85. As k increases, the time required for the object to reach the ground increases.

87. $y = \cosh x = \dfrac{e^x + e^{-x}}{2}$

$y' = \dfrac{e^x - e^{-x}}{2} = \sinh x$

89. $y = \cosh^{-1} x$

$\cosh y = x$

$(\sinh y)(y') = 1$

$y' = \dfrac{1}{\sinh y} = \dfrac{1}{\sqrt{\cosh^2 y - 1}} = \dfrac{1}{\sqrt{x^2 - 1}}$

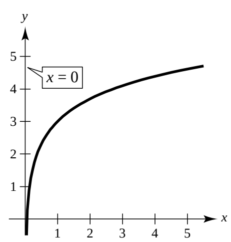
91. $y = \operatorname{sech} x = \dfrac{2}{e^x + e^{-x}}$

$y' = -2(e^x + e^{-x})^{-2}(e^x - e^{-x}) = \left(\dfrac{-2}{e^x + e^{-x}}\right)\left(\dfrac{e^x - e^{-x}}{e^x + e^{-x}}\right) = -\operatorname{sech} x \tanh x$

Review Exercises for Chapter 5

1. $f(x) = \ln x + 3$

Vertical shift 3 units upward
Vertical asymptote: $x = 0$



3. $\ln \sqrt[5]{\dfrac{4x^2 - 1}{4x^2 + 1}} = \dfrac{1}{5} \ln \dfrac{(2x - 1)(2x + 1)}{4x^2 + 1} = \dfrac{1}{5}[\ln(2x - 1) + \ln(2x + 1) - \ln(4x^2 + 1)]$

5. $\ln 3 + \dfrac{1}{3}\ln(4 - x^2) - \ln x = \ln 3 + \ln \sqrt[3]{4 - x^2} - \ln x = \ln\left(\dfrac{3\sqrt[3]{4 - x^2}}{x}\right)$

7. $\ln\sqrt{x + 1} = 2$

$\sqrt{x + 1} = e^2$

$x + 1 = e^4$

$x = e^4 - 1 \approx 53.598$

9. $g(x) = \ln\sqrt{x} = \dfrac{1}{2}\ln x$

$g'(x) = \dfrac{1}{2x}$

11. $f(x) = x\sqrt{\ln x}$

$f'(x) = \left(\dfrac{x}{2}\right)(\ln x)^{-1/2}\left(\dfrac{1}{x}\right) + \sqrt{\ln x}$

$= \dfrac{1}{2\sqrt{\ln x}} + \sqrt{\ln x} = \dfrac{1 + 2\ln x}{2\sqrt{\ln x}}$

13. $y = \dfrac{1}{b^2}\left[\ln(a + bx) + \dfrac{a}{a + bx}\right]$

$\dfrac{dy}{dx} = \dfrac{1}{b^2}\left[\dfrac{b}{a + bx} - \dfrac{ab}{(a + bx)^2}\right] = \dfrac{x}{(a + bx)^2}$

15. $y = -\dfrac{1}{a}\ln\left(\dfrac{a + bx}{x}\right) = -\dfrac{1}{a}[\ln(a + bx) - \ln x]$

$\dfrac{dy}{dx} = -\dfrac{1}{a}\left(\dfrac{b}{a + bx} - \dfrac{1}{x}\right) = \dfrac{1}{x(a + bx)}$

17. $u = 7x - 2,\ du = 7dx$

$\displaystyle\int \frac{1}{7x - 2}\,dx = \frac{1}{7}\int \frac{1}{7x - 2}(7)\,dx = \frac{1}{7}\ln|7x - 2| + C$

19. $\displaystyle\int \frac{\sin x}{1+\cos x}\,dx = -\int \frac{-\sin x}{1+\cos x}\,dx$

$= -\ln|1+\cos x| + C$

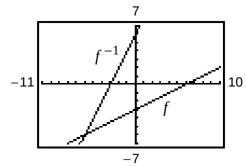
21. $\displaystyle\int_1^4 \frac{x+1}{x}\,dx = \int_1^4 \left(1+\frac{1}{x}\right)dx = \Big[x + \ln|x|\Big]_1^4 = 3 + \ln 4$

23. $\displaystyle\int_0^{\pi/3} \sec\theta\,d\theta = \Big[\ln|\sec\theta + \tan\theta|\Big]_0^{\pi/3} = \ln\left(2+\sqrt{3}\right)$

