac{(n+1)(2n+1)}{n^2}\end{aligned}$$

$$\begin{aligned}\int_1^3 3x^2 \, dx &= \lim_{n \rightarrow \infty} \left[6 + \frac{12(n+1)}{n} + \frac{4(n+1)(2n+1)}{n^2} \right] \\ &= 6 + 12 + 8 = 26\end{aligned}$$

8. $y = 3x^2 + 2$ on $[-1, 2]$. (Note: $\Delta x = \frac{2 - (-1)}{n} = \frac{3}{n}$, $\|\Delta\| \rightarrow 0$ as $n \rightarrow \infty$)

$$\begin{aligned}\sum_{i=1}^n f(c_i) \Delta x_i &= \sum_{i=1}^n f\left(-1 + \frac{3i}{n}\right) \left(\frac{3}{n}\right) \\ &= \frac{3}{n} \sum_{i=1}^n \left[3 \left(-1 + \frac{3i}{n}\right)^2 + 2 \right] \\ &= \frac{3}{n} \sum_{i=1}^n \left[3 \left(1 - \frac{6i}{n} + \frac{9i^2}{n^2}\right) + 2 \right] \\ &= \frac{3}{n} \left[3n - \frac{18}{n} \frac{n(n+1)}{2} + \frac{27}{n^2} \frac{n(n+1)(2n+1)}{6} + 2n \right] \\ &= 15 - \frac{27(n+1)}{n} + \frac{27}{2} \frac{(n+1)(2n+1)}{n^2}\end{aligned}$$

$$\begin{aligned}\int_{-1}^2 (3x^2 + 2) \, dx &= \lim_{n \rightarrow \infty} \left[15 - 27 \frac{(n+1)}{n} + \frac{27}{2} \frac{(n+1)(2n+1)}{n^2} \right] \\ &= 15 - 27 + 27 = 15\end{aligned}$$

10. $\lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n 6c_i(4 - c_i)^2 \Delta x_i = \int_0^4 6x(4 - x)^2 \, dx$

on the interval $[0, 4]$.

12. $\lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n \left(\frac{3}{c_i^2}\right) \Delta x_i = \int_1^3 \frac{3}{x^2} \, dx$

on the interval $[1, 3]$.

14. $\int_0^2 (4 - 2x) \, dx$

16. $\int_0^2 x^2 \, dx$

18. $\int_{-1}^1 \frac{1}{x^2 + 1} \, dx$

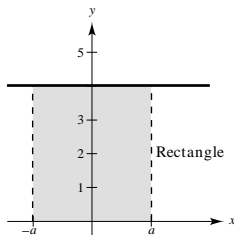
20. $\int_0^{\pi/4} \tan x \, dx$

22. $\int_0^2 (y - 2)^2 dy$

24. Rectangle

$$A = bh = 2(4)(a)$$

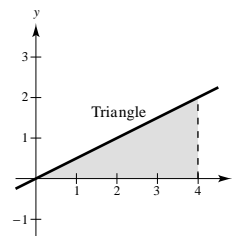
$$A = \int_{-a}^a 4 dx = 8a$$



26. Triangle

$$A = \frac{1}{2}bh = \frac{1}{2}(4)(2)$$

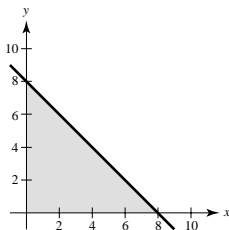
$$A = \int_0^4 \frac{x}{2} dx = 4$$



28. Triangle

$$A = \frac{1}{2}bh = \frac{1}{2}(8)(8) = 32$$

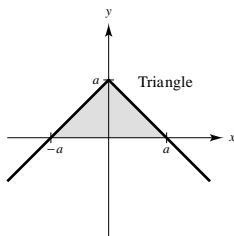
$$A = \int_0^8 (8 - x) dx = 32$$



30. Triangle

$$A = \frac{1}{2}bh = \frac{1}{2}(2a)a$$

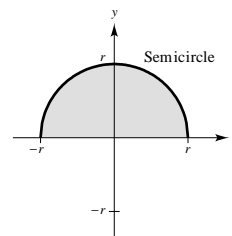
$$A = \int_{-a}^a (a - |x|) dx = a^2$$



32. Semicircle

$$A = \frac{1}{2}\pi r^2$$

$$A = \int_{-r}^r \sqrt{r^2 - x^2} dx = \frac{1}{2}\pi r^2$$



In Exercises 34–40, $\int_2^4 x^3 dx = 60$, $\int_2^4 x dx = 6$, $\int_2^4 dx = 2$.

34. $\int_2^2 x^3 dx = 0$

36. $\int_2^4 15 dx = 15 \int_2^4 dx = 15(2) = 30$

38. $\int_2^4 (x^3 + 4) dx = \int_2^4 x^3 dx + 4 \int_2^4 dx = 60 + 4(2) = 68$

40. $\int_2^4 (6 + 2x - x^3) dx = 6 \int_2^4 dx + 2 \int_2^4 x dx - \int_2^4 x^3 dx$
 $= 6(2) + 2(6) - 60 = -36$

42. (a) $\int_0^6 f(x) dx = \int_0^3 f(x) dx + \int_3^6 f(x) dx = 4 + (-1) = 3$

44. (a) $\int_{-1}^0 f(x) dx = \int_{-1}^1 f(x) dx - \int_0^1 f(x) dx = 0 - 5 = -5$

(b) $\int_6^3 f(x) dx = - \int_3^6 f(x) dx = -(-1) = 1$

(b) $\int_0^1 f(x) dx - \int_1^0 f(x) dx = 5 - (-5) = 10$

(c) $\int_3^3 f(x) dx = 0$

(c) $\int_{-1}^1 3f(x) dx = 3 \int_{-1}^1 f(x) dx = 3(0) = 0$

(d) $\int_3^6 -5f(x) dx = -5 \int_3^6 f(x) dx = -5(-1) = 5$

(d) $\int_0^1 3f(x) dx = 3 \int_0^1 f(x) dx = 3(5) = 15$

46. (a) $\int_0^5 [f(x) + 2] dx = \int_0^5 f(x) dx + \int_0^5 2 dx = 4 + 10 = 14$ (b) $\int_{-2}^3 f(x+2) dx = \int_0^5 f(x) dx = 4$ (Let $u = x + 2$.)

(c) $\int_{-5}^5 f(x) dx = 2 \int_0^5 f(x) dx = 2(4) = 8$ (f even) (d) $\int_{-5}^5 f(x) dx = 0$ (f odd)

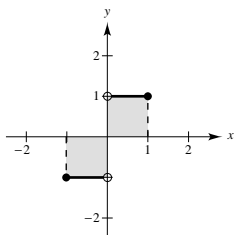
48. The right endpoint approximation will be less than the actual area: <

50. The average of Exercise 39 and Exercise 40 consists of a trapezoidal approximation, and is greater than the exact area: >

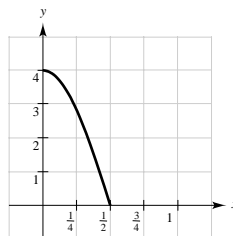
52. $f(x) = |x|/x$ is integrable on $[-1, 1]$, but is not continuous on $[-1, 1]$. There is discontinuity at $x = 0$. To see that

$$\int_{-1}^1 \frac{|x|}{x} dx$$

is integrable, sketch a graph of the region bounded by $f(x) = |x|/x$ and the x -axis for $-1 \leq x \leq 1$. You see that the integral equals 0.

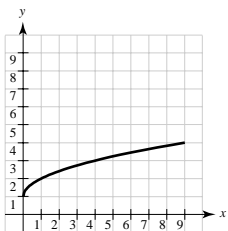


54.



b. $A \approx \frac{4}{3}$ square units

56.



c. Area ≈ 27 .

