# CHAPTER 4
# Integration

# C H A P T E R 4
# Integration

## Section 4.1 Antiderivatives and Indefinite Integration

Solutions to Odd-Numbered Exercises

**1.** $\frac{d}{dx}\left(\frac{3}{x^3} + C\right) = \frac{d}{dx}(3x^{-3} + C) = -9x^{-4} = \frac{-9}{x^4}$

**3.** $\frac{d}{dx}\left(\frac{1}{3}x^3 - 4x + C\right) = x^2 - 4 = (x - 2)(x + 2)$

**5.** $\frac{dy}{dt} = 3t^2$

$y = t^3 + C$

**Check:** $\frac{d}{dt}[t^3 + C] = 3t^2$

**7.** $\frac{dy}{dx} = x^{3/2}$

$y = \frac{2}{5}x^{5/2} + C$

**Check:** $\frac{d}{dx}\left[\frac{2}{5}x^{5/2} + C\right] = x^{3/2}$

| | *Given* | *Rewrite* | *Integrate* | *Simplify* |
|---|---|---|---|---|
| **9.** | $\int \sqrt[3]{x}\,dx$ | $\int x^{1/3}\,dx$ | $\frac{x^{4/3}}{4/3} + C$ | $\frac{3}{4}x^{4/3} + C$ |
| **11.** | $\int \frac{1}{x\sqrt{x}}\,dx$ | $\int x^{-3/2}\,dx$ | $\frac{x^{-1/2}}{-1/2} + C$ | $-\frac{2}{\sqrt{x}} + C$ |
| **13.** | $\int \frac{1}{2x^3}\,dx$ | $\frac{1}{2}\int x^{-3}\,dx$ | $\frac{1}{2}\left(\frac{x^{-2}}{-2}\right) + C$ | $-\frac{1}{4x^2} + C$ |

**15.** $\int (x + 3)dx = \frac{x^2}{2} + 3x + C$

**Check:** $\frac{d}{dx}\left[\frac{x^2}{2} + 3x + C\right] = x + 3$

**17.** $\int (2x - 3x^2)dx = x^2 - x^3 + C$

**Check:** $\frac{d}{dx}[x^2 - x^3 + C] = 2x - 3x^2$

**19.** $\int (x^3 + 2)\,dx = \frac{1}{4}x^4 + 2x + C$

**Check:** $\frac{d}{dx}\left(\frac{1}{4}x^4 + 2x + C\right) = x^3 + 2$

**21.** $\int (x^{3/2} + 2x + 1)\,dx = \frac{2}{5}x^{5/2} + x^2 + x + C$

**Check:** $\frac{d}{dx}\left(\frac{2}{5}x^{5/2} + x^2 + x + C\right) = x^{3/2} + 2x + 1$

**23.** $\int \sqrt[3]{x^2}\,dx = \int x^{2/3}\,dx = \frac{x^{5/3}}{5/3} + C = \frac{3}{5}x^{5/3} + C$

**Check:** $\frac{d}{dx}\left(\frac{3}{5}x^{5/3} + C\right) = x^{2/3} = \sqrt[3]{x^2}$

**25.** $\int \frac{1}{x^3}\,dx = \int x^{-3}\,dx = \frac{x^{-2}}{-2} + C = -\frac{1}{2x^2} + C$

**Check:** $\frac{d}{dx}\left(-\frac{1}{2x^2} + C\right) = \frac{1}{x^3}$

**27.** $\displaystyle\int \frac{x^2 + x + 1}{\sqrt{x}}\,dx = \int (x^{3/2} + x^{1/2} + x^{-1/2})\,dx = \frac{2}{5}x^{5/2} + \frac{2}{3}x^{3/2} + 2x^{1/2} + C = \frac{2}{15}x^{1/2}(3x^2 + 5x + 15) + C$

**Check:** $\displaystyle\frac{d}{dx}\left(\frac{2}{5}x^{5/2} + \frac{2}{3}x^{3/2} + 2x^{1/2} + C\right) = x^{3/2} + x^{1/2} + x^{-1/2} = \frac{x^2 + x + 1}{\sqrt{x}}$

**29.** $\displaystyle\int (x + 1)(3x - 2)\,dx = \int (3x^2 + x - 2)\,dx$

$\displaystyle = x^3 + \frac{1}{2}x^2 - 2x + C$

**Check:** $\displaystyle\frac{d}{dx}\left(x^3 + \frac{1}{2}x^2 - 2x + C\right) = 3x^2 + x - 2$

$= (x + 1)(3x - 2)$

**31.** $\displaystyle\int y^2\sqrt{y}\,dy = \int y^{5/2}\,dy = \frac{2}{7}y^{7/2} + C$

**Check:** $\displaystyle\frac{d}{dy}\left(\frac{2}{7}y^{7/2} + C\right) = y^{5/2} = y^2\sqrt{y}$

**33.** $\displaystyle\int dx = \int 1\,dx = x + C$

**Check:** $\displaystyle\frac{d}{dx}(x + C) = 1$

**35.** $\displaystyle\int (2\sin x + 3\cos x)\,dx = -2\cos x + 3\sin x + C$

**Check:** $\displaystyle\frac{d}{dx}(-2\cos x + 3\sin x + C) = 2\sin x + 3\cos x$

**37.** $\displaystyle\int (1 - \csc t\cot t)\,dt = t + \csc t + C$

**Check:** $\displaystyle\frac{d}{dt}(t + \csc t + C) = 1 - \csc t\cot t$

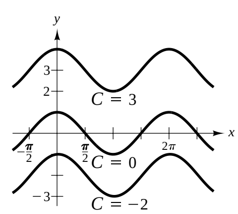
**39.** $\displaystyle\int (\sec^2\theta - \sin\theta)\,d\theta = \tan\theta + \cos\theta + C$

**Check:** $\displaystyle\frac{d}{d\theta}(\tan\theta + \cos\theta + C) = \sec^2\theta - \sin\theta$

**41.** $\displaystyle\int (\tan^2 y + 1)\,dy = \int \sec^2 y\,dy = \tan y + C$

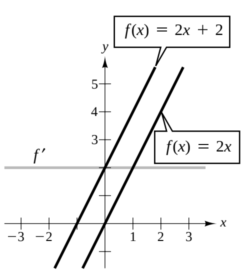
**Check:** $\displaystyle\frac{d}{dy}(\tan y + C) = \sec^2 y = \tan^2 y + 1$

**43.** $f(x) = \cos x$




**45.** $f'(x) = 2$

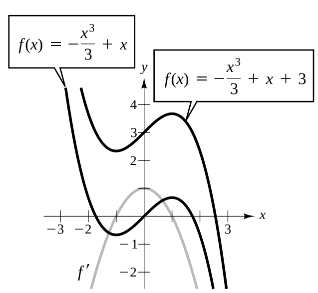
$f(x) = 2x + C$




Answers will vary.

**47.** $f'(x) = 1 - x^2$

$\displaystyle f(x) = x - \frac{x^3}{3} + C$




Answers will vary.

**49.** $\displaystyle\frac{dy}{dx} = 2x - 1,\ (1, 1)$

$\displaystyle y = \int (2x - 1)\,dx = x^2 - x + C$

$1 = (1)^2 - (1) + C \Rightarrow C = 1$

$y = x^2 - x + 1$

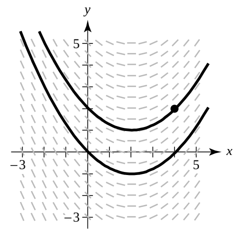
**51.** $\dfrac{dy}{dx} = \cos x, \ (0, 4)$

$$y = \int \cos x\, dx = \sin x + C$$

$$4 = \sin 0 + C \implies C = 4$$

$$y = \sin x + 4$$

**53.** (a) Answers will vary.



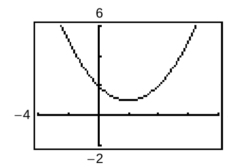

(b) $\dfrac{dy}{dx} = \dfrac{1}{2}x - 1, \ (4, 2)$

$$y = \frac{x^2}{4} - x + C$$

$$2 = \frac{4^2}{4} - 4 + C$$

$$2 = C$$

$$y = \frac{x^2}{4} - x + 2$$




**55.** $f'(x) = 4x, f(0) = 6$

$$f(x) = \int 4x\, dx = 2x^2 + C$$

$$f(0) = 6 = 2(0)^2 + C \implies C = 6$$

$$f(x) = 2x^2 + 6$$

**57.** $h'(t) = 8t^3 + 5, h(1) = -4$

$$h(t) = \int (8t^3 + 5) dt = 2t^4 + 5t + C$$

$$h(1) = -4 = 2 + 5 + C \implies C = -11$$

$$h(t) = 2t^4 + 5t - 11$$

**59.** $f''(x) = 2$

$f'(2) = 5$

$f(2) = 10$

$$f'(x) = \int 2\, dx = 2x + C_1$$

$$f'(2) = 4 + C_1 = 5 \implies C_1 = 1$$

$$f'(x) = 2x + 1$$

$$f(x) = \int (2x + 1)\, dx = x^2 + x + C_2$$

$$f(2) = 6 + C_2 = 10 \implies C_2 = 4$$

$$f(x) = x^2 + x + 4$$

**61.** $f''(x) = x^{-3/2}$

$f'(4) = 2$

$f(0) = 0$

$$f'(x) = \int x^{-3/2}\, dx = -2x^{-1/2} + C_1 = -\frac{2}{\sqrt{x}} + C_1$$

$$f'(4) = -\frac{2}{2} + C_1 = 2 \implies C_1 = 3$$

$$f'(x) = -\frac{2}{\sqrt{x}} + 3$$

$$f(x) = \int (-2x^{-1/2} + 3)\, dx = -4x^{1/2} + 3x + C_2$$

$$f(0) = 0 + 0 + C_2 = 0 \implies C_2 = 0$$

$$f(x) = -4x^{1/2} + 3x = -4\sqrt{x} + 3x$$

**63.** (a) $h(t) = \displaystyle\int (1.5t + 5)\, dt = 0.75t^2 + 5t + C$

$$h(0) = 0 + 0 + C = 12 \implies C = 12$$

$$h(t) = 0.75t^2 + 5t + 12$$

(b) $h(6) = 0.75(6)^2 + 5(6) + 12 = 69$ cm

**65.** $f(0) = -4$. Graph of $f'$ is given.

(a) $f'(4) \approx -1.0$

(b) No. The slopes of the tangent lines are greater than 2 on $[0, 2]$. Therefore, $f$ must increase more than 4 units on $[0, 4]$.

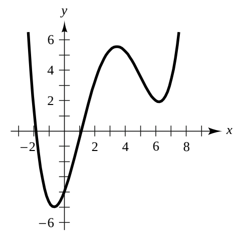
(c) No, $f(5) < f(4)$ because $f$ is decreasing on $[4, 5]$.

(d) $f$ is an maximum at $x = 3.5$ because $f'(3.5) \approx 0$ and the first derivative test.

(e) $f$ is concave upward when $f'$ is increasing on $(-\infty, 1)$ and $(5, \infty)$. $f$ is concave downward on $(1, 5)$. Points of inflection at $x = 1, 5$.

(f) $f''$ is a minimum at $x = 3$.

(g)




**67.** $a(t) = -32$ ft/sec$^2$

$v(t) = \int -32\,dt = -32t + C_1$

$v(0) = 60 = C_1$

$s(t) = \int (-32t + 60)dt = -16t^2 + 60t + C_2$

$s(0) = 6 = C_2$

$s(t) = -16t^2 + 60t + 6$ Position function

The ball reaches its maximim height when

$v(t) = -32t + 60 = 0$

$32t = 60$

$t = \frac{15}{8}$ seconds

$s\left(\frac{15}{18}\right) = -16\left(\frac{15}{8}\right)^2 + 60\left(\frac{15}{8}\right) + 6 \approx 62.26$ feet

**69.** From Exercise 68, we have:

$s(t) = -16t^2 + v_0 t$

$s'(t) = -32t + v_0 = 0$ when $t = \frac{v_0}{32}$ = time to reach maximum height.

$$s\left(\frac{v_0}{32}\right) = -16\left(\frac{v_0}{32}\right)^2 + v_0\left(\frac{v_0}{32}\right) = 550$$

$$-\frac{v_0^2}{64} + \frac{v_0^2}{32} = 550$$

$$v_0^2 = 35{,}200$$

$$v_0 \approx 187.617 \text{ ft/sec}$$

**71.** $a(t) = -9.8$

$v(t) = \int -9.8\,dt = -9.8t + C_1$

$v(0) = v_0 = C_1 \Rightarrow v(t) = -9.8t + v_0$

$f(t) = \int (-9.8t + v_0)\,dt = -4.9t^2 + v_0 t + C_2$

$f(0) = s_0 = C_2 \Rightarrow f(t) = -4.9t^2 + v_0 t + s_0$

**73.** From Exercise 71, $f(t) = -4.9t^2 + 10t + 2$.

$v(t) = -9.8t + 10 = 0$ (Maximum height when $v = 0$.)

$9.8t = 10$

$t = \frac{10}{9.8}$

$f\left(\frac{10}{9.8}\right) \approx 7.1$ m

**75.** $a = -1.6$

$v(t) = \int -1.6\,dt = -1.6t + v_0 = -1.6t$, since the stone was dropped, $v_0 = 0$.

$s(t) = \int (-1.6t)\,dt = -0.8t^2 + s_0$

$s(20) = 0 \Rightarrow -0.8(20)^2 + s_0 = 0$

$s_0 = 320$

Thus, the height of the cliff is 320 meters.

$v(t) = -1.6t$

$v(20) = -32$ m/sec

**77.** $x(t) = t^3 - 6t^2 + 9t - 2 \qquad 0 \le t \le 5$

(a) $v(t) = x'(t) = 3t^2 - 12t + 9$

$= 3(t^2 - 4t + 3) = 3(t - 1)(t - 3)$

$a(t) = v'(t) = 6t - 12 = 6(t - 2)$

(b) $v(t) > 0$ when $0 < t < 1$ or $3 < t < 5$.

(c) $a(t) = 6(t - 2) = 0$ when $t = 2$.

$v(2) = 3(1)(-1) = -3$

**79.** $v(t) = \dfrac{1}{\sqrt{t}} = t^{-1/2} \qquad t > 0$

$x(t) = \int v(t)\,dt = 2t^{1/2} + C$

$x(1) = 4 = 2(1) + C \implies C = 2$

$x(t) = 2t^{1/2} + 2$ position function

$a(t) = v'(t) = -\dfrac{1}{2}t^{-3/2} = \dfrac{-1}{2t^{3/2}}$ acceleration

**81.** (a) $v(0) = 25\text{ km/hr} = 25 \cdot \dfrac{1000}{3600} = \dfrac{250}{36}\text{ m/sec}$

$v(13) = 80\text{ km/hr} = 80 \cdot \dfrac{1000}{3600} = \dfrac{800}{36}\text{ m/sec}$

$a(t) = a$ (constant acceleration)

$v(t) = at + C$

$v(0) = \dfrac{250}{36} \implies v(t) = at + \dfrac{250}{36}$

$v(13) = \dfrac{800}{36} = 13a + \dfrac{250}{36}$

$\dfrac{550}{36} = 13a$

$a = \dfrac{550}{468} = \dfrac{275}{234} \approx 1.175\text{ m/sec}^2$

(b) $s(t) = a\dfrac{t^2}{2} + \dfrac{250}{36}t \qquad (s(0) = 0)$

$s(13) = \dfrac{275}{234}\dfrac{(13)^2}{2} + \dfrac{250}{36}(13) \approx 189.58\text{ m}$

**83.** Truck: $v(t) = 30$

$s(t) = 30t$ (Let $s(0) = 0$.)