58. $\int_0^3 \frac{5}{x^2 + 1} dx$

n	4	8	12	16	20
$L(n)$	7.9224	7.0855	6.8062	6.6662	6.5822
$M(n)$	6.2485	6.2470	7.2460	6.2457	6.2455
$R(n)$	4.5474	5.3980	5.6812	5.8225	5.9072

60. $\int_0^3 x \sin x dx$

n	4	8	12	16	20
$L(n)$	2.8186	2.9985	3.0434	3.0631	3.0740
$M(n)$	3.1784	3.1277	3.1185	3.1152	3.1138
$R(n)$	3.1361	3.1573	3.1493	3.1425	3.1375

62. False

$$\int_0^1 x\sqrt{x} dx \neq \left(\int_0^1 x dx \right) \left(\int_0^1 \sqrt{x} dx \right)$$

64. True

66. False

$$\int_{-2}^4 x dx = 6$$

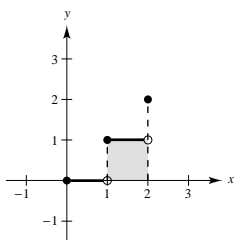
68. $f(x) = \sin x, [0, 2\pi]$

$$x_0 = 0, x_1 = \frac{\pi}{4}, x_2 = \frac{\pi}{3}, x_3 = \pi, x_4 = 2\pi$$

$$\Delta x_1 = \frac{\pi}{4}, \Delta x_2 = \frac{\pi}{12}, \Delta x_3 = \frac{2\pi}{3}, \Delta x_4 = \pi$$

$$c_1 = \frac{\pi}{6}, c_2 = \frac{\pi}{3}, c_3 = \frac{2\pi}{3}, c_4 = \frac{3\pi}{2}$$

$$\begin{aligned} \sum_{i=1}^4 f(c_i) \Delta x_i &= f\left(\frac{\pi}{6}\right) \Delta x_1 + f\left(\frac{\pi}{3}\right) \Delta x_2 + f\left(\frac{2\pi}{3}\right) \Delta x_3 + f\left(\frac{3\pi}{2}\right) \Delta x_4 \\ &= \left(\frac{1}{2}\right)\left(\frac{\pi}{4}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\pi}{12}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{2\pi}{3}\right) + (-1)(\pi) \approx -0.708 \end{aligned}$$

70. To find $\int_0^2 \llbracket x \rrbracket dx$, use a geometric approach.

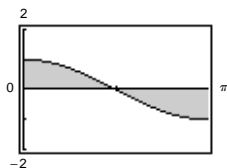
Thus,

$$\int_0^2 \llbracket x \rrbracket dx = 1(2 - 1) = 1.$$

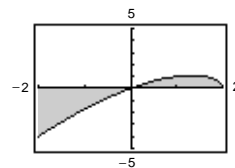
Section 4.4 The Fundamental Theorem of Calculus

2. $f(x) = \cos x$

$$\int_0^\pi \cos x dx = 0$$

4. $f(x) = x\sqrt{2-x}$

$$\int_{-2}^2 x\sqrt{2-x} dx \text{ is negative.}$$



$$6. \int_2^7 3 dv = [3v]_2^7 = 3(7) - 3(2) = 15$$

$$8. \int_2^5 (-3v + 4) dv = \left[-\frac{3}{2}v^2 + 4v\right]_2^5 = \left(-\frac{75}{2} + 20\right) - (-6 + 8) = -\frac{39}{2}$$

$$\begin{aligned} 10. \int_1^3 (3x^2 + 5x - 4) dx &= \left[x^3 + \frac{5x^2}{2} - 4x\right]_1^3 = \left(27 + \frac{45}{2} - 12\right) - \left(1 + \frac{5}{2} - 4\right) \\ &= 38 \end{aligned}$$

$$12. \int_{-1}^1 (t^3 - 9t) dt = \left[\frac{1}{4}t^4 - \frac{9}{2}t^2\right]_{-1}^1 = \left(\frac{1}{4} - \frac{9}{2}\right) - \left(\frac{1}{4} - \frac{9}{2}\right) = 0$$

$$14. \int_{-2}^{-1} \left(u - \frac{1}{u^2}\right) du = \left[\frac{u^2}{2} + \frac{1}{u}\right]_{-2}^{-1} = \left(\frac{1}{2} - 1\right) - \left(2 - \frac{1}{2}\right) = -2$$

$$16. \int_{-3}^3 v^{1/3} dv = \left[\frac{3}{4} v^{4/3} \right]_{-3}^3 = \frac{3}{4} [(\sqrt[3]{-3})^4] - (\sqrt[3]{-3})^4 = 0$$

$$18. \int_1^8 \sqrt{\frac{2}{x}} dx = \sqrt{2} \int_1^8 x^{-1/2} dx = \left[\sqrt{2}(2)x^{1/2} \right]_1^8 = \left[2\sqrt{2x} \right]_1^8 = 8 - 2\sqrt{2}$$

$$20. \int_0^2 (2-t)\sqrt{t} dt = \int_0^2 (2t^{1/2} - t^{3/2}) dt = \left[\frac{4}{3}t^{3/2} - \frac{2}{5}t^{5/2} \right]_0^2 = \left[\frac{t\sqrt{t}}{15}(20-6t) \right]_0^2 = \frac{2\sqrt{2}}{15}(20-12) = \frac{16\sqrt{2}}{15}$$

$$22. \int_{-8}^{-1} \frac{x-x^2}{2\sqrt[3]{x}} dx = \frac{1}{2} \int_{-8}^{-1} (x^{2/3} - x^{5/3}) dx \\ = \frac{1}{2} \left[\frac{3}{5}x^{5/3} - \frac{3}{8}x^{8/3} \right]_{-8}^{-1} = \left[\frac{x^{5/3}}{80}(24-15x) \right]_{-8}^{-1} = -\frac{1}{80}(39) + \frac{32}{80}(144) = \frac{4569}{80}$$

$$24. \int_1^4 (3-1x-31) dx = \int_1^3 [3+(x-3)] dx + \int_3^4 [3-(x-3)] dx \\ = \int_1^3 x dx + \int_3^4 (6-x) dx \\ = \left[\frac{x^2}{2} \right]_1^3 + \left[6x - \frac{x^2}{2} \right]_3^4 \\ = \left(\frac{9}{2} - \frac{1}{2} \right) + \left[(24-8) - \left(18 - \frac{9}{2} \right) \right] \\ = 4 + 16 - 18 + \frac{9}{2} = \frac{13}{2}$$

$$26. \int_0^4 |x^2 - 4x + 3| dx = \int_0^1 (x^2 - 4x + 3) dx - \int_1^3 (x^2 - 4x + 3) dx + \int_3^4 (x^2 - 4x + 3) dx \quad (\text{split up the integral at the zeros } x=1, 3) \\ = \left[\frac{x^3}{3} - 2x^2 + 3x \right]_0^1 - \left[\frac{x^3}{3} - 2x^2 + 3x \right]_1^3 + \left[\frac{x^3}{3} - 2x^2 + 3x \right]_3^4 \\ = \left(\frac{1}{3} - 2 + 3 \right) - (9 - 18 + 9) + \left(\frac{64}{3} - 32 + 12 \right) - (9 - 18 + 9) \\ = \frac{4}{3} - 0 + \frac{4}{3} + \frac{4}{3} - 0 = 4$$

$$28. \int_0^{\pi/4} \frac{1 - \sin^2 \theta}{\cos^2 \theta} d\theta = \int_0^{\pi/4} d\theta = \left[\theta \right]_0^{\pi/4} = \frac{\pi}{4}$$

$$30. \int_{\pi/4}^{\pi/2} (2 - \csc^2 x) dx = \left[2x + \cot x \right]_{\pi/4}^{\pi/2} = (\pi + 0) - \left(\frac{\pi}{2} + 1 \right) = \frac{\pi}{2} - 1 = \frac{\pi - 2}{2}$$

$$32. \int_{-\pi/2}^{\pi/2} (2t + \cos t) dt = \left[t^2 + \sin t \right]_{-\pi/2}^{\pi/2} = \left(\frac{\pi^2}{4} + 1 \right) - \left(\frac{\pi^2}{4} - 1 \right) = 2$$

$$34. P = \frac{2}{\pi} \int_0^{\pi/2} \sin \theta d\theta = \left[-\frac{2}{\pi} \cos \theta \right]_0^{\pi/2} = -\frac{2}{\pi}(0-1) = \frac{2}{\pi} \approx 63.7\%$$

$$36. A = \int_{-1}^1 (1-x^4) dx = \left[x - \frac{1}{5}x^5 \right]_{-1}^1 = \frac{8}{5}$$

$$38. A = \int_1^2 \frac{1}{x^2} dx = \left[-\frac{1}{x} \right]_1^2 = -\frac{1}{2} + 1 = \frac{1}{2}$$

$$40. A = \int_0^{\pi} (x + \sin x) dx = \left[\frac{x^2}{2} - \cos x \right]_0^{\pi} = \frac{\pi^2}{2} + 2 = \frac{\pi^2 + 4}{2}$$

42. Since $y \geq 0$ on $[0, 8]$,

$$\text{Area} = \int_0^8 (1 + x^{1/3}) dx = \left[x + \frac{3}{4}x^{4/3} \right]_0^8 = 8 + \frac{3}{4}(16) = 20$$

44. Since $y \geq 0$ on $[0, 3]$,

$$A = \int_0^3 (3x - x^2) dx = \left[\frac{3}{2}x^2 - \frac{x^3}{3} \right]_0^3 = \frac{9}{2}$$

$$46. \int_1^3 \frac{9}{x^3} dx = \left[-\frac{9}{2x^2} \right]_1^3 = -\frac{1}{2} + \frac{9}{2} = 4$$

$$f(c)(3 - 1) = 4$$

$$\frac{9}{c^3} = 2$$

$$c^3 = \frac{9}{2}$$

$$c = \sqrt[3]{\frac{9}{2}} \approx 1.6510$$

$$48. \int_{-\pi/3}^{\pi/3} \cos x dx = \left[\sin x \right]_{-\pi/3}^{\pi/3} = \sqrt{3}$$

$$f(c) \left[\frac{\pi}{3} - \left(-\frac{\pi}{3} \right) \right] = \sqrt{3}$$

$$\cos c = \frac{3\sqrt{3}}{2\pi}$$

$$c \approx \pm 0.5971$$

$$50. \frac{1}{3-1} \int_1^3 \frac{4(x^2+1)}{x^2} dx = 2 \int_1^3 (1+x^{-2}) dx = 2 \left[x - \frac{1}{x} \right]_1^3$$