Automobile: $a(t) = 6$

$v(t) = 6t$ (Let $v(0) = 0$.)

$s(t) = 3t^2$ (Let $s(0) = 0$.)

At the point where the automobile overtakes the truck:

$30t = 3t^2$

$0 = 3t^2 - 30t$

$0 = 3t(t - 10)$ when $t = 10$ sec.

(a) $s(10) = 3(10)^2 = 300$ ft

(b) $v(10) = 6(10) = 60$ ft/sec $\approx 41$ mph

**85.** $\dfrac{(1\text{ mi/hr})(5280\text{ ft/mi})}{(3600\text{ sec/hr})} = \dfrac{22}{15}\text{ ft/sec}$

(a)

| $t$ | 0 | 5 | 10 | 15 | 20 | 25 | 30 |
|---|---|---|---|---|---|---|---|
| $V_1$(ft/sec) | 0 | 3.67 | 10.27 | 23.47 | 42.53 | 66 | 95.33 |
| $V_2$(ft/sec) | 0 | 30.8 | 55.73 | 74.8 | 88 | 93.87 | 95.33 |

(b) $V_1(t) = 0.1068t^2 - 0.0416t + 0.3679$

$V_2(t) = -0.1208t^2 + 6.7991t - 0.0707$

(c) $S_1(t) = \int V_1(t)\,dt = \dfrac{0.1068}{3}t^3 - \dfrac{0.0416}{2}t^2 + 0.3679t$

$S_2(t) = \int V_2(t)\,dt = -\dfrac{0.1208t^3}{3} + \dfrac{6.7991t^2}{2} - 0.0707t$

[In both cases, the constant of integration is 0 because $S_1(0) = S_2(0) = 0$]

$S_1(30) \approx 953.5$ feet

$S_2(30) \approx 1970.3$ feet

The second car was going faster than the first until the end.

**87.** $a(t) = k$

$v(t) = kt$

$s(t) = \frac{k}{2}t^2$ since $v(0) = s(0) = 0$.

At the time of lift-off, $kt = 160$ and $(k/2)t^2 = 0.7$. Since $(k/2)t^2 = 0.7$,

$$t = \sqrt{\frac{1.4}{k}}$$

$$v\left(\sqrt{\frac{1.4}{k}}\right) = k\sqrt{\frac{1.4}{k}} = 160$$

$$1.4k = 160^2 \implies k = \frac{160^2}{1.4}$$

$$\approx 18{,}285.714 \text{ mi/hr}^2$$

$$\approx 7.45 \text{ ft/sec}^2.$$

**89.** True

**91.** True

**93.** False. For example, $\int x \cdot x\,dx \neq \int x\,dx \cdot \int x\,dx$ because $\frac{x^3}{3} + C \neq \left(\frac{x^2}{2} + C_1\right)\left(\frac{x^2}{2} + C_2\right)$

**95.** $f'(x) = \begin{cases} 1, & 0 \le x < 2 \\ 3x, & 2 \le x \le 5 \end{cases}$

$f(x) = \begin{cases} x + C_1, & 0 \le x < 2 \\ \frac{3x^2}{2} + C_2, & 2 \le x \le 5 \end{cases}$

$f(1) = 3 \implies 1 + C_1 = 3 \implies C_1 = 2$

$f$ is continuous: Values must agree at $x = 2$:

$4 = 6 + C_2 \implies C_2 = -2$

$f(x) = \begin{cases} x + 2, & 0 \le x < 2 \\ \frac{3x^2}{2} - 2, & 2 \le x \le 5 \end{cases}$

The left and right hand derivatives at $x = 2$ do not agree. Hence $f$ is not differentiable at $x = 2$.

## Section 4.2 Area

**1.** $\sum_{i=1}^{5}(2i + 1) = 2\sum_{i=1}^{5} i + \sum_{i=1}^{5} 1 = 2(1 + 2 + 3 + 4 + 5) + 5 = 35$

**3.** $\sum_{k=0}^{4} \frac{1}{k^2 + 1} = 1 + \frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} = \frac{158}{85}$

**5.** $\sum_{k=1}^{4} c = c + c + c + c = 4c$

**7.** $\sum_{i=1}^{9} \frac{1}{3i}$

**9.** $\sum_{j=1}^{8}\left[5\left(\frac{j}{8}\right) + 3\right]$

**11.** $\frac{2}{n}\sum_{i=1}^{n}\left[\left(\frac{2i}{n}\right)^3 - \left(\frac{2i}{n}\right)\right]$

**13.** $\frac{3}{n}\sum_{i=1}^{n}\left[2\left(1 + \frac{3i}{n}\right)^2\right]$

**15.** $\sum_{i=1}^{20} 2i = 2\sum_{i=1}^{20} i = 2\left[\frac{20(21)}{2}\right] = 420$

**17.** $\sum_{i=1}^{20}(i - 1)^2 = \sum_{i=1}^{19} i^2$

$$= \left[\frac{19(20)(39)}{6}\right] = 2470$$

**19.** $\sum_{i=1}^{15} i(i-1)^2 = \sum_{i=1}^{15} i^3 - 2\sum_{i=1}^{15} i^2 + \sum_{i=1}^{15} i$

$= \frac{15^2(16)^2}{4} - 2\frac{15(16)(31)}{6} + \frac{15(16)}{2}$

$= 14{,}400 - 2{,}480 + 120$

$= 12{,}040$

**21.** sum seq($x$ ^ 2 + 3, $x$, 1, 20, 1) = 2930 (*TI-82*)

$\sum_{i=1}^{20} (i^2 + 3) = \frac{20(20+1)(2(20)+1)}{6} + 3(20)$

$= \frac{(20)(21)(41)}{6} + 60 = 2930$

**23.** $S = \left[3 + 4 + \frac{9}{2} + 5\right](1) = \frac{33}{2} = 16.5$

$s = \left[1 + 3 + 4 + \frac{9}{2}\right](1) = \frac{25}{2} = 12.5$

**25.** $S = [3 + 3 + 5](1) = 11$

$s = [2 + 2 + 3](1) = 7$

**27.** $S(4) = \sqrt{\frac{1}{4}}\left(\frac{1}{4}\right) + \sqrt{\frac{1}{2}}\left(\frac{1}{4}\right) + \sqrt{\frac{3}{4}}\left(\frac{1}{4}\right) + \sqrt{1}\left(\frac{1}{4}\right) = \frac{1 + \sqrt{2} + \sqrt{3} + 2}{8} \approx 0.768$

$s(4) = 0\left(\frac{1}{4}\right) + \sqrt{\frac{1}{4}}\left(\frac{1}{4}\right) + \sqrt{\frac{1}{2}}\left(\frac{1}{4}\right) + \sqrt{\frac{3}{4}}\left(\frac{1}{4}\right) = \frac{1 + \sqrt{2} + \sqrt{3}}{8} \approx 0.518$

**29.** $S(5) = 1\left(\frac{1}{5}\right) + \frac{1}{6/5}\left(\frac{1}{5}\right) + \frac{1}{7/5}\left(\frac{1}{5}\right) + \frac{1}{8/5}\left(\frac{1}{5}\right) + \frac{1}{9/5}\left(\frac{1}{5}\right) = \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} \approx 0.746$

$s(5) = \frac{1}{6/5}\left(\frac{1}{5}\right) + \frac{1}{7/5}\left(\frac{1}{5}\right) + \frac{1}{8/5}\left(\frac{1}{5}\right) + \frac{1}{9/5}\left(\frac{1}{5}\right) + \frac{1}{2}\left(\frac{1}{5}\right) = \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} \approx 0.646$

**31.** $\lim_{n\to\infty}\left[\left(\frac{81}{n^4}\right)\frac{n^2(n+1)^2}{4}\right] = \frac{81}{4}\lim_{n\to\infty}\left[\frac{n^4 + 2n^3 + n^2}{n^4}\right] = \frac{81}{4}(1) = \frac{81}{4}$

**33.** $\lim_{n\to\infty}\left[\left(\frac{18}{n^2}\right)\frac{n(n+1)}{2}\right] = \frac{18}{2}\lim_{n\to\infty}\left[\frac{n^2 + n}{n^2}\right] = \frac{18}{2}(1) = 9$

**35.** $\sum_{i=1}^{n} \frac{2i+1}{n^2} = \frac{1}{n^2}\sum_{i=1}^{n}(2i+1) = \frac{1}{n^2}\left[2\frac{n(n+1)}{2} + n\right] = \frac{n+2}{n} = S(n)$

$S(10) = \frac{12}{10} = 1.2$

$S(100) = 1.02$

$S(1000) = 1.002$

$S(10{,}000) = 1.0002$

**37.** $\sum_{k=1}^{n} \frac{6k(k-1)}{n^3} = \frac{6}{n^3}\sum_{k=1}^{n}(k^2 - k) = \frac{6}{n^3}\left[\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}\right]$

$= \frac{6}{n^2}\left[\frac{2n^2 + 3n + 1 - 3n - 3}{6}\right] = \frac{1}{n^2}[2n^2 - 2] = S(n)$

$S(10) = 1.98$

$S(100) = 1.9998$

$S(1000) = 1.999998$

$S(10{,}000) = 1.99999998$

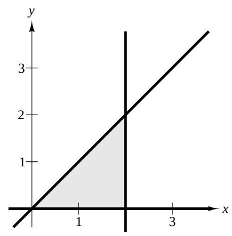
**39.** $\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{16i}{n^2}\right) = \lim_{n\to\infty}\frac{16}{n^2}\sum_{i=1}^{n} i = \lim_{n\to\infty}\frac{16}{n^2}\left(\frac{n(n+1)}{2}\right) = \lim_{n\to\infty}\left[8\left(\frac{n^2+n}{n^2}\right)\right] = 8\lim_{n\to\infty}\left(1 + \frac{1}{n}\right) = 8$

**41.** $\displaystyle \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n^3}(i-1)^2 = \lim_{n\to\infty} \frac{1}{n^3}\sum_{i=1}^{n-1} i^2 = \lim_{n\to\infty} \frac{1}{n^3}\left[\frac{(n-1)(n)(2n-1)}{6}\right]$

$$= \lim_{n\to\infty} \frac{1}{6}\left[\frac{2n^3 - 3n^2 + n}{n^3}\right] = \lim_{n\to\infty}\left[\frac{1}{6}\left(\frac{2 - (3/n) + (1/n^2)}{1}\right)\right] = \frac{1}{3}$$

**43.** $\displaystyle \lim_{n\to\infty} \sum_{i=1}^{n}\left(1 + \frac{i}{n}\right)\left(\frac{2}{n}\right) = 2\lim_{n\to\infty} \frac{1}{n}\left[\sum_{i=1}^{n} 1 + \frac{1}{n}\sum_{i=1}^{n} i\right] = 2\lim_{n\to\infty} \frac{1}{n}\left[n + \frac{1}{n}\left(\frac{n(n+1)}{2}\right)\right] = 2\lim_{n\to\infty}\left[1 + \frac{n^2+n}{2n^2}\right] = 2\left(1 + \frac{1}{2}\right) = 3$

**45.** (a)




(b) $\displaystyle \Delta x = \frac{2-0}{n} = \frac{2}{n}$

Endpoints:

$$0 < 1\left(\frac{2}{n}\right) < 2\left(\frac{2}{n}\right) < \cdots < (n-1)\left(\frac{2}{n}\right) < n\left(\frac{2}{n}\right) = 2$$

(c) Since $y = x$ is increasing, $f(m_i) = f(x_{i-1})$ on $[x_{i-1}, x_i]$.

$$s(n) = \sum_{i=1}^{n} f(x_{i-1})\,\Delta x$$

$$= \sum_{i=1}^{n} f\left(\frac{2i-2}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left[(i-1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$$

(d) $f(M_i) = f(x_i)$ on $[x_{i-1}, x_i]$

$$S(n) = \sum_{i=1}^{n} f(x_i)\,\Delta x = \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\frac{2}{n} = \sum_{i=1}^{n}\left[i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$$

(e)

| $x$ | 5 | 10 | 50 | 100 |
|---|---|---|---|---|
| $s(n)$ | 1.6 | 1.8 | 1.96 | 1.98 |
| $S(n)$ | 2.4 | 2.2 | 2.04 | 2.02 |

(f) $\displaystyle \lim_{n\to\infty} \sum_{i=1}^{n}\left[(i-1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right) = \lim_{n\to\infty} \frac{4}{n^2}\sum_{i=1}^{n}(i-1)$

$$= \lim_{n\to\infty} \frac{4}{n^2}\left[\frac{n(n+1)}{2} - n\right]$$

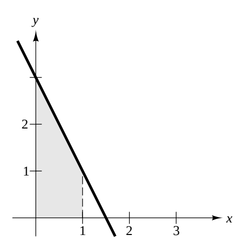
$$= \lim_{n\to\infty}\left[\frac{2(n+1)}{n} - \frac{4}{n}\right] = 2$$

$$\lim_{n\to\infty} \sum_{i=1}^{n}\left[i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right) = \lim_{n\to\infty} \frac{4}{n^2}\sum_{i=1}^{n} i$$

$$= \lim_{n\to\infty}\left(\frac{4}{n^2}\right)\frac{n(n+1)}{2}$$

$$= \lim_{n\to\infty} \frac{2(n+1)}{n} = 2$$

**47.** $y = -2x + 3$ on $[0, 1]$. $\left(\textbf{Note: } \Delta x = \dfrac{1-0}{n} = \dfrac{1}{n}\right)$

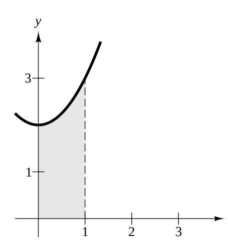



$$s(n) = \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\left[-2\left(\frac{i}{n}\right) + 3\right]\left(\frac{1}{n}\right)$$

$$= 3 - \frac{2}{n^2}\sum_{i=1}^{n} i = 3 - \frac{2(n+1)n}{2n^2} = 2 - \frac{1}{n}$$

$\text{Area} = \displaystyle\lim_{n\to\infty} s(n) = 2$

**49.** $y = x^2 + 2$ on $[0, 1]$. $\left(\textbf{Note: } \Delta x = \dfrac{1}{n}\right)$

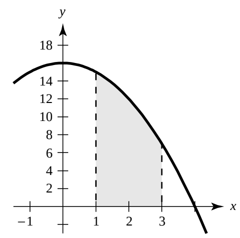



$$S(n) = \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\left[\left(\frac{i}{n}\right)^2 + 2\right]\left(\frac{1}{n}\right)$$

$$= \left[\frac{1}{n^3}\sum_{i=1}^{n} i^2\right] + 2 = \frac{n(n+1)(2n+1)}{6n^3} + 2 = \frac{1}{6}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) + 2$$

$\text{Area} = \displaystyle\lim_{n\to\infty} S(n) = \frac{7}{3}$

**51.** $y = 16 - x^2$ on $[1, 3]$. $\left(\textbf{Note: } \Delta x = \frac{2}{n}\right)$




$$s(n) = \sum_{i=1}^{n} f\left(1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^{n} \left[16 - \left(1 + \frac{2i}{n}\right)^2\right]\left(\frac{2}{n}\right)$$

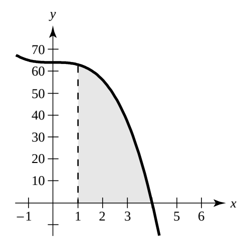
$$= \frac{2}{n}\sum_{i=1}^{n}\left[15 - \frac{4i^2}{n^2} - \frac{4i}{n}\right]$$

$$= \frac{2}{n}\left[15n - \frac{4}{n^2}\frac{n(n+1)(2n+1)}{6} - \frac{4}{n}\frac{n(n+1)}{2}\right]$$

$$= 30 - \frac{8}{6n^2}(n+1)(2n+1) - \frac{4}{n}(n+1)$$

$$\text{Area} = \lim_{n\to\infty} s(n) = 30 - \frac{8}{3} - 4 = \frac{70}{3} = 23\frac{1}{3}$$

**53.** $y = 64 - x^3$ on $[1, 4]$. $\left(\textbf{Note: } \Delta x = \frac{4-1}{n} = \frac{3}{n}\right)$




$$s(n) = \sum_{i=1}^{n} f\left(1 + \frac{3i}{n}\right)\left(\frac{3}{n}\right) = \sum_{i=1}^{n}\left[64 - \left(1 + \frac{3i}{n}\right)^3\right]\left(\frac{3}{n}\right)$$

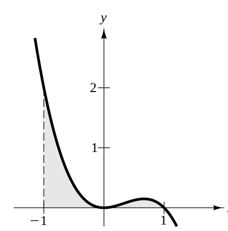
$$= \frac{3}{n}\sum_{i=1}^{n}\left[63 - \frac{27i^3}{n^3} - \frac{27i^2}{n^2} - \frac{9i}{n}\right]$$

$$= \frac{3}{n}\left[63n - \frac{27}{n^3}\frac{n^2(n+1)^2}{4} - \frac{27}{n^2}\frac{n(n+1)(2n+1)}{6} - \frac{9}{n}\frac{n(n+1)}{2}\right]$$

$$= 189 - \frac{81}{4n^2}(n+1)^2 - \frac{81}{6n^2}(n+1)(2n+1) - \frac{27}{2}\frac{n+1}{n}$$

$$\text{Area} = \lim_{n\to\infty} s(n) = 189 - \frac{81}{4} - 27 - \frac{27}{2} = \frac{513}{4} = 128.25$$

**55.** $y = x^2 - x^3$ on $[-1, 1]$. $\left(\textbf{Note: } \Delta x = \frac{1-(-1)}{n} = \frac{2}{n}\right)$




Again, $T(n)$ is neither an upper nor a lower sum.

$$T(n) = \sum_{i=1}^{n} f\left(-1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left[\left(-1 + \frac{2i}{n}\right)^2 - \left(-1 + \frac{2i}{n}\right)^3\right]\left(\frac{2}{n}\right)$$

$$= \sum_{i=1}^{n}\left[\left(1 - \frac{4i}{n} + \frac{4i^2}{n^2}\right) - \left(-1 + \frac{6i}{n} - \frac{12i^2}{n^2} + \frac{8i^3}{n^3}\right)\right]\left(\frac{2}{n}\right)$$

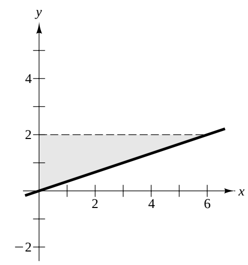
$$= \sum_{i=1}^{n}\left[2 - \frac{10i}{n} + \frac{16i^2}{n^2} - \frac{8i^3}{n^3}\right]\left(\frac{2}{n}\right) = \frac{4}{n}\sum_{i=1}^{n}1 - \frac{20}{n^2}\sum_{i=1}^{n}i + \frac{32}{n^3}\sum_{i=1}^{n}i^2 - \frac{16}{n^4}\sum_{i=1}^{n}i^3$$

$$= \frac{4}{n}(n) - \frac{20}{n^2}\cdot\frac{n(n+1)}{2} + \frac{32}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} - \frac{16}{n^4}\cdot\frac{n^2(n+1)^2}{4}$$

$$= 4 - 10\left(1 + \frac{1}{n}\right) + \frac{16}{3}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) - 4\left(1 + \frac{2}{n} + \frac{1}{n^2}\right)$$

$$\text{Area} = \lim_{n\to\infty} T(n) = 4 - 10 + \frac{32}{3} - 4 = \frac{2}{3}$$

**57.** $f(y) = 3y,\ 0 \le y \le 2 \qquad \left(\textbf{Note: } \Delta y = \frac{2-0}{n} = \frac{2}{n}\right)$

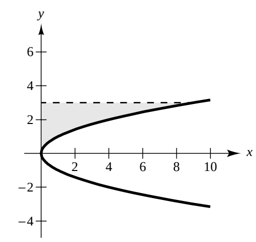



$$S(n) = \sum_{i=1}^{n} f(m_i)\,\Delta y = \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^{n} 3\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right)$$

$$= \frac{12}{n^2}\sum_{i=1}^{n} i = \left(\frac{12}{n^2}\right)\cdot\frac{n(n+1)}{2} = \frac{6(n+1)}{n} = 6 + \frac{6}{n}$$

$$\text{Area} = \lim_{n\to\infty} S(n) = \lim_{n\to\infty}\left(6 + \frac{6}{n}\right) = 6$$

**59.** $f(y) = y^2,\ 0 \le y \le 3 \qquad \left(\textbf{Note: } \Delta y = \frac{3-0}{n} = \frac{3}{n}\right)$

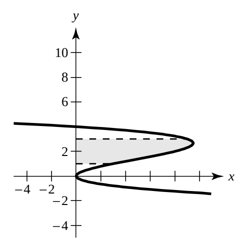



$$S(n) = \sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\left(\frac{3}{n}\right) = \sum_{i=1}^{n}\left(\frac{3i}{n}\right)^2\left(\frac{3}{n}\right) = \frac{27}{n^3}\sum_{i=1}^{n} i^2$$

$$= \frac{27}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} = \frac{9}{n^2}\left(\frac{2n^2+3n+1}{2}\right) = 9 + \frac{27}{2n} + \frac{9}{2n^2}$$

$$\text{Area} = \lim_{n\to\infty} S(n) = \lim_{n\to\infty}\left(9 + \frac{27}{2n} + \frac{9}{2n^2}\right) = 9$$

**61.** $g(y) = 4y^2 - y^3,\ 1 \le y \le 3. \qquad \left(\textbf{Note: } \Delta y = \frac{3-1}{n} = \frac{2}{n}\right)$




$$S(n) = \sum_{i=1}^{n} g\left(1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right)$$

$$= \sum_{i=1}^{n}\left[4\left(1 + \frac{2i}{n}\right)^2 - \left(1 + \frac{2i}{n}\right)^3\right]\frac{2}{n}$$

$$= \frac{2}{n}\sum_{i=1}^{n} 4\left[1 + \frac{4i}{n} + \frac{4i^2}{n^2}\right] - \left[1 + \frac{6i}{n} + \frac{12i^2}{n^2} + \frac{8i^3}{n^3}\right]$$

$$= \frac{2}{n}\sum_{i=1}^{n}\left[3 + \frac{10i}{n} + \frac{4i^2}{n^2} - \frac{8i^3}{n^3}\right] = \frac{2}{n}\left[3n + \frac{10}{n}\frac{n(n+1)}{2} + \frac{4}{n^2}\frac{n(n+1)(2n+1)}{6} - \frac{8}{n^2}\frac{n^2(n+1)^2}{4}\right]$$

$$\text{Area} = \lim_{n\to\infty} S(n) = 6 + 10 + \frac{8}{3} - 4 = \frac{44}{3}$$

**63.** $f(x) = x^2 + 3,\ 0 \le x \le 2,\ n = 4$

Let $c_i = \dfrac{x_i + x_{i-1}}{2}$.

$$\Delta x = \frac{1}{2},\ c_1 = \frac{1}{4},\ c_2 = \frac{3}{4},\ c_3 = \frac{5}{4},\ c_4 = \frac{7}{4}$$

$$\text{Area} \approx \sum_{i=1}^{n} f(c_i)\,\Delta x = \sum_{i=1}^{4}\left[c_i^2 + 3\right]\left(\frac{1}{2}\right)$$

$$= \frac{1}{2}\left[\left(\frac{1}{16} + 3\right) + \left(\frac{9}{16} + 3\right) + \left(\frac{25}{16} + 3\right) + \left(\frac{49}{16} + 3\right)\right]$$

$$= \frac{69}{8}$$

**65.** $f(x) = \tan x,\ 0 \le x \le \frac{\pi}{4},\ n = 4$

Let $c_i = \dfrac{x_i + x_{i-1}}{2}$.

$$\Delta x = \frac{\pi}{16},\ c_1 = \frac{\pi}{32},\ c_2 = \frac{3\pi}{32},\ c_3 = \frac{5\pi}{32},\ c_4 = \frac{7\pi}{32}$$

$$\text{Area} \approx \sum_{i=1}^{n} f(c_i)\,\Delta x = \sum_{i=1}^{4}(\tan c_i)\left(\frac{\pi}{16}\right)$$

$$= \frac{\pi}{16}\left(\tan\frac{\pi}{32} + \tan\frac{3\pi}{32} + \tan\frac{5\pi}{32} + \tan\frac{7\pi}{32}\right) \approx 0.345$$

**67.** $f(x) = \sqrt{x}$ on $[0, 4]$.

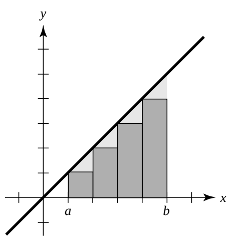
| $n$ | 4 | 8 | 12 | 16 | 20 |
|---|---|---|---|---|---|
| Approximate area | 5.3838 | 5.3523 | 5.3439 | 5.3403 | 5.3384 |

(Exact value is 16/3)

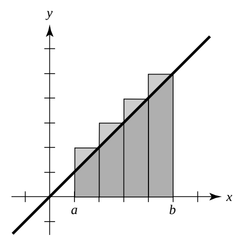
**69.** $f(x) = \tan\left(\dfrac{\pi x}{8}\right)$ on $[1, 3]$.

| $n$ | 4 | 8 | 12 | 16 | 20 |
|---|---|---|---|---|---|
| Approximate area | 2.2223 | 2.2387 | 2.2418 | 2.2430 | 2.2435 |

**71.** We can use the line $y = x$ bounded by $x = a$ and $x = b$. The sum of the areas of these inscribed rectangles is the lower sum.



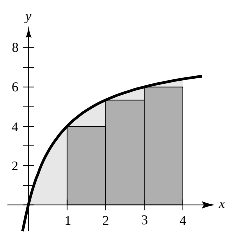

The sum of the areas of these circumscribed rectangles is the upper sum.




We can see that the rectangles do not contain all of the area in the first graph and the rectangles in the second graph cover more than the area of the region.

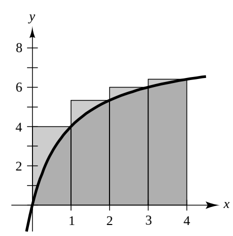
The exact value of the area lies between these two sums.

**73.** (a)



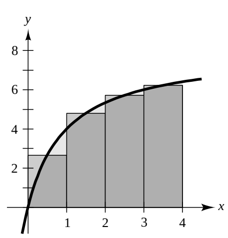

Lower sum:
$s(4) = 0 + 4 + 5\frac{1}{3} + 6 = 15\frac{1}{3} = \frac{46}{3} \approx 15.333$

(b)




Upper sum:
$S(4) = 4 + 5\frac{1}{3} + 6 + 6\frac{2}{5} = 21\frac{11}{15} = \frac{326}{15} \approx 21.733$

(c)




Midpoint Rule:
$M(4) = 2\frac{2}{3} + 4\frac{4}{5} + 5\frac{5}{7} + 6\frac{2}{9} = \frac{6112}{315} \approx 19.403$

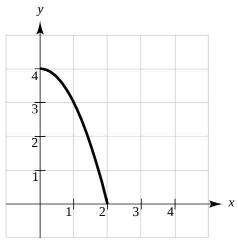
(d) In each case, $\Delta x = 4/n$. The lower sum uses left endpoints, $(i - 1)(4/n)$. The upper sum uses right endpoints, $(i)(4/n)$. The Midpoint Rule uses midpoints, $\left(i - \frac{1}{2}\right)(4/n)$.

(e)

| $n$ | 4 | 8 | 20 | 100 | 200 |
|---|---|---|---|---|---|
| $s(n)$ | 15.333 | 17.368 | 18.459 | 18.995 | 19.06 |
| $S(n)$ | 21.733 | 20.568 | 19.739 | 19.251 | 19.188 |
| $M(n)$ | 19.403 | 19.201 | 19.137 | 19.125 | 19.125 |

(f) $s(n)$ increases because the lower sum approaches the exact value as $n$ increases. $S(n)$ decreases because the upper sum approaches the exact value as $n$ increases. Because of the shape of the graph, the lower sum is always smaller than the exact value, whereas the upper sum is always larger.

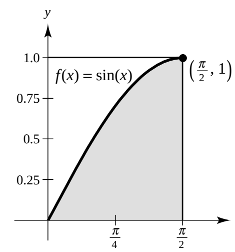
**75.**



b. $A \approx 6$ square units

**77.** True. (Theorem 4.2 (2))

**79.** $f(x) = \sin x, \left[0, \dfrac{\pi}{2}\right]$




Let $A_1 =$ area bounded by $f(x) = \sin x$, the $x$-axis, $x = 0$ and $x = \pi/2$. Let $A_2 =$ area of the rectangle bounded by $y = 1$, $y = 0$, $x = 0$, and $x = \pi/2$. Thus, $A_2 = (\pi/2)(1) \approx 1.570796$.
In this program, the computer is generating $N_2$ pairs of random points in the rectangle whose area is represented by $A_2$. It is keeping track of how many of these points, $N_1$, lie in the region whose area is represented by $A_1$. Since the points are randomly generated, we assume that

$$\frac{A_1}{A_2} \approx \frac{N_1}{N_2} \Rightarrow A_1 \approx \frac{N_1}{N_2}A_2.$$

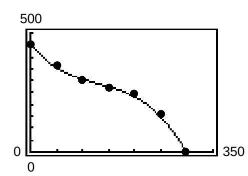
The larger $N_2$ is the better the approximation to $A_1$.