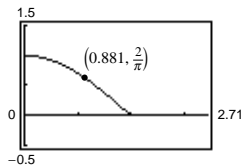
$$= 2 \left(3 - \frac{1}{3} \right) = \frac{16}{3}$$

$$52. \frac{1}{(\pi/2) - 0} \int_0^{\pi/2} \cos x dx = \left[\frac{2}{\pi} \sin x \right]_0^{\pi/2} = \frac{2}{\pi}$$

$$\text{Average value} = \frac{2}{\pi}$$

$$\cos x = \frac{2}{\pi}$$

$$x \approx 0.881$$



$$54. (a) \int_1^7 f(x) dx = \text{Sum of the areas}$$

$$= A_1 + A_2 + A_3 + A_4$$

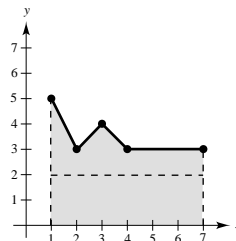
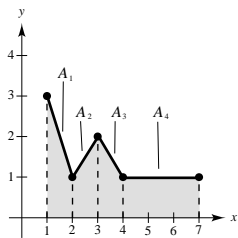
$$= \frac{1}{2}(3+1) + \frac{1}{2}(1+2) + \frac{1}{2}(2+1) + (3)(1)$$

$$= 8$$

$$(b) \text{Average value} = \frac{\int_1^7 f(x) dx}{7-1} = \frac{8}{6} = \frac{4}{3}$$

$$(c) A = 8 + (6)(2) = 20$$

$$\text{Average value} = \frac{20}{6} = \frac{10}{3}$$



$$56. \int_2^6 f(x) dx = (\text{area or region } B) = \int_0^6 f(x) dx - \int_0^2 f(x) dx = 3.5 - (-1.5) = 5.0$$

$$58. \int_0^2 -2f(x) dx = -2 \int_0^2 f(x) dx = -2(-1.5) = 3.0$$

$$60. \text{Average value} = \frac{1}{6} \int_0^6 f(x) dx = \frac{1}{6}(3.5) = 0.5833$$

$$62. \frac{1}{R-0} \int_0^R k(R^2 - r^2) dr = \frac{k}{R} \left[R^2 r - \frac{r^3}{3} \right]_0^R = \frac{2kR^2}{3}$$

$$64. P = 5(\sqrt{t} + 30)$$

(a)

t	1	2	3	4	5	6
P	155	157.071	158.660	160	161.180	162.247

$$\text{Average profit} \approx \frac{1}{6}(155 + 157.071 + 158.660 + 160 + 161.180 + 162.247) = \frac{954.158}{6} \approx 159.026$$

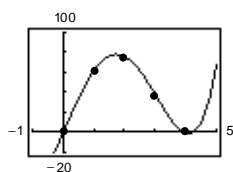
$$(b) \frac{1}{6} \int_{0.5}^{6.5} 5(\sqrt{t} + 30) dt = \frac{1}{6} \left[5 \left(\frac{2}{3} t^{3/2} + 30t \right) \right]_{0.5}^{6.5} \approx \frac{954.061}{6} \approx 159.010$$

(c) The definite integral yields a better approximation.

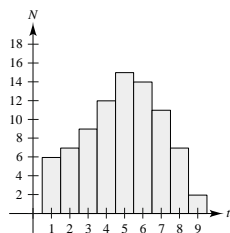
$$66. (a) R = 2.33t^4 - 14.67t^3 + 3.67t^2 + 70.67t$$

$$(c) \int_0^4 R(t) dt = \left[\frac{2.33t^5}{5} - \frac{14.67t^4}{4} + \frac{3.67t^3}{3} + \frac{70.67t^2}{2} \right]_0^4 = 181.957$$

(b)



68. (a) histogram

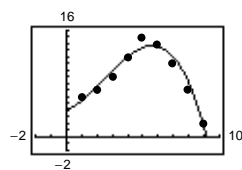


$$(b) [6 + 7 + 9 + 12 + 15 + 14 + 11 + 7 + 2]60 = (83)60 = 4980 \text{ customers}$$

(c) Using a graphing utility, you obtain

$$N(t) = -0.084175t^3 + 0.63492t^2 + 0.79052t + 4.10317.$$

(d)



$$(e) \int_0^9 N(t) dt \approx 85.162$$

The estimated number of customers is $(85.162)(60) \approx 5110$.

$$(f) \text{Between 3 P.M. and 7 P.M., the number of customers is approximately } \left(\int_3^7 N(t) dt \right) (60) \approx (50.28)(60) \approx 3017.$$

Hence, $3017/240 \approx 12.6$ per minute.

$$\begin{aligned}
 70. F(x) &= \int_2^x (t^3 + 2t - 2) dt = \left[\frac{t^4}{4} + t^2 - 2t \right]_2^x \\
 &= \left(\frac{x^4}{4} + x^2 - 2x \right) - (4 + 4 - 4) \\
 &= \frac{x^4}{4} + x^2 - 2x - 4
 \end{aligned}$$

$$F(2) = 4 + 4 - 4 - 4 = 0 \quad \left[\text{Note: } F(2) = \int_2^2 (t^3 + 2t - 2) dt = 0 \right]$$

$$F(5) = \frac{625}{4} + 25 - 10 - 4 = 167.25$$

$$F(8) = \frac{8^4}{4} + 64 - 16 - 4 = 1068$$

$$72. F(x) = \int_2^x \frac{-2}{t^3} dt = - \int_2^x 2t^{-3} dt = \left[\frac{1}{t^2} \right]_2^x = \frac{1}{x^2} - \frac{1}{4}$$

$$F(2) = \frac{1}{4} - \frac{1}{4} = 0$$

$$F(5) = \frac{1}{25} - \frac{1}{4} = -\frac{21}{100} = -0.21$$

$$F(8) = \frac{1}{64} - \frac{1}{4} = -\frac{15}{64}$$

$$74. F(x) = \int_0^x \sin \theta d\theta = -\cos \theta \Big|_0^x = -\cos x + \cos 0 = 1 - \cos x$$

$$F(2) = 1 - \cos 2 \approx 1.4161$$

$$F(5) = 1 - \cos 5 \approx 0.7163$$

$$F(8) = 1 - \cos 8 \approx 1.1455$$

$$76. (a) \int_0^x t(t^2 + 1) dt = \int_0^x (t^3 + t) dt = \left[\frac{1}{4}t^4 + \frac{1}{2}t^2 \right]_0^x = \frac{1}{4}x^4 + \frac{1}{2}x^2 = \frac{x^2}{4}(x^2 + 2)$$

$$(b) \frac{d}{dx} \left[\frac{1}{4}x^4 + \frac{1}{2}x^2 \right] = x^3 + x = x(x^2 + 1)$$

$$78. (a) \int_4^x \sqrt{t} dt = \left[\frac{2}{3}t^{3/2} \right]_4^x = \frac{2}{3}x^{3/2} - \frac{16}{3} = \frac{2}{3}(x^{3/2} - 8)$$

$$(b) \frac{d}{dx} \left[\frac{2}{3}x^{3/2} - \frac{16}{3} \right] = x^{1/2} = \sqrt{x}$$

$$80. (a) \int_{\pi/3}^x \sec t \tan t dt = \left[\sec t \right]_{\pi/3}^x = \sec x - 2$$

$$(b) \frac{d}{dx} [\sec x - 2] = \sec x \tan x$$

$$82. F(x) = \int_1^x \frac{t^2}{t^2 + 1} dt$$

$$F'(x) = \frac{x^2}{x^2 + 1}$$

$$84. F(x) = \int_1^x \sqrt[4]{t} dt$$

$$F'(x) = \sqrt[4]{x}$$

$$86. F(x) = \int_0^x \sec^3 t dt$$

$$F'(x) = \sec^3 x$$

$$88. F(x) = \int_{-x}^x t^3 dt = \left[\frac{t^4}{4} \right]_{-x}^x = 0$$

$$F'(x) = 0$$

Alternate solution

$$F(x) = \int_{-x}^x t^3 dt$$

$$= \int_{-x}^0 t^3 dt + \int_0^x t^3 dt$$

$$= -\int_0^{-x} t^3 dt + \int_0^x t^3 dt$$

$$F'(x) = -(-x)^3(-1) + (x^3) = 0$$

$$90. F(x) = \int_2^{x^2} t^{-3} dt = \left[\frac{t^{-2}}{-2} \right]_2^{x^2} = \left[-\frac{1}{2t^2} \right]_2^{x^2} = \frac{-1}{2x^4} + \frac{1}{8} \Rightarrow F'(x) = 2x^{-5}$$

Alternate solution: $F'(x) = (x^2)^{-3}(2x) = 2x^{-5}$

$$92. F(x) = \int_0^{x^2} \sin \theta^2 d\theta$$

$$F'(x) = \sin (x^2)^2 (2x) = 2x \sin x^4$$

94. (a)

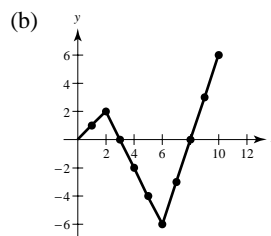
x	1	2	3	4	5	6	7	8	9	10
$g(x)$	1	2	0	-2	-4	-6	-3	0	3	6

(c) Minimum of g at $(6, -6)$.