**81.** Suppose there are $n$ rows in the figure. The stars on the left total $1 + 2 + \cdots + n$, as do the stars on the right. There are $n(n + 1)$ stars in total, hence

$$2[1 + 2 + \cdots + n] = n(n + 1)$$
$$1 + 2 + \cdots + n = \tfrac{1}{2}(n)(n + 1).$$

**83.** (a) $y = (-4.09 \times 10^{-5})x^3 + 0.016x^2 - 2.67x + 452.9$

(b)

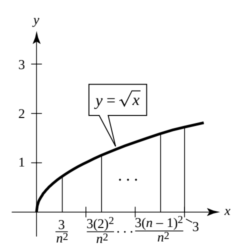


(c) Using the integration capability of a graphing utility, you obtain

$$A \approx 76{,}897.5 \text{ ft}^2.$$

## Section 4.3 Riemann Sums and Definite Integrals

**1.** $f(x) = \sqrt{x}, y = 0, x = 0, x = 3, c_i = \dfrac{3i^2}{n^2}$




$$\Delta x_i = \frac{3i^2}{n^2} - \frac{3(i - 1)^2}{n^2} = \frac{3}{n^2}(2i - 1)$$

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(c_i)\Delta x_i = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{\frac{3i^2}{n^2}}\,\frac{3}{n^2}(2i - 1)$$

$$= \lim_{n \to \infty} \frac{3\sqrt{3}}{n^3}\sum_{i=1}^{n}(2i^2 - i)$$

$$= \lim_{n \to \infty} \frac{3\sqrt{3}}{n^3}\left[2\frac{n(n + 1)(2n + 1)}{6} - \frac{n(n + 1)}{2}\right]$$

$$= \lim_{n \to \infty} 3\sqrt{3}\left[\frac{(n + 1)(2n + 1)}{3n^2} - \frac{n + 1}{2n^2}\right]$$

$$= 3\sqrt{3}\left[\frac{2}{3} - 0\right] = 2\sqrt{3} \approx 3.464$$

**3.** $y = 6$ on $[4, 10]$. $\left(\textbf{Note}: \Delta x = \dfrac{10-4}{n} = \dfrac{6}{n}, \|\Delta\| \to 0 \text{ as } n \to \infty\right)$

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i = \sum_{i=1}^{n} f\left(4 + \frac{6i}{n}\right)\left(\frac{6}{n}\right) = \sum_{i=1}^{n} 6\left(\frac{6}{n}\right) = \sum_{i=1}^{n} \frac{36}{n} = 36$$

$$\int_4^{10} 6\,dx = \lim_{n\to\infty} 36 = 36$$

**5.** $y = x^3$ on $[-1, 1]$. $\left(\textbf{Note:}\ \Delta x = \dfrac{1-(-1)}{n} = \dfrac{2}{n}, \|\Delta\| \to 0 \text{ as } n \to \infty\right)$

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i = \sum_{i=1}^{n} f\left(-1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left(-1 + \frac{2i}{n}\right)^3\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left[-1 + \frac{6i}{n} - \frac{12i^2}{n^2} + \frac{8i^3}{n^3}\right]\left(\frac{2}{n}\right)$$

$$= -2 + \frac{12}{n^2}\sum_{i=1}^{n} i - \frac{24}{n^3}\sum_{i=1}^{n} i^2 + \frac{16}{n^4}\sum_{i=1}^{n} i^3$$

$$= -2 + 6\left(1 + \frac{1}{n}\right) - 4\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) + 4\left(1 + \frac{2}{n} + \frac{1}{n^2}\right) = \frac{2}{n}$$

$$\int_{-1}^{1} x^3\,dx = \lim_{n\to\infty} \frac{2}{n} = 0$$

**7.** $y = x^2 + 1$ on $[1, 2]$. $\left(\textbf{Note:}\ \Delta x = \dfrac{2-1}{n} = \dfrac{1}{n}, \|\Delta\| \to 0 \text{ as } n \to \infty\right)$

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i = \sum_{i=1}^{n} f\left(1 + \frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\left[\left(1 + \frac{i}{n}\right)^2 + 1\right]\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\left[1 + \frac{2i}{n} + \frac{i^2}{n^2} + 1\right]\left(\frac{1}{n}\right)$$

$$= 2 + \frac{2}{n^2}\sum_{i=1}^{n} i + \frac{1}{n^3}\sum_{i=1}^{n} i^2 = 2 + \left(1 + \frac{1}{n}\right) + \frac{1}{6}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) = \frac{10}{3} + \frac{3}{2n} + \frac{1}{6n^2}$$

$$\int_1^2 (x^2 + 1)\,dx = \lim_{n\to\infty}\left(\frac{10}{3} + \frac{3}{2n} + \frac{1}{6n^2}\right) = \frac{10}{3}$$

**9.** $\displaystyle\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} (3c_i + 10)\,\Delta x_i = \int_{-1}^{5} (3x + 10)\,dx$

on the interval $[-1, 5]$.

**11.** $\displaystyle\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \sqrt{c_i^2 + 4}\,\Delta x_i = \int_0^3 \sqrt{x^2 + 4}\,dx$

on the interval $[0, 3]$.

**13.** $\displaystyle\int_0^5 3\,dx$

**15.** $\displaystyle\int_{-4}^{4} (4 - |x|)\,dx$

**17.** $\displaystyle\int_{-2}^{2} (4 - x^2)\,dx$

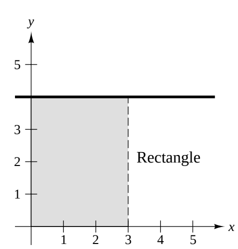
**19.** $\displaystyle\int_0^{\pi} \sin x\,dx$

**21.** $\displaystyle\int_0^2 y^3\,dy$

**23.** Rectangle

$A = bh = 3(4)$

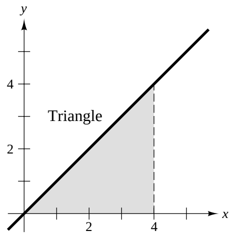
$A = \displaystyle\int_0^3 4\,dx = 12$



**25.** Triangle

$A = \frac{1}{2}bh = \frac{1}{2}(4)(4)$

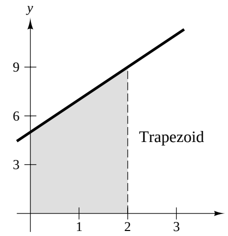
$A = \int_0^4 x\,dx = 8$




**27.** Trapezoid

$A = \frac{b_1 + b_2}{2}h = \left(\frac{5+9}{2}\right)2$

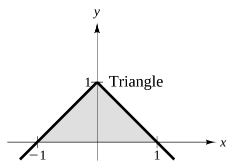
$A = \int_0^2 (2x + 5)\,dx = 14$




**29.** Triangle

$A = \frac{1}{2}bh = \frac{1}{2}(2)(1)$

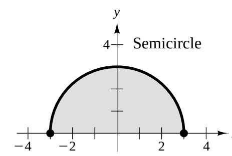
$A = \int_{-1}^1 (1 - |x|)\,dx = 1$




**31.** Semicircle

$A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(3)^2$

$A = \int_{-3}^3 \sqrt{9 - x^2}\,dx = \frac{9\pi}{2}$




**In Exercises 33–39, $\int_2^4 x^3\,dx = 60$, $\int_2^4 x\,dx = 6$, $\int_2^4 dx = 2$**

**33.** $\int_4^2 x\,dx = -\int_2^4 x\,dx = -6$

**35.** $\int_2^4 4x\,dx = 4\int_2^4 x\,dx = 4(6) = 24$

**37.** $\int_2^4 (x - 8)\,dx = \int_2^4 x\,dx - 8\int_2^4 dx = 6 - 8(2) = -10$

**39.** $\int_2^4 \left(\frac{1}{2}x^3 - 3x + 2\right) dx = \frac{1}{2}\int_2^4 x^3\,dx - 3\int_2^4 x\,dx + 2\int_2^4 dx$

$= \frac{1}{2}(60) - 3(6) + 2(2) = 16$

**41.** (a) $\int_0^7 f(x)\,dx = \int_0^5 f(x)\,dx + \int_5^7 f(x)\,dx = 10 + 3 = 13$

(b) $\int_5^0 f(x)\,dx = -\int_0^5 f(x)\,dx = -10$

(c) $\int_5^5 f(x)\,dx = 0$

(d) $\int_0^5 3f(x)\,dx = 3\int_0^5 f(x)\,dx = 3(10) = 30$

**43.** (a) $\int_2^6 [f(x) + g(x)]\,dx = \int_2^6 f(x)\,dx + \int_2^6 g(x)\,dx$

$= 10 + (-2) = 8$

(b) $\int_2^6 [g(x) - f(x)]\,dx = \int_2^6 g(x)\,dx - \int_2^6 f(x)\,dx$

$= -2 - 10 = -12$

(c) $\int_2^6 2g(x)\,dx = 2\int_2^6 g(x)\,dx = 2(-2) = -4$

(d) $\int_2^6 3f(x)\,dx = 3\int_2^6 f(x)\,dx = 3(10) = 30$

**45.** (a) Quarter circle below $x$-axis: $-\frac{1}{4}\pi r^2 = -\frac{1}{4}\pi(2)^2 = -\pi$

(b) Triangle: $\frac{1}{2}bh = \frac{1}{2}(4)(2) = 4$

(c) Triangle + Semicircle below $x$-axis: $-\frac{1}{2}(2)(1) - \frac{1}{2}\pi(2)^2 = -(1 + 2\pi)$

(d) Sum of parts (b) and (c): $4 - (1 + 2\pi) = 3 - 2\pi$

(e) Sum of absolute values of (b) and (c): $4 + (1 + 2\pi) = 5 + 2\pi$

(f) Answer to (d) plus $2(10) = 20$: $(3 - 2\pi) + 20 = 23 - 2\pi$

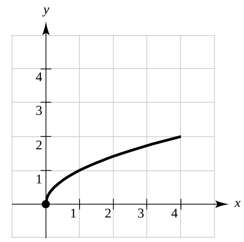
**47.** The left endpoint approximation will be greater than the actual area: $>$

**49.** Because the curve is concave upward, the midpoint approximation will be less than the actual area: $<$

**51.** $f(x) = \dfrac{1}{x - 4}$

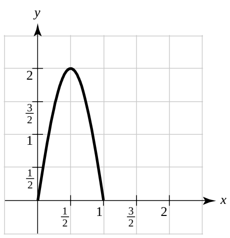
is not integrable on the interval $[3, 5]$ and $f$ has a discontinuity at $x = 4$.

**53.**




a. $A \approx 5$ square units

**55.**




d. $\displaystyle\int_0^1 2 \sin \pi x \, dx \approx \frac{1}{2}(1)(2) \approx 1$

**57.** $\displaystyle\int_0^3 x\sqrt{3 - x}\, dx$

| $n$ | 4 | 8 | 12 | 16 | 20 |
|---|---|---|---|---|---|
| $L(n)$ | 3.6830 | 3.9956 | 4.0707 | 4.1016 | 4.1177 |
| $M(n)$ | 4.3082 | 4.2076 | 4.1838 | 4.1740 | 4.1690 |
| $R(n)$ | 3.6830 | 3.9956 | 4.0707 | 4.1016 | 4.1177 |

**59.** $\displaystyle\int_0^{\pi/2} \sin^2 x \, dx$

| $n$ | 4 | 8 | 12 | 16 | 20 |
|---|---|---|---|---|---|
| $L(n)$ | 0.5890 | 0.6872 | 0.7199 | 0.7363 | 0.7461 |
| $M(n)$ | 0.7854 | 0.7854 | 0.7854 | 0.7854 | 0.7854 |
| $R(n)$ | 0.9817 | 0.8836 | 0.8508 | 0.8345 | 0.8247 |

**61.** True

**63.** True

**65.** False

$\displaystyle\int_0^2 (-x)\, dx = -2$

**67.** $f(x) = x^2 + 3x, [0, 8]$

$x_0 = 0, x_1 = 1, x_2 = 3, x_3 = 7, x_4 = 8$

$\Delta x_1 = 1, \Delta x_2 = 2, \Delta x_3 = 4, \Delta x_4 = 1$

$c_1 = 1, c_2 = 2, c_3 = 5, c_4 = 8$

$$\sum_{i=1}^{4} f(c_i)\, \Delta x = f(1)\, \Delta x_1 + f(2)\, \Delta x_2 + f(5)\, \Delta x_3 + f(8)\, \Delta x_4$$

$$= (4)(1) + (10)(2) + (40)(4) + (88)(1) = 272$$

**69.** $f(x) = \begin{cases} 1, & x \text{ is rational} \\ 0, & x \text{ is irrational} \end{cases}$

is not integrable on the interval $[0, 1]$. As $\|\Delta\| \to 0$, $f(c_i) = 1$ or $f(c_i) = 0$ in each subinterval since there are an infinite number of both rational and irrational numbers in any interval, no matter how small.

**71.** Let $f(x) = x^2$, $0 \le x \le 1$, and $\Delta x_i = 1/n$. The appropriate Riemann Sum is

$$\sum_{i=1}^{n} f(c_i)\Delta x_i = \sum_{i=1}^{n}\left(\frac{i}{n}\right)^2\frac{1}{n} = \frac{1}{n^3}\sum_{i=1}^{n} i^2.$$

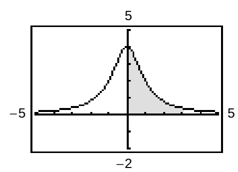
$$\lim_{n\to\infty}\frac{1}{n^3}[1^2 + 2^2 + 3^2 + \cdots + n^2] = \lim_{n\to\infty}\frac{1}{n^3}\cdot\frac{n(2n+1)(n+1)}{6}$$

$$= \lim_{n\to\infty}\frac{2n^2+3n+1}{6n^2} = \lim_{n\to\infty}\left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right) = \frac{1}{3}$$

## Section 4.4 The Fundamental Theorem of Calculus

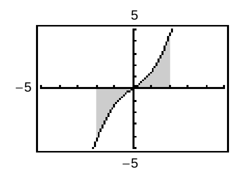
**1.** $f(x) = \dfrac{4}{x^2+1}$

$\displaystyle\int_0^{\pi} \frac{4}{x^2+1}\,dx$ is positive.




**3.** $f(x) = x\sqrt{x^2+1}$

$\displaystyle\int_{-2}^{2} x\sqrt{x^2+1}\,dx = 0$




**5.** $\displaystyle\int_0^1 2x\,dx = \Big[x^2\Big]_0^1 = 1 - 0 = 1$

**7.** $\displaystyle\int_{-1}^{0} (x-2)\,dx = \left[\frac{x^2}{2} - 2x\right]_{-1}^{0} = 0 - \left(\frac{1}{2} + 2\right) = -\frac{5}{2}$

**9.** $\displaystyle\int_{-1}^{1} (t^2-2)\,dt = \left[\frac{t^3}{3} - 2t\right]_{-1}^{1} = \left(\frac{1}{3} - 2\right) - \left(-\frac{1}{3} + 2\right) = -\frac{10}{3}$

**11.** $\displaystyle\int_0^1 (2t-1)^2\,dt = \int_0^1 (4t^2 - 4t + 1)\,dt = \left[\frac{4}{3}t^3 - 2t^2 + t\right]_0^1 = \frac{4}{3} - 2 + 1 = \frac{1}{3}$

**13.** $\displaystyle\int_1^2 \left(\frac{3}{x^2} - 1\right)dx = \left[-\frac{3}{x} - x\right]_1^2 = \left(-\frac{3}{2} - 2\right) - (-3 - 1) = \frac{1}{2}$

**15.** $\displaystyle\int_1^4 \frac{u-2}{\sqrt{u}}\,du = \int_1^4 (u^{1/2} - 2u^{-1/2})\,du = \left[\frac{2}{3}u^{3/2} - 4u^{1/2}\right]_1^4 = \left[\frac{2}{3}(\sqrt{4})^3 - 4\sqrt{4}\right] - \left[\frac{2}{3} - 4\right] = \frac{2}{3}$

**17.** $\displaystyle\int_{-1}^{1} (\sqrt[3]{t} - 2)\,dt = \left[\frac{3}{4}t^{4/3} - 2t\right]_{-1}^{1} = \left(\frac{3}{4} - 2\right) - \left(\frac{3}{4} + 2\right) = -4$

**19.** $\displaystyle\int_0^1 \frac{x-\sqrt{x}}{3}\,dx = \frac{1}{3}\int_0^1 (x - x^{1/2})\,dx = \frac{1}{3}\left[\frac{x^2}{2} - \frac{2}{3}x^{3/2}\right]_0^1 = \frac{1}{3}\left(\frac{1}{2} - \frac{2}{3}\right) = -\frac{1}{18}$

**21.** $\displaystyle\int_{-1}^{0} (t^{1/3} - t^{2/3})\,dt = \left[\frac{3}{4}t^{4/3} - \frac{3}{5}t^{5/3}\right]_{-1}^{0} = 0 - \left(\frac{3}{4} + \frac{3}{5}\right) = -\frac{27}{20}$

**23.** $\displaystyle\int_0^3 |2x-3|\,dx = \int_0^{3/2} (3-2x)\,dx + \int_{3/2}^{3} (2x-3)\,dx$ $\left(\text{split up the integral at the zero } x = \frac{3}{2}\right)$

$$= \Big[3x - x^2\Big]_0^{3/2} + \Big[x^2 - 3x\Big]_{3/2}^{3} = \left(\frac{9}{2} - \frac{9}{4}\right) - 0 + (9 - 9) - \left(\frac{9}{4} - \frac{9}{2}\right) = 2\left(\frac{9}{2} - \frac{9}{4}\right) = \frac{9}{2}$$

**25.** $\displaystyle\int_0^3 |x^2 - 4|\, dx = \int_0^2 (4 - x^2)\, dx + \int_2^3 (x^2 - 4)\, dx$

$$= \left[4x - \frac{x^3}{3}\right]_0^2 + \left[\frac{x^3}{3} - 4x\right]_2^3$$

$$= \left(8 - \frac{8}{3}\right) + (9 - 12) - \left(\frac{8}{3} - 8\right)$$

$$= \frac{23}{3}$$

**27.** $\displaystyle\int_0^{\pi} (1 + \sin x)\, dx = \Big[x - \cos x\Big]_0^{\pi} = (\pi + 1) - (0 - 1) = 2 + \pi$

**29.** $\displaystyle\int_{-\pi/6}^{\pi/6} \sec^2 x\, dx = \Big[\tan x\Big]_{-\pi/6}^{\pi/6} = \frac{\sqrt{3}}{3} - \left(-\frac{\sqrt{3}}{3}\right) = \frac{2\sqrt{3}}{3}$

**31.** $\displaystyle\int_{-\pi/3}^{\pi/3} 4 \sec\theta \tan\theta\, d\theta = \Big[4 \sec\theta\Big]_{-\pi/3}^{\pi/3} = 4(2) - 4(2) = 0$

**33.** $\displaystyle\int_0^3 10{,}000(t - 6)\, dt = 10{,}000\left[\frac{t^2}{2} - 6t\right]_0^3 = -\$135{,}000$

**35.** $\displaystyle A = \int_0^1 (x - x^2)\, dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{6}$

**37.** $\displaystyle A = \int_0^3 (3 - x)\sqrt{x}\, dx = \int_0^3 (3x^{1/2} - x^{3/2})\, dx = \left[2x^{3/2} - \frac{2}{5}x^{5/2}\right]_0^3 = \left[\frac{x\sqrt{x}}{5}(10 - 2x)\right]_0^3 = \frac{12\sqrt{3}}{5}$

**39.** $\displaystyle A = \int_0^{\pi/2} \cos x\, dx = \Big[\sin x\Big]_0^{\pi/2} = 1$

**41.** Since $y \geq 0$ on $[0, 2]$,

$$A = \int_0^2 (3x^2 + 1)\, dx = \Big[x^3 + x\Big]_0^2 = 8 + 2 = 10.$$

**43.** Since $y \geq 0$ on $[0, 2]$,

$$A = \int_0^2 (x^3 + x)\, dx = \left[\frac{x^4}{4} + \frac{x^2}{2}\right]_0^2 = 4 + 2 = 6.$$

**45.** $\displaystyle\int_0^2 \left(x - 2\sqrt{x}\right) dx = \left[\frac{x^2}{2} - \frac{4x^{3/2}}{3}\right]_0^2 = 2 - \frac{8\sqrt{2}}{3}$

$$f(c)(2 - 0) = \frac{6 - 8\sqrt{2}}{3}$$

$$c - 2\sqrt{c} = \frac{3 - 4\sqrt{2}}{3}$$

$$c - 2\sqrt{c} + 1 = \frac{3 - 4\sqrt{2}}{3} + 1$$

$$\left(\sqrt{c} - 1\right)^2 = \frac{6 - 4\sqrt{2}}{3}$$

$$\sqrt{c} - 1 = \pm\sqrt{\frac{6 - 4\sqrt{2}}{3}}$$

$$c = \left[1 \pm \sqrt{\frac{6 - 4\sqrt{2}}{3}}\right]^2$$

$$c \approx 0.4380 \text{ or } c \approx 1.7908$$

**47.** $\displaystyle\int_{-\pi/4}^{\pi/4} 2\sec^2 x\,dx = \Big[2\tan x\Big]_{-\pi/4}^{\pi/4} = 2(1) - 2(-1) = 4$

$$f(c)\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = 4$$

$$2\sec^2 c = \frac{8}{\pi}$$

$$\sec^2 c = \frac{4}{\pi}$$

$$\sec c = \pm\frac{2}{\sqrt{\pi}}$$

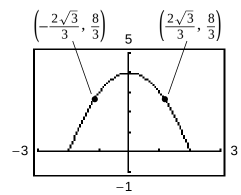
$$c = \pm\operatorname{arcsec}\left(\frac{2}{\sqrt{\pi}}\right)$$

$$= \pm\arccos\frac{\sqrt{\pi}}{2} \approx \pm 0.4817$$

**49.** $\displaystyle\frac{1}{2-(-2)}\int_{-2}^{2}(4 - x^2)\,dx = \frac{1}{4}\left[4x - \frac{1}{3}x^3\right]_{-2}^{2} = \frac{1}{4}\left[\left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right)\right] = \frac{8}{3}$

Average value $= \dfrac{8}{3}$

$4 - x^2 = \dfrac{8}{3}$ when $x^2 = 4 - \dfrac{8}{3}$ or $x = \pm\dfrac{2\sqrt{3}}{3} \approx \pm 1.155.$

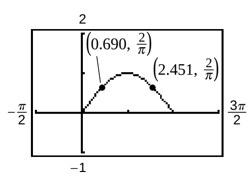



**51.** $\displaystyle\frac{1}{\pi - 0}\int_0^{\pi}\sin x\,dx = \left[-\frac{1}{\pi}\cos x\right]_0^{\pi} = \frac{2}{\pi}$

Average value $= \dfrac{2}{\pi}$

$$\sin x = \frac{2}{\pi}$$

$$x \approx 0.690,\ 2.451$$




**53.** If $f$ is continuous on $[a, b]$ and $F'(x) = f(x)$ on $[a, b]$, then $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$.

**55.** $\displaystyle\int_0^2 f(x)\,dx = -(\text{area of region } A) = -1.5$

**57.** $\displaystyle\int_0^6 |f(x)|\,dx = -\int_0^2 f(x)\,dx + \int_2^6 f(x)\,dx = 1.5 + 5.0 = 6.5$

**59.** $\displaystyle\int_0^6 [2 + f(x)]\,dx = \int_0^6 2\,dx + \int_0^6 f(x)\,dx$

$$= 12 + 3.5 = 15.5$$

**61.** (a) $F(x) = k\sec^2 x$

$F(0) = k = 500$

$F(x) = 500\sec^2 x$

(b) $\displaystyle\frac{1}{\pi/3 - 0}\int_0^{\pi/3} 500\sec^2 x\,dx = \frac{1500}{\pi}\Big[\tan x\Big]_0^{\pi/3}$

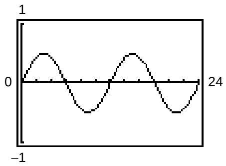
$$= \frac{1500}{\pi}\left(\sqrt{3} - 0\right)$$

$$\approx 826.99 \text{ newtons}$$

$$\approx 827 \text{ newtons}$$

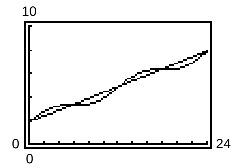
**63.** $\displaystyle\frac{1}{5-0}\int_0^5 (0.1729t + 0.1552t^2 - 0.0374t^3)\,dt \approx \frac{1}{5}\Big[0.08645t^2 + 0.05073t^3 - 0.00935t^4\Big]_0^5 \approx 0.5318 \text{ liter}$

**65.** (a)




The area above the $x$-axis equals the area below the $x$-axis. Thus, the average value is zero.

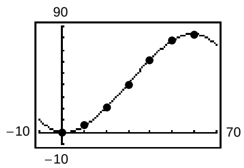
(b)




The average value of $S$ appears to be $g$.

**67.** (a) $v = -8.61 \times 10^{-4}t^3 + 0.0782t^2 - 0.208t + 0.0952$

(b)




(c) $\displaystyle\int_0^{60} v(t)\,dt = \left[\frac{-8.61 \times 10^{-4}t^4}{4} + \frac{0.0782t^3}{3} - \frac{0.208t^2}{2} + 0.0952t\right]_0^{60} \approx 2476 \text{ meters}$

**69.** $\displaystyle F(x) = \int_0^x (t-5)\,dt = \left[\frac{t^2}{2} - 5t\right]_0^x = \frac{x^2}{2} - 5x$

$F(2) = \dfrac{4}{2} - 5(2) = -8$

$F(5) = \dfrac{25}{2} - 5(5) = -\dfrac{25}{2}$

$F(8) = \dfrac{64}{2} - 5(8) = -8$

**71.** $\displaystyle F(x) = \int_1^x \frac{10}{v^2}\,dv = \int_1^x 10v^{-2}\,dv = \frac{-10}{v}\bigg]_1^x$

$\displaystyle = -\frac{10}{x} + 10 = 10\left(1 - \frac{1}{x}\right)$

$F(2) = 10\left(\dfrac{1}{2}\right) = 5$

$F(5) = 10\left(\dfrac{4}{5}\right) = 8$

$F(8) = 10\left(\dfrac{7}{8}\right) = \dfrac{35}{4}$

**73.** $\displaystyle F(x) = \int_1^x \cos\theta\,d\theta = \sin\theta\bigg]_1^x = \sin x - \sin 1$

$F(2) = \sin 2 - \sin 1 = 0.0678$

$F(5) = \sin 5 - \sin 1 \approx -1.8004$

$F(8) = \sin 8 - \sin 1 \approx 0.1479$

**75.** (a) $\displaystyle\int_0^x (t+2)\,dt = \left[\frac{t^2}{2} + 2t\right]_0^x = \frac{1}{2}x^2 + 2x$

(b) $\dfrac{d}{dx}\left[\dfrac{1}{2}x^2 + 2x\right] = x + 2$

**77.** (a) $\displaystyle\int_8^x \sqrt[3]{t}\,dt = \left[\frac{3}{4}t^{4/3}\right]_8^x = \frac{3}{4}(x^{4/3} - 16) = \frac{3}{4}x^{4/3} - 12$

(b) $\dfrac{d}{dx}\left[\dfrac{3}{4}x^{4/3} - 12\right] = x^{1/3} = \sqrt[3]{x}$

**79.** (a) $\displaystyle\int_{\pi/4}^x \sec^2 t\,dt = \Big[\tan t\Big]_{\pi/4}^x = \tan x - 1$

(b) $\dfrac{d}{dx}[\tan x - 1] = \sec^2 x$

**81.** $\displaystyle F(x) = \int_{-2}^x (t^2 - 2t)\,dt$

$F'(x) = x^2 - 2x$

**83.** $\displaystyle F(x) = \int_{-1}^x \sqrt{t^4 + 1}\,dt$

$F'(x) = \sqrt{x^4 + 1}$

**85.** $\displaystyle F(x) = \int_0^x t\cos t\,dt$

$F'(x) = x\cos x$

**87.** $F(x) = \int_x^{x+2} (4t + 1)\,dt$

$= \left[2t^2 + t\right]_x^{x+2}$

$= [2(x+2)^2 + (x+2)] - [2x^2 + x]$

$= 8x + 10$

$F'(x) = 8$

**Alternate solution:**

$F(x) = \int_x^{x+2} (4t + 1)\,dt$

$= \int_x^0 (4t + 1)\,dt + \int_0^{x+2} (4t + 1)\,dt$

$= -\int_0^x (4t + 1)\,dt + \int_0^{x+2} (4t + 1)\,dt$

$F'(x) = -(4x + 1) + 4(x + 2) + 1 = 8$

**89.** $F(x) = \int_0^{\sin x} \sqrt{t}\,dt = \left[\frac{2}{3}t^{3/2}\right]_0^{\sin x} = \frac{2}{3}(\sin x)^{3/2}$

$F'(x) = (\sin x)^{1/2}\cos x = \cos x\sqrt{\sin x}$

**Alternate solution**

$F(x) = \int_0^{\sin x} \sqrt{t}\,dt$

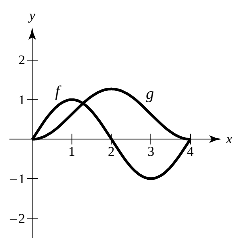
$F'(x) = \sqrt{\sin x}\,\frac{d}{dx}(\sin x) = \sqrt{\sin x}(\cos x)$

**91.** $F(x) = \int_0^{x^3} \sin t^2\,dt$

$F'(x) = \sin(x^3)^2 \cdot 3x^2 = 3x^2 \sin x^6$

**93.** $g(x) = \int_0^x f(t)\,dt$

$g(0) = 0, g(1) \approx \frac{1}{2}, g(2) \approx 1, g(3) \approx \frac{1}{2}, g(4) = 0$




$g$ has a relative maximum at $x = 2$.

**95.** (a) $C(x) = 5000\left(25 + 3\int_0^x t^{1/4}\,dt\right)$

$= 5000\left(25 + 3\left[\frac{4}{5}t^{5/4}\right]_0^x\right)$

$= 5000\left(25 + \frac{12}{5}x^{5/4}\right) = 1000(125 + 12x^{5/4})$

(b) $C(1) = 1000(125 + 12(1)) = \$137{,}000$

$C(5) = 1000(125 + 12(5)^{5/4}) \approx \$214{,}721$

$C(10) = 1000(125 + 12(10)^{5/4}) \approx \$338{,}394$

**97.** True

**99.** False; $\int_{-1}^{1} x^{-2}\,dx = \int_{-1}^{0} x^{-2}\,dx + \int_0^1 x^{-2}\,dx$

Each of these integrals is infinite. $f(x) = x^{-2}$ has a nonremovable discontinuity at $x = 0$.