(d) Minimum at $(10, 6)$. Relative maximum at $(2, 2)$.

(e) On $[6, 10]$ g increases at a rate of $\frac{12}{4} = 3$.

(f) Zeros of g : $x = 3, x = 8$.



$$96. (a) g(t) = 4 - \frac{4}{t^2}$$

$$\lim_{t \rightarrow \infty} g(t) = 4$$

Horizontal asymptote: $y = 4$

$$(b) A(x) = \int_1^x \left(4 - \frac{4}{t^2} \right) dt$$

$$= \left[4t + \frac{4}{t} \right]_1^x = 4x + \frac{4}{x} - 8$$

$$= \frac{4x^2 - 8x + 4}{x} = \frac{4(x-1)^2}{x}$$

$$\lim_{x \rightarrow \infty} A(x) = \lim_{x \rightarrow \infty} \left(4x + \frac{4}{x} - 8 \right) = \infty + 0 - 8 = \infty$$

The graph of $A(x)$ does not have a horizontal asymptote.

98. True

100. Let $F(t)$ be an antiderivative of $f(t)$. Then,

$$\int_{u(x)}^{v(x)} f(t) dt = \left[F(t) \right]_{u(x)}^{v(x)} = F(v(x)) - F(u(x))$$

$$\frac{d}{dx} \left[\int_{u(x)}^{v(x)} f(t) dt \right] = \frac{d}{dx} [F(v(x)) - F(u(x))]$$

$$= F'(v(x))v'(x) - F'(u(x))u'(x)$$

$$= f(v(x))v'(x) - f(u(x))u'(x).$$

$$102. G(x) = \int_0^x \left[s \int_0^s f(t) dt \right] ds$$

$$(a) G(0) = \int_0^0 \left[s \int_0^s f(t) dt \right] ds = 0$$

$$(c) G''(x) = x \cdot f(x) + \int_0^x f(t) dt$$

$$(d) G''(0) = 0 \cdot f(0) + \int_0^0 f(t) dt = 0$$

$$(b) \text{ Let } F(s) = s \int_0^s f(t) dt.$$

$$G(x) = \int_0^x F(s) ds$$

$$G'(x) = F(x) = x \int_0^x f(t) dt$$

$$G'(0) = 0 \int_0^0 f(t) dt = 0$$

$$104. x(t) = (t-1)(t-3)^2 = t^3 - 7t^2 + 15t - 9$$

$$x'(t) = 3t^2 - 14t + 15$$

Using a graphing utility,

$$\text{Total distance} = \int_0^5 |x'(t)| dt \approx 27.37 \text{ units}$$

Section 4.5 Integration by Substitution

$$\int f(g(x))g'(x) dx \quad u = g(x) \quad du = g'(x) dx$$

$$2. \int x^2 \sqrt{x^3 + 1} dx \quad u = x^3 + 1 \quad du = 3x^2 dx$$

$$4. \int \sec 2x \tan 2x dx \quad u = 2x \quad du = 2 dx$$

$$6. \int \frac{\cos x}{\sin^2 x} dx \quad u = \sin x \quad du = \cos x dx$$

$$8. \int (x^2 - 9)^3 (2x) dx = \frac{(x^2 - 9)^4}{4} + C$$

$$\text{Check: } \frac{d}{dx} \left[\frac{(x^2 - 9)^4}{4} + C \right] = \frac{4(x^2 - 9)^3}{4} (2x) = (x^2 - 9)^3 (2x)$$

$$10. \int (1 - 2x^2)^{1/3} (-4x) dx = \frac{3}{4} (1 - 2x^2)^{4/3} + C$$

$$\text{Check: } \frac{d}{dx} \left[\frac{3}{4} (1 - 2x^2)^{4/3} + C \right] = \frac{3}{4} \cdot \frac{4}{3} (1 - 2x^2)^{1/3} (-4x) = (1 - 2x^2)^{1/3} (-4x)$$

$$12. \int x^2 (x^3 + 5)^4 dx = \frac{1}{3} \int (x^3 + 5)^4 (3x^2) dx = \frac{1}{3} \frac{(x^3 + 5)^5}{5} + C = \frac{(x^3 + 5)^5}{15} + C$$

$$\text{Check: } \frac{d}{dx} \left[\frac{(x^3 + 5)^5}{15} + C \right] = \frac{5(x^3 + 5)^4 (3x^2)}{15} = (x^3 + 5)^4 x^2$$

$$14. \int x(4x^2 + 3)^3 dx = \frac{1}{8} \int (4x^2 + 3)^3 (8x) dx = \frac{1}{8} \left[\frac{(4x^2 + 3)^4}{4} \right] + C = \frac{(4x^2 + 3)^4}{32} + C$$

$$\text{Check: } \frac{d}{dx} \left[\frac{(4x^2 + 3)^4}{32} + C \right] = \frac{4(4x^2 + 3)^3 (8x)}{32} = x(4x^2 + 3)^3$$

$$16. \int t^3 \sqrt{t^4 + 5} dt = \frac{1}{4} \int (t^4 + 5)^{1/2} (4t^3) dt = \frac{1}{4} \frac{(t^4 + 5)^{3/2}}{3/2} + C = \frac{1}{6} (t^4 + 5)^{3/2} + C$$

$$\text{Check: } \frac{d}{dt} \left[\frac{1}{6} (t^4 + 5)^{3/2} + C \right] = \frac{1}{6} \cdot \frac{3}{2} (t^4 + 5)^{1/2} (4t^3) = (t^4 + 5)^{1/2} (t^3)$$

$$18. \int u^2 \sqrt{u^3 + 2} du = \frac{1}{3} \int (u^3 + 2)^{1/2} (3u^2) du = \frac{1}{3} \frac{(u^3 + 2)^{3/2}}{3/2} + C = \frac{2(u^3 + 2)^{3/2}}{9} + C$$

$$\text{Check: } \frac{d}{du} \left[\frac{2(u^3 + 2)^{3/2}}{9} + C \right] = \frac{2}{9} \cdot \frac{3}{2} (u^3 + 2)^{1/2} (3u^2) = (u^3 + 2)^{1/2} (u^2)$$

$$20. \int \frac{x^3}{(1 + x^4)^2} dx = \frac{1}{4} \int (1 + x^4)^{-2} (4x^3) dx = -\frac{1}{4} (1 + x^4)^{-1} + C = \frac{-1}{4(1 + x^4)} + C$$

$$\text{Check: } \frac{d}{dx} \left[\frac{-1}{4(1 + x^4)} + C \right] = \frac{1}{4} (1 + x^4)^{-2} (4x^3) = \frac{x^3}{(1 + x^4)^2}$$

$$22. \int \frac{x^2}{(16 - x^3)^2} dx = -\frac{1}{3} \int (16 - x^3)^{-2} (-3x^2) dx = -\frac{1}{3} \left[\frac{(16 - x^3)^{-1}}{-1} \right] + C = \frac{1}{3(16 - x^3)} + C$$

$$\text{Check: } \frac{d}{dx} \left[\frac{1}{3(16 - x^3)} + C \right] = \frac{1}{3} (-1) (16 - x^3)^{-2} (-3x^2) = \frac{x^2}{(16 - x^3)^2}$$

$$24. \int \frac{x^3}{\sqrt{1 + x^4}} dx = \frac{1}{4} \int (1 + x^4)^{-1/2} (4x^3) dx = \frac{1}{4} \frac{(1 + x^4)^{1/2}}{1/2} + C = \frac{\sqrt{1 + x^4}}{2} + C$$

$$\text{Check: } \frac{d}{dx} \left[\frac{\sqrt{1 + x^4}}{2} + C \right] = \frac{1}{2} \cdot \frac{1}{2} (1 + x^4)^{-1/2} (4x^3) = \frac{x^3}{\sqrt{1 + x^4}}$$

$$26. \int \left[x^2 + \frac{1}{(3x)^2} \right] dx = \int \left(x^2 + \frac{1}{9} x^{-2} \right) dx = \frac{x^3}{3} + \frac{1}{9} \left(\frac{x^{-1}}{-1} \right) + C = \frac{x^3}{3} - \frac{1}{9x} + C = \frac{3x^4 - 1}{9x} + C$$