**101.** $f(x) = \int_0^{1/x} \frac{1}{t^2 + 1}\,dt + \int_0^x \frac{1}{t^2 + 1}\,dt$

By the Second Fundamental Theorem of Calculus, we have

$f'(x) = \frac{1}{(1/x)^2 + 1}\left(-\frac{1}{x^2}\right) + \frac{1}{x^2 + 1}$

$= -\frac{1}{1 + x^2} + \frac{1}{x^2 + 1} = 0.$

Since $f'(x) = 0$, $f(x)$ must be constant.

**103.** $x(t) = t^3 - 6t^2 + 9t - 2$

$$\begin{aligned} x'(t) &= 3t^2 - 12t + 9 \\ &= 3(t^2 - 4t + 3) \\ &= 3(t - 3)(t - 1) \end{aligned}$$

$$\begin{aligned} \text{Total distance} &= \int_0^5 |x'(t)|\,dt \\ &= \int_0^5 3|(t-3)(t-1)|\,dt \\ &= 3\int_0^1 (t^2 - 4t + 3)\,dt - 3\int_1^3 (t^2 - 4t + 3)\,dt + 3\int_3^5 (t^2 - 4t + 3)\,dt \\ &= 4 + 4 + 20 \\ &= 28 \text{ units} \end{aligned}$$

**105.** 
$$\begin{aligned} \text{Total distance} &= \int_1^4 |x'(t)|\,dt \\ &= \int_1^4 |v(t)|\,dt \\ &= \int_1^4 \frac{1}{\sqrt{t}}\,dt \\ &= 2t^{1/2}\Big]_1^4 \\ &= 2(2 - 1) = 2 \text{ units} \end{aligned}$$

## Section 4.5 Integration by Substitution

| $\int f(g(x))g'(x)\,dx$ | $u = g(x)$ | $du = g'(x)\,dx$ |
|---|---|---|
| **1.** $\int (5x^2 + 1)^2(10x)\,dx$ | $5x^2 + 1$ | $10x\,dx$ |
| **3.** $\int \frac{x}{\sqrt{x^2 + 1}}\,dx$ | $x^2 + 1$ | $2x\,dx$ |
| **5.** $\int \tan^2 x \sec^2 x\,dx$ | $\tan x$ | $\sec^2 x\,dx$ |

**7.** $\int (1 + 2x)^4\, 2\,dx = \frac{(1 + 2x)^5}{5} + C$

**Check:** $\frac{d}{dx}\left[\frac{(1 + 2x)^5}{5} + C\right] = 2(1 + 2x)^4$

**9.** $\int (9 - x^2)^{1/2}(-2x)\,dx = \frac{(9 - x^2)^{3/2}}{3/2} + C = \frac{2}{3}(9 - x^2)^{3/2} + C$

**Check:** $\frac{d}{dx}\left[\frac{2}{3}(9 - x^2)^{3/2} + C\right] = \frac{2}{3} \cdot \frac{3}{2}(9 - x^2)^{1/2}(-2x) = \sqrt{9 - x^2}(-2x)$

**11.** $\int x^3(x^4+3)^2\,dx = \frac{1}{4}\int (x^4+3)^2(4x^3)\,dx = \frac{1}{4}\frac{(x^4+3)^3}{3} + C = \frac{(x^4+3)^3}{12} + C$

**Check:** $\frac{d}{dx}\left[\frac{(x^4+3)^3}{12} + C\right] = \frac{3(x^4+3)^2}{12}(4x^3) = (x^4+3)^2(x^3)$

**13.** $\int x^2(x^3-1)^4\,dx = \frac{1}{3}\int (x^3-1)^4(3x^2)\,dx = \frac{1}{3}\left[\frac{(x^3-1)^5}{5}\right] + C = \frac{(x^3-1)^5}{15} + C$

**Check:** $\frac{d}{dx}\left[\frac{(x^3-1)^5}{15} + C\right] = \frac{5(x^3-1)^4(3x^2)}{15} = x^2(x^3-1)^4$

**15.** $\int t\sqrt{t^2+2}\,dt = \frac{1}{2}\int (t^2+2)^{1/2}(2t)\,dt = \frac{1}{2}\frac{(t^2+2)^{3/2}}{3/2} + C = \frac{(t^2+2)^{3/2}}{3} + C$

**Check:** $\frac{d}{dt}\left[\frac{(t^2+2)^{3/2}}{3} + C\right] = \frac{3/2(t^2+2)^{1/2}(2t)}{3} = (t^2+2)^{1/2}t$

**17.** $\int 5x(1-x^2)^{1/3}\,dx = -\frac{5}{2}\int (1-x^2)^{1/3}(-2x)\,dx = -\frac{5}{2}\cdot\frac{(1-x^2)^{4/3}}{4/3} + C = -\frac{15}{8}(1-x^2)^{4/3} + C$

**Check:** $\frac{d}{dx}\left[-\frac{15}{8}(1-x^2)^{4/3} + C\right] = -\frac{15}{8}\cdot\frac{4}{3}(1-x^2)^{1/3}(-2x) = 5x(1-x^2)^{1/3} = 5x\sqrt[3]{1-x^2}$

**19.** $\int \frac{x}{(1-x^2)^3}\,dx = -\frac{1}{2}\int (1-x^2)^{-3}(-2x)\,dx = -\frac{1}{2}\frac{(1-x^2)^{-2}}{-2} + C = \frac{1}{4(1-x^2)^2} + C$

**Check:** $\frac{d}{dx}\left[\frac{1}{4(1-x^2)^2} + C\right] = \frac{1}{4}(-2)(1-x^2)^{-3}(-2x) = \frac{x}{(1-x^2)^3}$

**21.** $\int \frac{x^2}{(1+x^3)^2}\,dx = \frac{1}{3}\int (1+x^3)^{-2}(3x^2)\,dx = \frac{1}{3}\left[\frac{(1+x^3)^{-1}}{-1}\right] + C = -\frac{1}{3(1+x^3)} + C$

**Check:** $\frac{d}{dx}\left[-\frac{1}{3(1+x^3)} + C\right] = -\frac{1}{3}(-1)(1+x^3)^{-2}(3x^2) = \frac{x^2}{(1+x^3)^2}$

**23.** $\int \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int (1-x^2)^{-1/2}(-2x)\,dx = -\frac{1}{2}\frac{(1-x^2)^{1/2}}{1/2} + C = -\sqrt{1-x^2} + C$

**Check:** $\frac{d}{dx}[-(1-x^2)^{1/2} + C] = -\frac{1}{2}(1-x^2)^{-1/2}(-2x) = \frac{x}{\sqrt{1-x^2}}$

**25.** $\int \left(1+\frac{1}{t}\right)^3\left(\frac{1}{t^2}\right)dt = -\int \left(1+\frac{1}{t}\right)^3\left(-\frac{1}{t^2}\right)dt = -\frac{[1+(1/t)]^4}{4} + C$

**Check:** $\frac{d}{dt}\left[-\frac{[1+(1/t)]^4}{4} + C\right] = -\frac{1}{4}(4)\left(1+\frac{1}{t}\right)^3\left(-\frac{1}{t^2}\right) = \frac{1}{t^2}\left(1+\frac{1}{t}\right)^3$

**27.** $\int \frac{1}{\sqrt{2x}}\,dx = \frac{1}{2}\int (2x)^{-1/2}\,2\,dx = \frac{1}{2}\left[\frac{(2x)^{1/2}}{1/2}\right] + C = \sqrt{2x} + C$

**Check:** $\frac{d}{dx}\left[\sqrt{2x} + C\right] = \frac{1}{2}(2x)^{-1/2}(2) = \frac{1}{\sqrt{2x}}$

**29.** $\int \frac{x^2 + 3x + 7}{\sqrt{x}}\,dx = \int (x^{3/2} + 3x^{1/2} + 7x^{-1/2})\,dx = \frac{2}{5}x^{5/2} + 2x^{3/2} + 14x^{1/2} + C = \frac{2}{5}\sqrt{x}(x^2 + 5x + 35) + C$

**Check:** $\frac{d}{dx}\left[\frac{2}{5}x^{5/2} + 2x^{3/2} + 14x^{1/2} + C\right] = \frac{x^2 + 3x + 7}{\sqrt{x}}$

**31.** $\int t^2\left(t - \frac{2}{t}\right)dt = \int (t^3 - 2t)\,dt = \frac{1}{4}t^4 - t^2 + C$

**Check:** $\frac{d}{dt}\left[\frac{1}{4}t^4 - t^2 + C\right] = t^3 - 2t = t^2\left(t - \frac{2}{t}\right)$

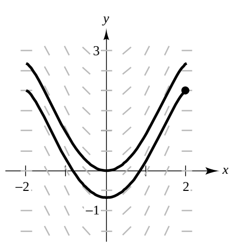
**33.** $\int (9 - y)\sqrt{y}\,dy = \int (9y^{1/2} - y^{3/2})\,dy = 9\left(\frac{2}{3}y^{3/2}\right) - \frac{2}{5}y^{5/2} + C = \frac{2}{5}y^{3/2}(15 - y) + C$

**Check:** $\frac{d}{dy}\left[\frac{2}{5}y^{3/2}(15 - y) + C\right] = \frac{d}{dy}\left[6y^{3/2} - \frac{2}{5}y^{5/2} + C\right] = 9y^{1/2} - y^{3/2} = (9 - y)\sqrt{y}$

**35.** 
$$\begin{aligned} y &= \int\left[4x + \frac{4x}{\sqrt{16 - x^2}}\right]dx \\ &= 4\int x\,dx - 2\int (16 - x^2)^{-1/2}(-2x)\,dx \\ &= 4\left(\frac{x^2}{2}\right) - 2\left[\frac{(16 - x^2)^{1/2}}{1/2}\right] + C \\ &= 2x^2 - 4\sqrt{16 - x^2} + C \end{aligned}$$

**37.** 
$$\begin{aligned} y &= \int \frac{x + 1}{(x^2 + 2x - 3)^2}\,dx \\ &= \frac{1}{2}\int (x^2 + 2x - 3)^{-2}(2x + 2)\,dx \\ &= \frac{1}{2}\left[\frac{(x^2 + 2x - 3)^{-1}}{-1}\right] + C \\ &= -\frac{1}{2(x^2 + 2x - 3)} + C \end{aligned}$$

**39.** (a)



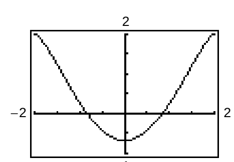

(b) $\frac{dy}{dx} = x\sqrt{4 - x^2}$, $(2, 2)$

$$y = \int x\sqrt{4 - x^2}\,dx = -\frac{1}{2}\int (4 - x^2)^{1/2}(-2x\,dx)$$

$$= -\frac{1}{2}\cdot\frac{2}{3}(4 - x^2)^{3/2} + C = -\frac{1}{3}(4 - x^2)^{3/2} + C$$

$(2, 2)$: $2 = -\frac{1}{3}(4 - 2^2)^{3/2} + C \Longrightarrow C = 2$

$$y = -\frac{1}{3}(4 - x^2)^{3/2} + 2$$




**41.** $\int \pi \sin \pi x\,dx = -\cos \pi x + C$

**43.** $\int \sin 2x\,dx = \frac{1}{2}\int (\sin 2x)(2x)\,dx = -\frac{1}{2}\cos 2x + C$

**45.** $\int \frac{1}{\theta^2}\cos\frac{1}{\theta}\,d\theta = -\int \cos\frac{1}{\theta}\left(-\frac{1}{\theta^2}\right)d\theta = -\sin\frac{1}{\theta} + C$

**47.** $\displaystyle\int \sin 2x \cos 2x\,dx = \frac{1}{2}\int (\sin 2x)(2\cos 2x)\,dx = \frac{1}{2}\frac{(\sin 2x)^2}{2} + C = \frac{1}{4}\sin^2 2x + C$ OR

$\displaystyle\int \sin 2x \cos 2x\,dx = -\frac{1}{2}\int (\cos 2x)(-2\sin 2x)\,dx = -\frac{1}{2}\frac{(\cos 2x)^2}{2} + C_1 = -\frac{1}{4}\cos^2 2x + C_1$ OR

$\displaystyle\int \sin 2x \cos 2x\,dx = \frac{1}{2}\int 2\sin 2x \cos 2x\,dx = \frac{1}{2}\int \sin 4x\,dx - \frac{1}{8}\cos 4x + C_2$

**49.** $\displaystyle\int \tan^4 x \sec^2 x\,dx = \frac{\tan^5 x}{5} + C = \frac{1}{5}\tan^5 x + C$

**51.** $\displaystyle\int \frac{\csc^2 x}{\cot^3 x}\,dx = -\int (\cot x)^{-3}(-\csc^2 x)\,dx$

$\displaystyle = -\frac{(\cot x)^{-2}}{-2} + C = \frac{1}{2\cot^2 x} + C = \frac{1}{2}\tan^2 x + C = \frac{1}{2}(\sec^2 x - 1) + C = \frac{1}{2}\sec^2 x + C_1$

**53.** $\displaystyle\int \cot^2 x\,dx = \int (\csc^2 x - 1)\,dx = -\cot x - x + C$

**55.** $\displaystyle f(x) = \int \cos\frac{x}{2}\,dx = 2\sin\frac{x}{2} + C$

Since $f(0) = 3 = 2\sin 0 + C$, $C = 3$. Thus,

$\displaystyle f(x) = 2\sin\frac{x}{2} + 3.$

**57.** $u = x + 2,\ x = u - 2,\ dx = du$

$$\begin{aligned}\int x\sqrt{x+2}\,dx &= \int (u-2)\sqrt{u}\,du\\ &= \int (u^{3/2} - 2u^{1/2})\,du\\ &= \frac{2}{5}u^{5/2} - \frac{4}{3}u^{3/2} + C\\ &= \frac{2u^{3/2}}{15}(3u - 10) + C\\ &= \frac{2}{15}(x+2)^{3/2}[3(x+2) - 10] + C\\ &= \frac{2}{15}(x+2)^{3/2}(3x - 4) + C\end{aligned}$$

**59.** $u = 1 - x,\ x = 1 - u,\ dx = -du$

$$\begin{aligned}\int x^2\sqrt{1-x}\,dx &= -\int (1-u)^2\sqrt{u}\,du\\ &= -\int (u^{1/2} - 2u^{3/2} + u^{5/2})\,du\\ &= -\left(\frac{2}{3}u^{3/2} - \frac{4}{5}u^{5/2} + \frac{2}{7}u^{7/2}\right) + C\\ &= -\frac{2u^{3/2}}{105}(35 - 42u + 15u^2) + C\\ &= -\frac{2}{105}(1-x)^{3/2}[35 - 42(1-x) + 15(1-x)^2] + C\\ &= -\frac{2}{105}(1-x)^{3/2}(15x^2 + 12x + 8) + C\end{aligned}$$

**61.** $u = 2x - 1,\ x = \frac{1}{2}(u + 1),\ dx = \frac{1}{2}\,du$

$$\int \frac{x^2 - 1}{\sqrt{2x - 1}}\,dx = \int \frac{[(1/2)(u + 1)]^2 - 1}{\sqrt{u}}\,\frac{1}{2}\,du$$
$$= \frac{1}{8}\int u^{-1/2}[(u^2 + 2u + 1) - 4]\,du$$
$$= \frac{1}{8}\int (u^{3/2} + 2u^{1/2} - 3u^{-1/2})\,du$$
$$= \frac{1}{8}\left(\frac{2}{5}u^{5/2} + \frac{4}{3}u^{3/2} - 6u^{1/2}\right) + C$$
$$= \frac{u^{1/2}}{60}(3u^2 + 10u - 45) + C$$
$$= \frac{\sqrt{2x - 1}}{60}[3(2x - 1)^2 + 10(2x - 1) - 45] + C$$
$$= \frac{1}{60}\sqrt{2x - 1}(12x^2 + 8x - 52) + C$$
$$= \frac{1}{15}\sqrt{2x - 1}(3x^2 + 2x - 13) + C$$

**63.** $u = x + 1,\ x = u - 1,\ dx = du$

$$\int \frac{-x}{(x + 1) - \sqrt{x + 1}}\,dx = \int \frac{-(u - 1)}{u - \sqrt{u}}\,du$$
$$= -\int \frac{\left(\sqrt{u} + 1\right)\left(\sqrt{u} - 1\right)}{\sqrt{u}\left(\sqrt{u} - 1\right)}\,du$$
$$= -\int (1 + u^{-1/2})\,du$$
$$= -(u + 2u^{1/2}) + C$$
$$= -u - 2\sqrt{u} + C$$
$$= -(x + 1) - 2\sqrt{x + 1} + C$$
$$= -x - 2\sqrt{x + 1} - 1 + C$$
$$= -\left(x + 2\sqrt{x + 1}\right) + C_1$$

where $C_1 = -1 + C$.