$$\text{Check: } \frac{d}{dx} \left[\frac{1}{3} x^3 - \frac{1}{9} x^{-1} + C \right] = x^2 + \frac{1}{9} x^{-2} = x^2 + \frac{1}{(3x)^2}$$

$$28. \int \frac{1}{2\sqrt{x}} dx = \frac{1}{2} \int x^{-1/2} dx = \frac{1}{2} \left(\frac{x^{1/2}}{1/2} \right) + C = \sqrt{x} + C$$

$$\text{Check: } \frac{d}{dx} [\sqrt{x} + C] = \frac{1}{2\sqrt{x}}$$

$$30. \int \frac{t + 2t^2}{\sqrt{t}} dt = \int (t^{1/2} + 2t^{3/2}) dt = \frac{2}{3} t^{3/2} + \frac{4}{5} t^{5/2} + C = \frac{2}{15} t^{3/2} (5 + 6t) + C$$

$$\text{Check: } \frac{d}{dt} \left[\frac{2}{3} t^{3/2} + \frac{4}{5} t^{5/2} + C \right] = t^{1/2} + 2t^{3/2} = \frac{t + 2t^2}{\sqrt{t}}$$

$$32. \int \left(\frac{t^3}{3} + \frac{1}{4t^2} \right) dt = \int \left(\frac{1}{3} t^3 + \frac{1}{4} t^{-2} \right) dt = \frac{1}{3} \left(\frac{t^4}{4} \right) + \frac{1}{4} \left(\frac{t^{-1}}{-1} \right) + C = \frac{1}{12} t^4 - \frac{1}{4t} + C$$

$$\text{Check: } \frac{d}{dt} \left[\frac{1}{12} t^4 - \frac{1}{4t} + C \right] = \frac{1}{3} t^3 + \frac{1}{4t^2}$$

$$34. \int 2\pi y(8 - y^{3/2}) dy = 2\pi \int (8y - y^{5/2}) dy = 2\pi \left(4y^2 - \frac{2}{7}y^{7/2} \right) + C = \frac{4\pi y^2}{7}(14 - y^{3/2}) + C$$

$$\text{Check: } \frac{d}{dy} \left[\frac{4\pi y^2}{7}(14 - y^{3/2}) + C \right] = \frac{d}{dy} \left[2\pi \left(4y^2 - \frac{2}{7}y^{7/2} \right) + C \right] = 16\pi y - 2\pi y^{5/2} = (2\pi y)(8 - y^{3/2})$$

$$36. y = \int \frac{10x^2}{\sqrt{1+x^3}} dx$$

$$= \frac{10}{3} \int (1+x^3)^{-1/2} (3x^2) dx$$

$$= \frac{10}{3} \left[\frac{(1+x^3)^{1/2}}{1/2} \right] + C$$

$$= \frac{20}{3} \sqrt{1+x^3} + C$$

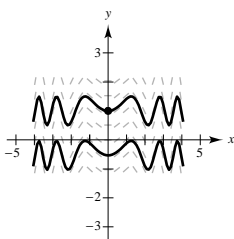
$$38. y = \int \frac{x-4}{\sqrt{x^2-8x+1}} dx$$

$$= \frac{1}{2} \int (x^2-8x+1)^{-1/2} (2x-8) dx$$

$$= \frac{1}{2} \left[\frac{(x^2-8x+1)^{1/2}}{1/2} \right] + C$$

$$= \sqrt{x^2-8x+1} + C$$

40. (a)



$$(b) \frac{dy}{dx} = x \cos x^2, (0, 1)$$

$$y = \int x \cos x^2 dx = \frac{1}{2} \int \cos(x^2) 2x dx$$

$$= \frac{1}{2} \sin(x^2) + C$$

$$(0, 1): 1 = \frac{1}{2} \sin(0) + C \Rightarrow C = 1$$

$$y = \frac{1}{2} \sin(x^2) + 1$$

$$42. \int 4x^3 \sin x^4 dx = \int \sin x^4 (4x^3) dx = -\cos x^4 + C$$

$$44. \int \cos 6x dx = \frac{1}{6} \int (\cos 6x)(6) dx = \frac{1}{6} \sin 6x + C$$

$$46. \int x \sin x^2 dx = \frac{1}{2} \int (\sin x^2)(2x) dx = -\frac{1}{2} \cos x^2 + C$$

$$48. \int \sec(1-x) \tan(1-x) dx = - \int [\sec(1-x) \tan(1-x)](-1) dx = -\sec(1-x) + C$$

$$50. \int \sqrt{\tan x} \sec^2 x dx = \frac{(\tan x)^{3/2}}{3/2} + C = \frac{2}{3}(\tan x)^{3/2} + C$$

$$52. \int \frac{\sin x}{\cos^3 x} dx = - \int (\cos x)^{-3} (-\sin x) dx = -\frac{(\cos x)^{-2}}{-2} + C = \frac{1}{2 \cos^2 x} + C = \frac{1}{2} \sec^2 x + C$$

$$54. \int \csc^2\left(\frac{x}{2}\right) dx = 2 \int \csc^2\left(\frac{x}{2}\right) \left(\frac{1}{2}\right) dx = -2 \cot\left(\frac{x}{2}\right) + C$$

$$56. f(x) = \int \pi \sec \pi x \tan \pi x dx = \sec \pi x + C$$

Since $f(1/3) = 1 = \sec(\pi/3) + C$, $C = -1$. Thus

$$f(x) = \sec \pi x - 1.$$

58. $u = 2x + 1, x = \frac{1}{2}(u - 1), dx = \frac{1}{2} du$

$$\begin{aligned} \int x\sqrt{2x+1} dx &= \int \frac{1}{2}(u-1)\sqrt{u} \frac{1}{2} du \\ &= \frac{1}{4} \int (u^{3/2} - u^{1/2}) du \\ &= \frac{1}{4} \left(\frac{2}{5} u^{5/2} - \frac{2}{3} u^{3/2} \right) + C \\ &= \frac{u^{3/2}}{30} (3u - 5) + C \\ &= \frac{1}{30} (2x+1)^{3/2} [3(2x+1) - 5] + C \\ &= \frac{1}{30} (2x+1)^{3/2} (6x-2) + C \\ &= \frac{1}{15} (2x+1)^{3/2} (3x-1) + C \end{aligned}$$

60. $u = 2 - x, x = 2 - u, dx = -du$

$$\begin{aligned} \int (x+1)\sqrt{2-x} dx &= - \int (3-u)\sqrt{u} du \\ &= - \int (3u^{1/2} - u^{3/2}) du \\ &= - \left(2u^{3/2} - \frac{2}{5} u^{5/2} \right) + C \\ &= -\frac{2u^{3/2}}{5} (5-u) + C \\ &= -\frac{2}{5} (2-x)^{3/2} [5 - (2-x)] + C \\ &= -\frac{2}{5} (2-x)^{3/2} (x+3) + C \end{aligned}$$

62. Let $u = x + 4, x = u - 4, du = dx$.

$$\begin{aligned} \int \frac{2x+1}{\sqrt{x+4}} dx &= \int \frac{2(u-4)+1}{\sqrt{u}} du \\ &= \int (2u^{1/2} - 7u^{-1/2}) du \\ &= \frac{4}{3} u^{3/2} - 14u^{1/2} + C \\ &= \frac{2}{3} u^{1/2} (2u - 21) + C \\ &= \frac{2}{3} \sqrt{x+4} [2(x+4) - 21] + C \\ &= \frac{2}{3} \sqrt{x+4} (2x-13) + C \end{aligned}$$

64. $u = t - 4, t = u + 4, dt = du$

$$\begin{aligned} \int t\sqrt[3]{t-4} dt &= \int (u+4)u^{1/3} du \\ &= \int (u^{4/3} + 4u^{1/3}) du \\ &= \frac{3}{7} u^{7/3} + 3u^{4/3} + C \\ &= \frac{3u^{4/3}}{7} (u+7) + C \\ &= \frac{3}{7} (t-4)^{4/3} [(t-4) + 7] + C \\ &= \frac{3}{7} (t-4)^{4/3} (t+3) + C \end{aligned}$$

66. Let $u = x^3 + 8, du = 3x^2 dx$.

$$\begin{aligned} \int_{-2}^4 x^2(x^3+8)^2 dx &= \frac{1}{3} \int_{-2}^4 (x^3+8)^2 (3x^2) dx = \left[\frac{1}{3} \frac{(x^3+8)^3}{3} \right]_{-2}^4 \\ &= \frac{1}{9} [(64+8)^3 - (-8+8)^3] = 41,472 \end{aligned}$$

68. Let $u = 1 - x^2, du = -2x dx$.

$$\int_0^1 x\sqrt{1-x^2} dx = -\frac{1}{2} \int_0^1 (1-x^2)^{1/2} (-2x) dx = \left[-\frac{1}{3} (1-x^2)^{3/2} \right]_0^1 = 0 + \frac{1}{3} = \frac{1}{3}$$

70. Let $u = 1 + 2x^2, du = 4x dx$.

$$\int_0^2 \frac{x}{\sqrt{1+2x^2}} dx = \frac{1}{4} \int_0^2 (1+2x^2)^{-1/2} (4x) dx = \left[\frac{1}{2} \sqrt{1+2x^2} \right]_0^2 = \frac{3}{2} - \frac{1}{2} = 1$$

72. Let $u = 4 + x^2$, $du = 2x dx$.

$$\int_0^2 x \sqrt[3]{4 + x^2} dx = \frac{1}{2} \int_0^2 (4 + x^2)^{1/3} (2x) dx = \left[\frac{3}{8} (4 + x^2)^{4/3} \right]_0^2 = \frac{3}{8} (8^{4/3} - 4^{4/3}) = 6 - \frac{3}{2} \sqrt[3]{4} \approx 3.619$$

74. Let $u = 2x - 1$, $du = 2 dx$, $x = \frac{1}{2}(u + 1)$.

When $x = 1$, $u = 1$. When $x = 5$, $u = 9$.

$$\begin{aligned} \int_1^5 \frac{x}{\sqrt{2x-1}} dx &= \int_1^9 \frac{1/2(u+1)}{\sqrt{u}} \frac{1}{2} du = \frac{1}{4} \int_1^9 (u^{1/2} + u^{-1/2}) du \\ &= \frac{1}{4} \left[\frac{2}{3} u^{3/2} + 2u^{1/2} \right]_1^9 \\ &= \frac{1}{4} \left[\left(\frac{2}{3}(27) + 2(3) \right) - \left(\frac{2}{3} + 2 \right) \right] \\ &= \frac{16}{3} \end{aligned}$$

76. $\int_{\pi/3}^{\pi/2} (x + \cos x) dx = \left[\frac{x^2}{2} + \sin x \right]_{\pi/3}^{\pi/2} = \left(\frac{\pi^2}{8} + 1 \right) - \left(\frac{\pi^2}{18} + \frac{\sqrt{3}}{2} \right) = \frac{5\pi^2}{72} + \frac{2 - \sqrt{3}}{2}$

78. $u = x + 2$, $x = u - 2$, $dx = du$

When $x = -2$, $u = 0$. When $x = 6$, $u = 8$.

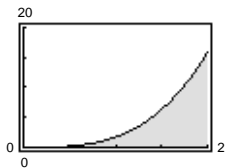
$$\text{Area} = \int_{-2}^6 x^2 \sqrt[3]{x+2} dx = \int_0^8 (u-2)^2 \sqrt[3]{u} du = \int_0^8 (u^{7/3} - 4u^{4/3} + 4u^{1/3}) du = \left[\frac{3}{10} u^{10/3} - \frac{12}{7} u^{7/3} + 3u^{4/3} \right]_0^8 = \frac{4752}{35}$$

80. $A = \int_0^\pi (\sin x + \cos 2x) dx = \left[-\cos x + \frac{1}{2} \sin 2x \right]_0^\pi = 2$

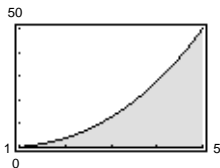
82. Let $u = 2x$, $du = 2 dx$.

$$\text{Area} = \int_{\pi/12}^{\pi/4} \csc 2x \cot 2x dx = \frac{1}{2} \int_{\pi/12}^{\pi/4} \csc 2x \cot 2x (2) dx = \left[-\frac{1}{2} \csc 2x \right]_{\pi/12}^{\pi/4} = \frac{1}{2}$$

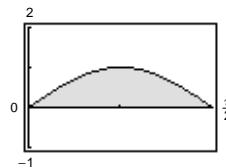
84. $\int_0^2 x^3 \sqrt{x+2} dx \approx 7.581$



86. $\int_1^5 x^2 \sqrt{x-1} dx \approx 67.505$



88. $\int_0^{\pi/2} \sin 2x dx \approx 1.0$



90. $\int \sin x \cos x dx = \int (\sin x)^1 (\cos x) dx = \frac{\sin^2 x}{2} + C_1$

$$\int \sin x \cos x dx = -\int (\cos x)^1 (-\sin x) dx = -\frac{\cos^2 x}{2} + C_2$$

$$-\frac{\cos^2 x}{2} + C_2 = -\frac{(1 - \sin^2 x)}{2} + C_2 = \frac{\sin^2 x}{2} - \frac{1}{2} + C_2$$

They differ by a constant: $C_2 = C_1 + \frac{1}{2}$.

92. $f(x) = \sin^2 x \cos x$ is even.

$$\begin{aligned} \int_{-\pi/2}^{\pi/2} \sin^2 x \cos x \, dx &= \int_0^{\pi/2} \sin^2 x (\cos x) \, dx \\ &= 2 \left[\frac{\sin^3 x}{3} \right]_0^{\pi/2} \\ &= \frac{2}{3} \end{aligned}$$

94. $f(x) = \sin x \cos x$ is odd.

$$\int_{-\pi/2}^{\pi/2} \sin x \cos x \, dx = 0$$

96. (a) $\int_{-\pi/4}^{\pi/4} \sin x \, dx = 0$ since $\sin x$ is symmetric to the origin.

(b) $\int_{-\pi/4}^{\pi/4} \cos x \, dx = 2 \int_0^{\pi/4} \cos x \, dx = \left[2 \sin x \right]_0^{\pi/4} = \sqrt{2}$ since $\cos x$ is symmetric to the y-axis.

(c) $\int_{-\pi/2}^{\pi/2} \cos x \, dx = 2 \int_0^{\pi/2} \cos x \, dx = \left[2 \sin x \right]_0^{\pi/2} = 2$

(d) $\int_{-\pi/2}^{\pi/2} \sin x \cos x \, dx = 0$ since $\sin(-x) \cos(-x) = -\sin x \cos x$ and hence, is symmetric to the origin.

98. $\int_{-\pi}^{\pi} (\sin 3x + \cos 3x) \, dx = \int_{-\pi}^{\pi} \sin 3x \, dx + \int_{-\pi}^{\pi} \cos 3x \, dx = 0 + 2 \int_0^{\pi} \cos 3x \, dx = \left[\frac{2}{3} \sin 3x \right]_0^{\pi} = 0$

100. If $u = 5 - x^2$, then $du = -2x \, dx$ and $\int x(5 - x^2)^3 \, dx = -\frac{1}{2} \int (5 - x^2)^3 (-2x) \, dx = -\frac{1}{2} \int u^3 \, du$.

102. $\frac{dQ}{dt} = k(100 - t)^2$

$$Q(t) = \int k(100 - t)^2 \, dt = -\frac{k}{3}(100 - t)^3 + C$$

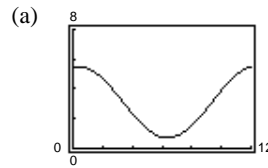
$$Q(100) = C = 0$$

$$Q(t) = -\frac{k}{3}(100 - t)^3$$

$$Q(0) = -\frac{k}{3}(100)^3 = 2,000,000 \Rightarrow k = -6$$

Thus, $Q(t) = 2(100 - t)^3$. When $t = 50$, $Q(50) = \$250,000$.

104. $R = 3.121 + 2.399 \sin(0.524t + 1.377)$

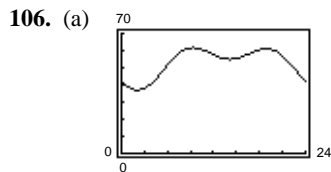


Relative minimum: (6.4, 0.7) or June

Relative maximum: (0.4, 5.5) or January

(b) $\int_0^{12} R(t) \, dt \approx 37.47$ inches

(c) $\frac{1}{3} \int_9^{12} R(t) \, dt \approx \frac{1}{3}(13) = 4.33$ inches

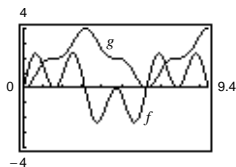


Maximum flow: $R \approx 61.713$ at $t = 9.36$.

[(18.861, 61.178) is a relative maximum.]

(b) Volume = $\int_0^{24} R(t) \, dt \approx 1272$ (5 thousand of gallons)

108. (a)

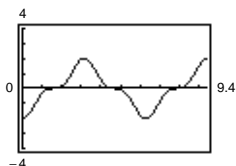


(b) g is nonnegative because the graph of f is positive at the beginning, and generally has more positive sections than negative ones.

(c) The points on g that correspond to the extrema of f are points of inflection of g .

(d) No, some zeros of f , like $x = \pi/2$, do not correspond to an extrema of g . The graph of g continues to increase after $x = \pi/2$ because f remains above the x -axis.