**65.** Let $u = x^2 + 1,\ du = 2x\,dx$.

$$\int_{-1}^{1} x(x^2 + 1)^3\,dx = \frac{1}{2}\int_{-1}^{1} (x^2 + 1)^3(2x)\,dx = \left[\frac{1}{8}(x^2 + 1)^4\right]_{-1}^{1} = 0$$

**67.** Let $u = x^3 + 1,\ du = 3x^2\,dx$

$$\int_{1}^{2} 2x^2\sqrt{x^3 + 1}\,dx = 2 \cdot \frac{1}{3}\int_{1}^{2} (x^3 + 1)^{1/2}(3x^2)\,dx$$
$$= \left[\frac{2}{3}\,\frac{(x^3 + 1)^{3/2}}{3/2}\right]_{1}^{2}$$
$$= \frac{4}{9}\left[(x^3 + 1)^{3/2}\right]_{1}^{2}$$
$$= \frac{4}{9}\left[27 - 2\sqrt{2}\right] = 12 - \frac{8}{9}\sqrt{2}$$

**69.** Let $u = 2x + 1$, $du = 2\,dx$.

$$\int_0^4 \frac{1}{\sqrt{2x+1}}\,dx = \frac{1}{2}\int_0^4 (2x+1)^{-1/2}(2)\,dx = \Big[\sqrt{2x+1}\Big]_0^4 = \sqrt{9} - \sqrt{1} = 2$$

**71.** Let $u = 1 + \sqrt{x}$, $du = \dfrac{1}{2\sqrt{x}}\,dx$.

$$\int_1^9 \frac{1}{\sqrt{x}\left(1+\sqrt{x}\right)^2}\,dx = 2\int_1^9 \left(1+\sqrt{x}\right)^{-2}\left(\frac{1}{2\sqrt{x}}\right)dx = \left[-\frac{2}{1+\sqrt{x}}\right]_1^9 = -\frac{1}{2} + 1 = \frac{1}{2}$$

**73.** $u = 2 - x$, $x = 2 - u$, $dx = -du$

When $x = 1$, $u = 1$. When $x = 2$, $u = 0$.

$$\int_1^2 (x-1)\sqrt{2-x}\,dx = \int_1^0 -\left[(2-u)-1\right]\sqrt{u}\,du = \int_1^0 \left(u^{3/2} - u^{1/2}\right)du = \left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_1^0 = -\left[\frac{2}{5} - \frac{2}{3}\right] = \frac{4}{15}$$

**75.** $\displaystyle\int_0^{\pi/2} \cos\left(\frac{2}{3}x\right)dx = \left[\frac{3}{2}\sin\left(\frac{2}{3}x\right)\right]_0^{\pi/2} = \frac{3}{2}\left(\frac{\sqrt{3}}{2}\right) = \frac{3\sqrt{3}}{4}$

**77.** $u = x + 1$, $x = u - 1$, $dx = du$

When $x = 0$, $u = 1$. When $x = 7$, $u = 8$.

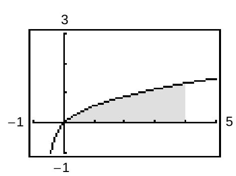
$$\text{Area} = \int_0^7 x\sqrt[3]{x+1}\,dx = \int_1^8 (u-1)\sqrt[3]{u}\,du$$

$$= \int_1^8 \left(u^{4/3} - u^{1/3}\right)du = \left[\frac{3}{7}u^{7/3} - \frac{3}{4}u^{4/3}\right]_1^8 = \left(\frac{384}{7} - 12\right) - \left(\frac{3}{7} - \frac{3}{4}\right) = \frac{1209}{28}$$

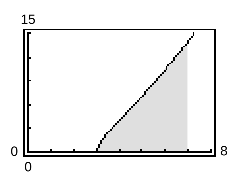
**79.** $\displaystyle A = \int_0^{\pi} (2\sin x + \sin 2x)\,dx = -\left[2\cos x + \frac{1}{2}\cos 2x\right]_0^{\pi} = 4$

**81.** $\displaystyle \text{Area} = \int_{\pi/2}^{2\pi/3} \sec^2\left(\frac{x}{2}\right)dx = 2\int_{\pi/2}^{2\pi/3} \sec^2\left(\frac{x}{2}\right)\left(\frac{1}{2}\right)dx = \left[2\tan\left(\frac{x}{2}\right)\right]_{\pi/2}^{2\pi/3} = 2\left(\sqrt{3} - 1\right)$

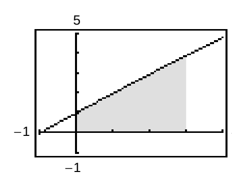
**83.** $\displaystyle\int_0^4 \frac{x}{\sqrt{2x+1}}\,dx \approx 3.333 = \frac{10}{3}$




**85.** $\displaystyle\int_3^7 x\sqrt{x-3}\,dx \approx 28.8 = \frac{144}{5}$




**87.** $\displaystyle\int_0^3 \left(\theta + \cos\frac{\theta}{6}\right)d\theta \approx 7.377$




**89.** $\displaystyle\int (2x-1)^2\,dx = \frac{1}{2}\int (2x-1)^2\,2\,dx = \frac{1}{6}(2x-1)^3 + C_1 = \frac{4}{3}x^3 - 2x^2 + x - \frac{1}{6} + C_1$

$\displaystyle\int (2x-1)^2\,dx = \int (4x^2 - 4x + 1)\,dx = \frac{4}{3}x^3 - 2x^2 + x + C_2$

They differ by a constant: $C_2 = C_1 - \dfrac{1}{6}$.

**91.** $f(x) = x^2(x^2 + 1)$ is even.

$$\int_{-2}^{2} x^2(x^2 + 1)\,dx = 2\int_0^2 (x^4 + x^2)\,dx = 2\left[\frac{x^5}{5} + \frac{x^3}{3}\right]_0^2$$
$$= 2\left[\frac{32}{5} + \frac{8}{3}\right] = \frac{272}{15}$$

**93.** $f(x) = x(x^2 + 1)^3$ is odd.

$$\int_{-2}^{2} x(x^2 + 1)^3\,dx = 0$$

**95.** $\int_0^2 x^2\,dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3}$; the function $x^2$ is an even function.

(a) $\int_{-2}^{0} x^2\,dx = \int_0^2 x^2\,dx = \frac{8}{3}$

(b) $\int_{-2}^{2} x^2\,dx = 2\int_0^2 x^2\,dx = \frac{16}{3}$

(c) $\int_0^2 (-x^2)\,dx = -\int_0^2 x^2\,dx = -\frac{8}{3}$

(d) $\int_{-2}^{0} 3x^2\,dx = 3\int_0^2 x^2\,dx = 8$

**97.** $\int_{-4}^{4}(x^3 + 6x^2 - 2x - 3)\,dx = \int_{-4}^{4}(x^3 - 2x)\,dx + \int_{-4}^{4}(6x^2 - 3)\,dx = 0 + 2\int_0^4 (6x^2 - 3)\,dx = 2\Big[2x^3 - 3x\Big]_0^4 = 232$

**99.** Answers will vary. See "Guidelines for Making a Change of Variables" on page 292.

**101.** $f(x) = x(x^2 + 1)^2$ is odd. Hence, $\int_{-2}^{2} x(x^2 + 1)^2\,dx = 0.$

**103.** $\frac{dV}{dt} = \frac{k}{(t + 1)^2}$

$$V(t) = \int \frac{k}{(t + 1)^2}\,dt = -\frac{k}{t + 1} + C$$

$$V(0) = -k + C = 500{,}000$$

$$V(1) = -\frac{1}{2}k + C = 400{,}000$$

Solving this system yields $k = -200{,}000$ and $C = 300{,}000$. Thus,

$$V(t) = \frac{200{,}000}{t + 1} + 300{,}000.$$

When $t = 4$, $V(4) = \$340{,}000$.

**105.** $\frac{1}{b - a}\int_a^b \left[74.50 + 43.75 \sin\frac{\pi t}{6}\right] dt = \frac{1}{b - a}\left[74.50t - \frac{262.5}{\pi}\cos\frac{\pi t}{6}\right]_a^b$

(a) $\frac{1}{3}\left[74.50t - \frac{262.5}{\pi}\cos\frac{\pi t}{6}\right]_0^3 = \frac{1}{3}\left(223.5 + \frac{262.5}{\pi}\right) \approx 102.352$ thousand units

(b) $\frac{1}{3}\left[74.50t - \frac{262.5}{\pi}\cos\frac{\pi t}{6}\right]_3^6 = \frac{1}{3}\left(447 + \frac{262.5}{\pi} - 223.5\right) \approx 102.352$ thousand units

(c) $\frac{1}{12}\left[74.50t - \frac{262.5}{\pi}\cos\frac{\pi t}{6}\right]_0^{12} = \frac{1}{12}\left(894 - \frac{262.5}{\pi} + \frac{262.5}{\pi}\right) = 74.5$ thousand units

**107.** $\dfrac{1}{b-a}\displaystyle\int_a^b [2\sin(60\pi t) + \cos(120\pi t)]\,dt = \dfrac{1}{b-a}\left[-\dfrac{1}{30\pi}\cos(60\pi t) + \dfrac{1}{120\,\pi}\sin(120\pi t)\right]_a^b$

(a) $\dfrac{1}{(1/60)-0}\left[-\dfrac{1}{30\pi}\cos(60\pi t) + \dfrac{1}{120\pi}\sin(120\pi t)\right]_0^{1/60} = 60\left[\left(\dfrac{1}{30\pi}+0\right) - \left(-\dfrac{1}{30\pi}\right)\right] = \dfrac{4}{\pi} \approx 1.273$ amps

(b) $\dfrac{1}{(1/240)-0}\left[-\dfrac{1}{30\pi}\cos(60\pi t) + \dfrac{1}{120\pi}\sin(120\pi t)\right]_0^{1/240} = 240\left[\left(-\dfrac{1}{30\sqrt{2}\pi} + \dfrac{1}{120\pi}\right) - \left(-\dfrac{1}{30\pi}\right)\right]$

$= \dfrac{2}{\pi}\left(5 - 2\sqrt{2}\right) \approx 1.382$ amps

(c) $\dfrac{1}{(1/30)-0}\left[-\dfrac{1}{30\pi}\cos(60\pi t) + \dfrac{1}{120\pi}\sin(120\pi t)\right]_0^{1/30} = 30\left[\left(\dfrac{1}{30\pi}\right) - \left(-\dfrac{1}{30\pi}\right)\right] = 0$ amps

**109.** False

$$\int (2x+1)^2\,dx = \frac{1}{2}\int (2x+1)^2\,2\,dx = \frac{1}{6}(2x+1)^3 + C$$

**111.** True

$$\int_{-10}^{10} (ax^3 + bx^2 + cx + d)\,dx = \underbrace{\int_{-10}^{10} (ax^3 + cx)\,dx}_{\text{Odd}} + \underbrace{\int_{-10}^{10} (bx^2 + d)\,dx}_{\text{Even}} = 0 + 2\int_0^{10} (bx^2 + d)\,dx$$

**113.** True

$$4\int \sin x \cos x\,dx = 2\int \sin 2x\,dx = -\cos 2x + C$$

**115.** Let $u = x + h$, then $du = dx$. When $x = a$, $u = a + h$. When $x = b$, $u = b + h$. Thus,

$$\int_a^b f(x+h)\,dx = \int_{a+h}^{b+h} f(u)\,du = \int_{a+h}^{b+h} f(x)\,dx.$$

## Section 4.6 Numerical Integration

**1.** Exact: $\displaystyle\int_0^2 x^2\,dx = \left[\frac{1}{3}x^3\right]_0^2 = \frac{8}{3} \approx 2.6667$

Trapezoidal: $\displaystyle\int_0^2 x^2\,dx \approx \frac{1}{4}\left[0 + 2\left(\frac{1}{2}\right)^2 + 2(1)^2 + 2\left(\frac{3}{2}\right)^2 + (2)^2\right] = \frac{11}{4} = 2.7500$

Simpson's: $\displaystyle\int_0^2 x^2\,dx \approx \frac{1}{6}\left[0 + 4\left(\frac{1}{2}\right)^2 + 2(1)^2 + 4\left(\frac{3}{2}\right)^2 + (2)^2\right] = \frac{8}{3} \approx 2.6667$

**3.** Exact: $\displaystyle\int_0^2 x^3\,dx = \left[\frac{x^4}{4}\right]_0^2 = 4.000$

Trapezoidal: $\displaystyle\int_0^2 x^3\,dx \approx \frac{1}{4}\left[0 + 2\left(\frac{1}{2}\right)^3 + 2(1)^3 + 2\left(\frac{3}{2}\right)^3 + (2)^3\right] = \frac{17}{4} = 4.2500$

Simpson's: $\displaystyle\int_0^2 x^3\,dx \approx \frac{1}{6}\left[0 + 4\left(\frac{1}{2}\right)^3 + 2(1)^3 + 4\left(\frac{3}{2}\right)^3 + (2)^3\right] = \frac{24}{6} = 4.0000$

**5.** Exact: $\int_0^2 x^3\,dx = \left[\frac{1}{4}x^4\right]_0^2 = 4.0000$

Trapezoidal: $\int_0^2 x^3\,dx \approx \frac{1}{8}\left[0 + 2\left(\frac{1}{4}\right)^3 + 2\left(\frac{2}{4}\right)^3 + 2\left(\frac{3}{4}\right)^3 + 2(1)^3 + 2\left(\frac{5}{4}\right)^3 + 2\left(\frac{6}{4}\right)^3 + 2\left(\frac{7}{4}\right)^3 + 8\right] = 4.0625$