(e)



The graph of h is that of g shifted 2 units downward.

$$\begin{aligned} g(t) &= \int_0^t f(x) dx \\ &= \int_0^{\pi/2} f(x) dx + \int_{\pi/2}^t f(x) dx = 2 + h(t). \end{aligned}$$

110. False

$$\int x(x^2 + 1)^2 dx = \frac{1}{2} \int (x^2 + 1)(2x) dx = \frac{1}{4}(x^2 + 1)^2 + C$$

112. True

$$\int_a^b \sin x dx = \left[-\cos x \right]_a^b = -\cos b + \cos a = -\cos(b + 2\pi) + \cos a = \int_a^{b+2\pi} \sin x dx$$

114. False

$$\int \sin^2 2x \cos 2x dx = \frac{1}{2} \int (\sin 2x)^2 (2 \cos 2x) dx = \frac{1}{2} \frac{(\sin 2x)^3}{3} + C = \frac{1}{6} \sin^3 2x + C$$

116. Because f is odd, $f(-x) = -f(x)$. Then

$$\begin{aligned} \int_{-a}^a f(x) dx &= \int_{-a}^0 f(x) dx + \int_0^a f(x) dx \\ &= -\int_0^{-a} f(x) dx + \int_0^a f(x) dx. \end{aligned}$$

Let $x = -u$, $dx = -du$ in the first integral.

When $x = 0$, $u = 0$. When $x = -a$, $u = a$.

$$\begin{aligned} \int_{-a}^a f(x) dx &= -\int_0^a f(-u)(-du) + \int_0^a f(x) dx \\ &= -\int_0^a f(u) du + \int_0^a f(x) dx = 0 \end{aligned}$$

Section 4.6 Numerical Integration

2. Exact: $\int_0^1 \left(\frac{x^2}{2} + 1\right) dx = \left[\frac{x^3}{6} + x\right]_0^1 = \frac{7}{6} \approx 1.1667$

Trapezoidal: $\int_0^1 \left(\frac{x^2}{2} + 1\right) dx \approx \frac{1}{8} \left[1 + 2\left(\frac{(1/4)^2}{2} + 1\right) + 2\left(\frac{(1/2)^2}{2} + 1\right) + 2\left(\frac{(3/4)^2}{2} + 1\right) + \left(\frac{1^2}{2} + 1\right)\right] = \frac{75}{64} \approx 1.1719$

Simpson's: $\int_0^1 \left(\frac{x^2}{2} + 1\right) dx \approx \frac{1}{12} \left[1 + 4\left(\frac{(1/4)^2}{2} + 1\right) + 2\left(\frac{(1/2)^2}{2} + 1\right) + 4\left(\frac{(3/4)^2}{2} + 1\right) + \left(\frac{1^2}{2} + 1\right)\right] = \frac{7}{6} \approx 1.1667$

4. Exact: $\int_1^2 \frac{1}{x^2} dx = \left[\frac{-1}{x}\right]_1^2 = 0.5000$

Trapezoidal: $\int_1^2 \frac{1}{x^2} dx \approx \frac{1}{8} \left[1 + 2\left(\frac{4}{5}\right)^2 + 2\left(\frac{4}{6}\right)^2 + 2\left(\frac{4}{7}\right)^2 + \frac{1}{4}\right] \approx 0.5090$

Simpson's: $\int_1^2 \frac{1}{x^2} dx \approx \frac{1}{12} \left[1 + 4\left(\frac{4}{5}\right)^2 + 2\left(\frac{4}{6}\right)^2 + 4\left(\frac{4}{7}\right)^2 + \frac{1}{4}\right] \approx 0.5004$

6. Exact: $\int_0^8 \sqrt[3]{x} dx = \left[\frac{3}{4}x^{4/3}\right]_0^8 = 12.0000$

Trapezoidal: $\int_0^8 \sqrt[3]{x} dx \approx \frac{1}{2} [0 + 2 + 2\sqrt[3]{2} + 2\sqrt[3]{3} + 2\sqrt[3]{4} + 2\sqrt[3]{5} + 2\sqrt[3]{6} + 2\sqrt[3]{7} + 2] \approx 11.7296$

Simpson's: $\int_0^8 \sqrt[3]{x} dx \approx \frac{1}{3} [0 + 4 + 2\sqrt[3]{2} + 4\sqrt[3]{3} + 2\sqrt[3]{4} + 4\sqrt[3]{5} + 2\sqrt[3]{6} + 4\sqrt[3]{7} + 2] \approx 11.8632$

8. Exact: $\int_1^3 (4 - x^2) dx = \left[4x - \frac{x^3}{3}\right]_1^3 = 3 - \frac{11}{3} = -\frac{2}{3} \approx -0.6667$

Trapezoidal: $\int_1^3 (4 - x^2) dx \approx \frac{1}{4} \left\{3 + 2\left[4 - \left(\frac{3}{2}\right)^2\right] + 2(0) + 2\left[4 - \left(\frac{5}{2}\right)^2\right] - 5\right\} = -0.7500$

Simpson's: $\int_1^3 (4 - x^2) dx \approx \frac{1}{6} \left[3 + 4\left(4 - \frac{9}{4}\right) + 0 + 4\left(4 - \frac{25}{4}\right) - 5\right] \approx -0.6667$

10. Exact: $\int_0^2 x\sqrt{x^2 + 1} dx = \frac{1}{3} \left[(x^2 + 1)^{3/2}\right]_0^2 = \frac{1}{3}(5^{3/2} - 1) \approx 3.393$

Trapezoidal: $\int_0^2 x\sqrt{x^2 + 1} dx \approx \frac{1}{4} \left[0 + 2\left(\frac{1}{2}\right)\sqrt{(1/2)^2 + 1} + 2(1)\sqrt{1^2 + 1} + 2\left(\frac{3}{2}\right)\sqrt{(3/2)^2 + 1} + 2\sqrt{2^2 + 1}\right] \approx 3.457$

Simpson's: $\int_0^2 x\sqrt{x^2 + 1} dx \approx \frac{1}{6} \left[0 + 4\left(\frac{1}{2}\right)\sqrt{(1/2)^2 + 1} + 2(1)\sqrt{1^2 + 1} + 4\left(\frac{3}{2}\right)\sqrt{(3/2)^2 + 1} + 2\sqrt{2^2 + 1}\right] \approx 3.392$

12. Trapezoidal: $\int_0^2 \frac{1}{\sqrt{1 + x^3}} dx \approx \frac{1}{4} \left[1 + 2\left(\frac{1}{\sqrt{1 + (1/2)^3}}\right) + 2\left(\frac{1}{\sqrt{1 + 1^3}}\right) + 2\left(\frac{1}{\sqrt{1 + (3/2)^3}}\right) + \frac{1}{3}\right] \approx 1.397$

Simpson's: $\int_0^2 \frac{1}{\sqrt{1 + x^3}} dx \approx \frac{1}{6} \left[1 + 4\left(\frac{1}{\sqrt{1 + (1/2)^3}}\right) + 2\left(\frac{1}{\sqrt{1 + 1^3}}\right) + 4\left(\frac{1}{\sqrt{1 + (3/2)^3}}\right) + \frac{1}{3}\right] \approx 1.405$

Graphing utility: 1.402

14. Trapezoidal: $\int_{\pi/2}^{\pi} \sqrt{x} \sin x \, dx \approx \frac{\pi}{16} \left[\sqrt{\frac{\pi}{2}}(1) + 2\sqrt{\frac{5\pi}{8}} \sin\left(\frac{5\pi}{8}\right) + 2\sqrt{\frac{3\pi}{4}} \sin\left(\frac{3\pi}{4}\right) + 2\sqrt{\frac{7\pi}{8}} \sin\left(\frac{7\pi}{8}\right) + 0 \right] \approx 1.430$

Simpson's: $\int_{\pi/2}^{\pi} \sqrt{x} \sin x \, dx \approx \frac{\pi}{24} \left[\sqrt{\frac{\pi}{2}} + 4\sqrt{\frac{5\pi}{8}} \sin\left(\frac{5\pi}{8}\right) + 2\sqrt{\frac{3\pi}{4}} \sin\left(\frac{3\pi}{4}\right) + 4\sqrt{\frac{7\pi}{8}} \sin\left(\frac{7\pi}{8}\right) + 0 \right] \approx 1.458$

Graphing utility: 1.458

16. Trapezoidal: $\int_0^{\sqrt{\pi/4}} \tan(x^2) \, dx \approx \frac{\sqrt{\pi/4}}{8} \left[\tan 0 + 2 \tan\left(\frac{\sqrt{\pi/4}}{4}\right)^2 + 2 \tan\left(\frac{\sqrt{\pi/4}}{2}\right)^2 + 2 \tan\left(\frac{3\sqrt{\pi/4}}{4}\right)^2 + \tan\left(\sqrt{\frac{\pi}{4}}\right)^2 \right]$
 ≈ 0.271

Simpson's: $\int_0^{\sqrt{\pi/4}} \tan(x^2) \, dx \approx \frac{\sqrt{\pi/4}}{12} \left[\tan 0 + 4 \tan\left(\frac{\sqrt{\pi/4}}{4}\right)^2 + 2 \tan\left(\frac{\sqrt{\pi/4}}{2}\right)^2 + 4 \tan\left(\frac{3\sqrt{\pi/4}}{4}\right)^2 + \tan\left(\sqrt{\frac{\pi}{4}}\right)^2 \right]$
 ≈ 0.257

Graphing utility: 0.256

18. Trapezoidal: $\int_0^{\pi/2} \sqrt{1 + \cos^2 x} \, dx \approx \frac{\pi}{16} \left[\sqrt{2} + 2\sqrt{1 + \cos^2(\pi/8)} + 2\sqrt{1 + \cos^2(\pi/4)} + 2\sqrt{1 + \cos^2(3\pi/8)} + 1 \right] \approx 1.910$

Simpson's: $\int_0^{\pi/2} \sqrt{1 + \cos^2 x} \, dx \approx \frac{\pi}{24} \left[\sqrt{2} + 4\sqrt{1 + \cos^2(\pi/8)} + 2\sqrt{1 + \cos^2(\pi/4)} + 4\sqrt{1 + \cos^2(3\pi/8)} + 1 \right] \approx 1.910$

Graphing utility: 1.910

20. Trapezoidal: $\int_0^{\pi} \frac{\sin x}{x} \, dx \approx \frac{\pi}{8} \left[1 + \frac{2 \sin(\pi/4)}{\pi/4} + \frac{2 \sin(\pi/2)}{\pi/2} + \frac{2 \sin(3\pi/4)}{3\pi/4} + 0 \right] \approx 1.836$

Simpson's: $\int_0^{\pi} \frac{\sin x}{x} \, dx \approx \frac{\pi}{12} \left[1 + \frac{4 \sin(\pi/4)}{\pi/4} + \frac{2 \sin(\pi/2)}{\pi/2} + \frac{4 \sin(3\pi/4)}{3\pi/4} + 0 \right] \approx 1.852$

Graphing utility: 1.852

22. Trapezoidal: Linear polynomials

Simpson's: Quadratic polynomials

24. $f(x) = \frac{1}{x+1}$

$$f'(x) = \frac{-1}{(x+1)^2}$$

$$f''(x) = \frac{2}{(x+1)^3}$$

$$f'''(x) = \frac{-6}{(x+1)^4}$$

$$f^{(4)}(x) = \frac{24}{(x+1)^5}$$

(a) Trapezoidal: Error $\leq \frac{(1-0)^2}{12(4^2)}(2) = \frac{1}{96} \approx 0.01$ since

$f''(x)$ is maximum in $[0, 1]$ when $x = 0$.

(b) Simpson's: Error $\leq \frac{(1-0)^5}{180(4^4)}(24) = \frac{1}{1920} \approx 0.0005$

since $f^{(4)}(x)$ is maximum in $[0, 1]$ when $x = 0$.

26. $f''(x) = \frac{2}{(1+x)^3}$ in $[0, 1]$.

(a) $|f''(x)|$ is maximum when $x = 0$ and $|f''(0)| = 2$.

Trapezoidal: Error $\leq \frac{1}{12n^2}(2) < 0.00001, n^2 > 16,666.67, n > 129.10$; let $n = 130$.

$$f^{(4)}(x) = \frac{24}{(1+x)^5} \text{ in } [0, 1]$$

(b) $|f^{(4)}(x)|$ is maximum when $x = 0$ and $|f^{(4)}(0)| = 24$.

Simpson's: Error $\leq \frac{1}{180n^4}(24) < 0.00001, n^4 > 13,333.33, n > 10.75$; let $n = 12$. (In Simpson's Rule n must be even.)

28. $f(x) = (x+1)^{2/3}$

(a) $f''(x) = -\frac{2}{9(x+1)^{4/3}}$ in $[0, 2]$.

$|f''(x)|$ is maximum when $x = 0$ and $|f''(0)| = \frac{2}{9}$.

Trapezoidal: Error $\leq \frac{8}{12n^4}\left(\frac{2}{9}\right) < 0.00001, n^2 > 14,814.81, n > 121.72$; let $n = 122$.

(b) $f^{(4)}(x) = -\frac{56}{81(x+1)^{10/3}}$ in $[0, 2]$

$|f^{(4)}(x)|$ is maximum when $x = 0$ and $|f^{(4)}(0)| = \frac{56}{81}$.

Simpson's: Error $\leq \frac{32}{180n^4}\left(\frac{56}{81}\right) < 0.00001, n^4 > 12,290.81, n > 10.53$; let $n = 12$. (In Simpson's Rule n must be even.)

30. $f(x) = \sin(x^2)$

(a) $f''(x) = 2[-2x^2 \sin(x^2) + \cos(x^2)]$ in $[0, 1]$.

$|f''(x)|$ is maximum when $x = 1$ and $|f''(1)| \approx 2.2853$.

Trapezoidal: Error $\leq \frac{(1-0)^3}{12n^2}(2.2853) < 0.00001, n^2 > 19,044.17, n > 138.00$; let $n = 139$.

(b) $f^{(4)}(x) = (16x^4 - 12) \sin(x^2) - 48x^2 \cos(x^2)$ in $[0, 1]$

$|f^{(4)}(x)|$ is maximum when $x \approx 0.852$ and $|f^{(4)}(0.852)| \approx 28.4285$.

Simpson's: Error $\leq \frac{(1-0)^5}{180n^4}(28.4285) < 0.00001, n^4 > 15,793.61, n > 11.21$; let $n = 12$.

32. The program will vary depending upon the computer or programmable calculator that you use.

34. $f(x) = \sqrt{1-x^2}$ on $[0, 1]$.

n	$L(n)$	$M(n)$	$R(n)$	$T(n)$	$S(n)$
4	0.8739	0.7960	0.6239	0.7489	0.7709
8	0.8350	0.7892	0.7100	0.7725	0.7803
10	0.8261	0.7881	0.7261	0.7761	0.7818
12	0.8200	0.7875	0.7367	0.7783	0.7826
16	0.8121	0.7867	0.7496	0.7808	0.7836
20	0.8071	0.7864	0.7571	0.7821	0.7841

36. $f(x) = \frac{\sin x}{x}$ on $[1, 2]$.

n	$L(n)$	$M(n)$	$R(n)$	$T(n)$	$S(n)$
4	0.7070	0.6597	0.6103	0.6586	0.6593
8	0.6833	0.6594	0.6350	0.6592	0.6593
10	0.6786	0.6594	0.6399	0.6592	0.6593
12	0.6754	0.6594	0.6431	0.6593	0.6593
16	0.6714	0.6594	0.6472	0.6593	0.6593
20	0.6690	0.6593	0.6496	0.6593	0.6593

38. Simpson's Rule: $n = 8$

$$8\sqrt{3} \int_0^{\pi/2} \sqrt{1 - \frac{2}{3} \sin^2 \theta} d\theta \approx \frac{\sqrt{3}\pi}{6} \left[\sqrt{1 - \frac{2}{3} \sin^2 0} + 4\sqrt{1 - \frac{2}{3} \sin^2 \frac{\pi}{16}} + 2\sqrt{1 - \frac{2}{3} \sin^2 \frac{\pi}{8}} + \cdots + \sqrt{1 - \frac{2}{3} \sin^2 \frac{\pi}{2}} \right] \\ \approx 17.476$$

40. (a) Trapezoidal:

$$\int_0^2 f(x) dx \approx \frac{2}{2(8)} [4.32 + 2(4.36) + 2(4.58) + 2(5.79) + 2(6.14) + 2(7.25) + 2(7.64) + 2(8.08) + 8.14] \approx 12.518$$

Simpson's:

$$\int_0^2 f(x) dx \approx \frac{2}{3(8)} [4.32 + 4(4.36) + 2(4.58) + 4(5.79) + 2(6.14) + 4(7.25) + 2(7.64) + 4(8.08) + 8.14] \approx 12.592$$

(b) Using a graphing utility,

$$y = -1.3727x^3 + 4.0092x^2 - 0.6202x + 4.2844$$

$$\text{Integrating, } \int_0^2 y dx \approx 12.53$$

42. Simpson's Rule: $n = 6$

$$\pi = 4 \int_0^1 \frac{1}{1+x^2} dx \approx \frac{4}{3(6)} \left[1 + \frac{4}{1+(1/6)^2} + \frac{2}{1+(2/6)^2} + \frac{4}{1+(3/6)^2} + \frac{2}{1+(4/6)^2} + \frac{4}{1+(5/6)^2} + \frac{1}{2} \right] \\ \approx 3.14159$$

44. Area $\approx \frac{120}{2(12)} [75 + 2(81) + 2(84) + 2(76) + 2(67) + 2(68) + 2(69) + 2(72) + 2(68) + 2(56) + 2(42) + 2(23) + 0]$
 $= 7435$ sq m

46. The quadratic polynomial

$$p(x) = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} y_1 + \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)} y_3$$

passes through the three points.