Simpson's: $\int_0^2 x^3\,dx \approx \frac{1}{12}\left[0 + 4\left(\frac{1}{4}\right)^3 + 2\left(\frac{2}{4}\right)^3 + 4\left(\frac{3}{4}\right)^3 + 2(1)^3 + 4\left(\frac{5}{4}\right)^3 + 2\left(\frac{6}{4}\right)^3 + 4\left(\frac{7}{4}\right)^3 + 8\right] = 4.0000$

**7.** Exact: $\int_4^9 \sqrt{x}\,dx = \left[\frac{2}{3}x^{3/2}\right]_4^9 = 18 - \frac{16}{3} = \frac{38}{3} \approx 12.6667$

Trapezoidal: $\int_4^9 \sqrt{x}\,dx \approx \frac{5}{16}\left[2 + 2\sqrt{\frac{37}{8}} + 2\sqrt{\frac{21}{4}} + 2\sqrt{\frac{47}{8}} + 2\sqrt{\frac{26}{4}} + 2\sqrt{\frac{57}{8}} + 2\sqrt{\frac{31}{4}} + 2\sqrt{\frac{67}{8}} + 3\right]$

$\approx 12.6640$

Simpson's: $\int_4^9 \sqrt{x}\,dx \approx \frac{5}{24}\left[2 + 4\sqrt{\frac{37}{8}} + \sqrt{21} + 4\sqrt{\frac{47}{8}} + \sqrt{26} + 4\sqrt{\frac{57}{8}} + \sqrt{31} + 4\sqrt{\frac{67}{8}} + 3\right] \approx 12.6667$

**9.** Exact: $\int_1^2 \frac{1}{(x+1)^2}\,dx = \left[-\frac{1}{x+1}\right]_1^2 = -\frac{1}{3} + \frac{1}{2} = \frac{1}{6} \approx 0.1667$

Trapezoidal: $\int_1^2 \frac{1}{(x+1)^2}\,dx \approx \frac{1}{8}\left[\frac{1}{4} + 2\left(\frac{1}{((5/4)+1)^2}\right) + 2\left(\frac{1}{((3/2)+1)^2}\right) + 2\left(\frac{1}{((7/4)+1)^2}\right) + \frac{1}{9}\right]$

$= \frac{1}{8}\left(\frac{1}{4} + \frac{32}{81} + \frac{8}{25} + \frac{32}{121} + \frac{1}{9}\right) \approx 0.1676$

Simpson's: $\int_1^2 \frac{1}{(x+1)^2}\,dx \approx \frac{1}{12}\left[\frac{1}{4} + 4\left(\frac{1}{((5/4)+1)^2}\right) + 2\left(\frac{1}{((3/2)+1)^2}\right) + 4\left(\frac{1}{((7/4)+1)^2}\right) + \frac{1}{9}\right]$

$= \frac{1}{12}\left(\frac{1}{4} + \frac{64}{81} + \frac{8}{25} + \frac{64}{121} + \frac{1}{9}\right) \approx 0.1667$

**11.** Trapezoidal: $\int_0^2 \sqrt{1+x^3}\,dx \approx \frac{1}{4}[1 + 2\sqrt{1+(1/8)} + 2\sqrt{2} + 2\sqrt{1+(27/8)} + 3] \approx 3.283$

Simpson's: $\int_0^2 \sqrt{1+x^3}\,dx \approx \frac{1}{6}[1 + 4\sqrt{1+(1/8)} + 2\sqrt{2} + 4\sqrt{1+(27/8)} + 3] \approx 3.240$

Graphing utility: 3.241

**13.** $\int_0^1 \sqrt{x}\sqrt{1-x}\,dx = \int_0^1 \sqrt{x(1-x)}\,dx$

Trapezoidal: $\int_0^1 \sqrt{x(1-x)}\,dx \approx \frac{1}{8}\left[0 + 2\sqrt{\frac{1}{4}\left(1-\frac{1}{4}\right)} + 2\sqrt{\frac{1}{2}\left(1-\frac{1}{2}\right)} + 2\sqrt{\frac{3}{4}\left(1-\frac{3}{4}\right)}\right] \approx 0.342$

Simpson's: $\int_0^1 \sqrt{x(1-x)}\,dx \approx \frac{1}{12}\left[0 + 4\sqrt{\frac{1}{4}\left(1-\frac{1}{4}\right)} + 2\sqrt{\frac{1}{2}\left(1-\frac{1}{2}\right)} + 4\sqrt{\frac{3}{4}\left(1-\frac{3}{4}\right)}\right] \approx 0.372$

Graphing utility: 0.393

**15.** Trapezoidal: $\int_0^{\sqrt{\pi/2}} \cos(x^2)\,dx \approx \frac{\sqrt{\pi/2}}{8}\left[\cos 0 + 2\cos\left(\frac{\sqrt{\pi/2}}{4}\right)^2 + 2\cos\left(\frac{\sqrt{\pi/2}}{2}\right)^2 + 2\cos\left(\frac{\sqrt{\pi/2}}{4}\right)^2 + \cos\left(\sqrt{\frac{\pi}{2}}\right)^2\right]$

$\approx 0.957$

Simpson's: $\int_0^{\sqrt{\pi/2}} \cos(x^2)\,dx \approx \frac{\sqrt{\pi/2}}{12}\left[\cos 0 + 4\cos\left(\frac{\sqrt{\pi/2}}{4}\right)^2 + 2\cos\left(\frac{\sqrt{\pi/2}}{2}\right)^2 + 4\cos\left(\frac{\sqrt{\pi/2}}{4}\right)^2 + \cos\left(\sqrt{\frac{\pi}{2}}\right)^2\right]$

$\approx 0.978$

Graphing utility: 0.977

**17.** Trapezoidal: $\int_1^{1.1} \sin x^2\,dx \approx \frac{1}{80}[\sin(1) + 2\sin(1.025)^2 + 2\sin(1.05)^2 + 2\sin(1.075)^2 + \sin(1.1)^2] \approx 0.089$

Simpson's: $\int_1^{1.1} \sin x^2\,dx \approx \frac{1}{120}[\sin(1) + 4\sin(1.025)^2 + 2\sin(1.05)^2 + 4\sin(1.075)^2 + \sin(1.1)^2] \approx 0.089$

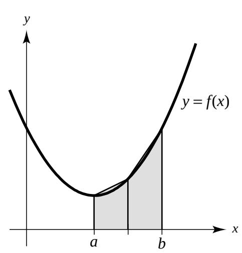
Graphing utility: 0.089

**19.** Trapezoidal: $\int_0^{\pi/4} x\tan x\,dx \approx \frac{\pi}{32}\left[0 + 2\left(\frac{\pi}{16}\right)\tan\left(\frac{\pi}{16}\right) + 2\left(\frac{2\pi}{16}\right)\tan\left(\frac{2\pi}{16}\right) + 2\left(\frac{3\pi}{16}\right)\tan\left(\frac{3\pi}{16}\right) + \frac{\pi}{4}\right] \approx 0.194$

Simpson's: $\int_0^{\pi/4} x\tan x\,dx \approx \frac{\pi}{48}\left[0 + 4\left(\frac{\pi}{16}\right)\tan\left(\frac{\pi}{16}\right) + 2\left(\frac{2\pi}{16}\right)\tan\left(\frac{2\pi}{16}\right) + 4\left(\frac{3\pi}{16}\right)\tan\left(\frac{3\pi}{16}\right) + \frac{\pi}{4}\right] \approx 0.186$

Graphing utility: 0.186

**21.** (a)




The Trapezoidal Rule overestimates the area if the graph of the integrand is concave up.

**23.** $f(x) = x^3$

$f'(x) = 3x^2$

$f''(x) = 6x$

$f'''(x) = 6$

$f^{(4)}(x) = 0$

(a) Trapezoidal: Error $\le \frac{(2-0)^3}{12(4^2)}(12) = 0.5$ since $f''(x)$ is maximum in $[0, 2]$ when $x = 2$.

(b) Simpson's: Error $\le \frac{(2-0)^5}{180(4^4)}(0) = 0$ since $f^{(4)}(x) = 0$.

**25.** $f''(x) = \frac{2}{x^3}$ in $[1, 3]$.

(a) $|f''(x)|$ is maximum when $x = 1$ and $|f''(1)| = 2$.

Trapezoidal: Error $\le \frac{2^3}{12n^2}(2) < 0.00001$, $n^2 > 133{,}333.33$, $n > 365.15$; let $n = 366$.

$f^{(4)}(x) = \frac{24}{x^5}$ in $[1, 3]$

(b) $|f^{(4)}(x)|$ is maximum when $x = 1$ and when $|f^{(4)}(1)| = 24$.

Simpson's: Error $\le \frac{2^5}{180n^4}(24) < 0.00001$, $n^4 > 426{,}666.67$, $n > 25.56$; let $n = 26$.

**27.** $f(x) = \sqrt{1 + x}$

(a) $f''(x) = -\dfrac{1}{4(1 + x)^{3/2}}$ in $[0, 2]$.

$|f''(x)|$ is maximum when $x = 0$ and $|f''(0)| = \dfrac{1}{4}$.

Trapezoidal: Error $\leq \dfrac{8}{12n^2}\left(\dfrac{1}{4}\right) < 0.00001$, $n^2 > 16{,}666.67$, $n > 129.10$; let $n = 130$.

(b) $f^{(4)}(x) = \dfrac{-15}{16(1 + x)^{7/2}}$ in $[0, 2]$

$|f^{(4)}(x)|$ is maximum when $x = 0$ and $|f^{(4)}(0)| = \dfrac{15}{16}$.

Simpson's: Error $\leq \dfrac{32}{180n^4}\left(\dfrac{15}{16}\right) < 0.00001$, $n^4 > 16{,}666.67$, $n > 11.36$; let $n = 12$.

**29.** $f(x) = \tan(x^2)$

(a) $f''(x) = 2\sec^2(x^2)[1 + 4x^2\tan(x^2)]$ in $[0, 1]$.

$|f''(x)|$ is maximum when $x = 1$ and $|f''(1)| \approx 49.5305$.

Trapezoidal: Error $\leq \dfrac{(1 - 0)^3}{12n^2}(49.5305) < 0.00001$, $n^2 > 412{,}754.17$, $n > 642.46$; let $n = 643$.

(b) $f^{(4)}(x) = 8\sec^2(x^2)[12x^2 + (3 + 32x^4)\tan(x^2) + 36x^2\tan^2(x^2) + 48x^4\tan^3(x^2)]$ in $[0, 1]$

$|f^{(4)}(x)|$ is maximum when $x = 1$ and $|f^{(4)}(1)| \approx 9184.4734$.

Simpson's: Error $\leq \dfrac{(1 - 0)^5}{180n^4}(9184.4734) < 0.00001$, $n^4 > 5{,}102{,}485.22$, $n > 47.53$; let $n = 48$.

**31.** Let $f(x) = Ax^3 + Bx^2 + Cx + D$. Then $f^{(4)}(x) = 0$.

Simpson's: Error $\leq \dfrac{(b - a)^5}{180n^4}(0) = 0$

Therefore, Simpson's Rule is exact when approximating the integral of a cubic polynomial.

Example: $\displaystyle\int_0^1 x^3\,dx = \frac{1}{6}\left[0 + 4\left(\frac{1}{2}\right)^3 + 1\right] = \frac{1}{4}$

This is the exact value of the integral.

**33.** $f(x) = \sqrt{2 + 3x^2}$ on $[0, 4]$.

| $n$ | $L(n)$ | $M(n)$ | $R(n)$ | $T(n)$ | $S(n)$ |
|---|---|---|---|---|---|
| 4 | 12.7771 | 15.3965 | 18.4340 | 15.6055 | 15.4845 |
| 8 | 14.0868 | 15.4480 | 16.9152 | 15.5010 | 15.4662 |
| 10 | 14.3569 | 15.4544 | 16.6197 | 15.4883 | 15.4658 |
| 12 | 14.5386 | 15.4578 | 16.4242 | 15.4814 | 15.4657 |
| 16 | 14.7674 | 15.4613 | 16.1816 | 15.4745 | 15.4657 |
| 20 | 14.9056 | 15.4628 | 16.0370 | 15.4713 | 15.4657 |

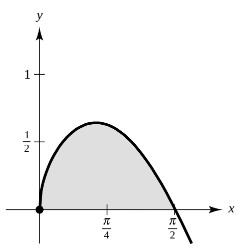
**35.** $f(x) = \sin\sqrt{x}$ on $[0, 4]$.

| $n$ | $L(n)$ | $M(n)$ | $R(n)$ | $T(n)$ | $S(n)$ |
|---|---|---|---|---|---|
| 4 | 2.8163 | 3.5456 | 3.7256 | 3.2709 | 3.3996 |
| 8 | 3.1809 | 3.5053 | 3.6356 | 3.4083 | 3.4541 |
| 10 | 3.2478 | 3.4990 | 3.6115 | 3.4296 | 3.4624 |
| 12 | 3.2909 | 3.4952 | 3.5940 | 3.4425 | 3.4674 |
| 16 | 3.3431 | 3.4910 | 3.5704 | 3.4568 | 3.4730 |
| 20 | 3.3734 | 3.4888 | 3.5552 | 3.4643 | 3.4759 |

**37.** $A = \int_0^{\pi/2} \sqrt{x}\cos x\, dx$

Simpson's Rule: $n = 14$

$$\int_0^{\pi/2} \sqrt{x}\cos x\, dx \approx \frac{\pi}{84}\left[\sqrt{0}\cos 0 + 4\sqrt{\frac{\pi}{28}}\cos\frac{\pi}{28} + 2\sqrt{\frac{\pi}{14}}\cos\frac{\pi}{14} + 4\sqrt{\frac{3\pi}{28}}\cos\frac{3\pi}{28} + \cdots + \sqrt{\frac{\pi}{2}}\cos\frac{\pi}{2}\right]$$
$$\approx 0.701$$




**39.** $W = \int_0^5 100x\sqrt{125 - x^3}\, dx$

Simpson's Rule: $n = 12$

$$\int_0^5 100x\sqrt{125 - x^3}\, dx \approx \frac{5}{3(12)}\left[0 + 400\left(\frac{5}{12}\right)\sqrt{125 - \left(\frac{5}{12}\right)^3} + 200\left(\frac{10}{12}\right)\sqrt{125 - \left(\frac{10}{12}\right)^3} + 400\left(\frac{15}{12}\right)\sqrt{125 - \left(\frac{15}{12}\right)^3} + \cdots + 0\right] \approx 10{,}233.58 \text{ ft} \cdot \text{lb}$$

**41.** $\int_0^{1/2} \frac{6}{\sqrt{1 - x^2}}\, dx$ Simpson's Rule, $n = 6$

$$\pi \approx \frac{\left(\frac{1}{2} - 0\right)}{3(6)}[6 + 4(6.0209) + 2(6.0851) + 4(6.1968) + 2(6.3640) + 4(6.6002) + 6.9282]$$
$$\approx \frac{1}{36}[113.098] \approx 3.1416$$

**43.** Area $\approx \frac{1000}{2(10)}[125 + 2(125) + 2(120) + 2(112) + 2(90) + 2(90) + 2(95) + 2(88) + 2(75) + 2(35)] = 89{,}250$ sq m

**45.** $\int_0^t \sin\sqrt{x}\, dx = 2,\ n = 10$

By trial and error, we obtain $t \approx 2.477$.