25. (a)

$$\begin{aligned} f(x) &= \tfrac{1}{2}x - 3 \\ y &= \tfrac{1}{2}x - 3 \\ 2(y+3) &= x \\ 2(x+3) &= y \\ f^{-1}(x) &= 2x + 6 \end{aligned}$$

(b)



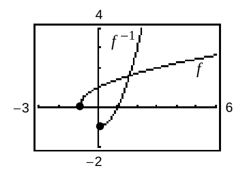
(c) $f^{-1}(f(x)) = f^{-1}\left(\tfrac{1}{2}x - 3\right) = 2\left(\tfrac{1}{2}x - 3\right) + 6 = x$

$f(f^{-1}(x)) = f(2x+6) = \tfrac{1}{2}(2x+6) - 3 = x$

27. (a)

$$\begin{aligned} f(x) &= \sqrt{x+1} \\ y &= \sqrt{x+1} \\ y^2 - 1 &= x \\ x^2 - 1 &= y \\ f^{-1}(x) &= x^2 - 1, x \ge 0 \end{aligned}$$

(b)



(c) $f^{-1}(f(x)) = f^{-1}\left(\sqrt{x+1}\right) = \sqrt{(x^2-1)^2} - 1 = x$

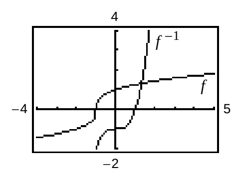
$f(f^{-1}(x)) = f(x^2-1) = \sqrt{(x^2-1)+1}$

$= \sqrt{x^2} = x$ for $x \ge 0$.

29. (a)

$$\begin{aligned} f(x) &= \sqrt[3]{x+1} \\ y &= \sqrt[3]{x+1} \\ y^3 - 1 &= x \\ x^3 - 1 &= y \\ f^{-1}(x) &= x^3 - 1 \end{aligned}$$

(b)



(c) $f^{-1}(f(x)) = f^{-1}\left(\sqrt[3]{x+1}\right) = \left(\sqrt[3]{x+1}\right)^3 - 1 = x$

$f(f^{-1}(x)) = f(x^3-1) = \sqrt[3]{(x^3-1)+1} = x$

31.

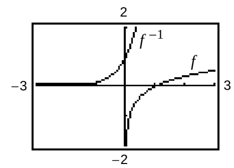
$$\begin{aligned} f(x) &= x^3 + 2 \\ f^{-1}(x) &= (x-2)^{1/3} \\ (f^{-1})'(x) &= \frac{1}{3}(x-2)^{-2/3} \\ (f^{-1})'(-1) &= \frac{1}{3}(-1-2)^{-2/3} = \frac{1}{3(-3)^{2/3}} \\ &= \frac{1}{3^{5/3}} \approx 0.160 \end{aligned}$$

33.

$$\begin{aligned} f(x) &= \tan x \\ f\left(\frac{\pi}{6}\right) &= \frac{\sqrt{3}}{3} \\ f'(x) &= \sec^2 x \\ f'\left(\frac{\pi}{6}\right) &= \frac{4}{3} \\ (f^{-1})'\left(\frac{\sqrt{3}}{3}\right) &= \frac{1}{f'(\pi/6)} = \frac{3}{4} \end{aligned}$$

35. (a)
$$f(x) = \ln \sqrt{x}$$
$$y = \ln\sqrt{x}$$
$$e^y = \sqrt{x}$$
$$e^{2y} = x$$
$$e^{2x} = y$$
$$f^{-1}(x) = e^{2x}$$

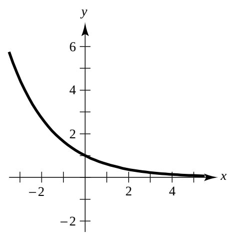
(b)



(c) $f^{-1}(f(x)) = f^{-1}(\ln \sqrt{x}) = e^{2\ln\sqrt{x}} = e^{\ln x} = x$

$f(f^{-1}(x)) = f(e^{2x}) = \ln \sqrt{e^{2x}} = \ln e^x = x$

37. $y = e^{-x/2}$



39. $f(x) = \ln(e^{-x^2}) = -x^2$

$f'(x) = -2x$

41. $g(t) = t^2e^t$

$g'(x) = t^2e^t + 2te^t = te^t(t + 2)$

43. $y = \sqrt{e^{2x} + e^{-2x}}$

$$y' = \frac{1}{2}(e^{2x} + e^{-2x})^{-1/2}(2e^{2x} - 2e^{-2x}) = \frac{e^{2x} - e^{-2x}}{\sqrt{e^{2x} + e^{-2x}}}$$

45. $g(x) = \dfrac{x^2}{e^x}$

$$g'(x) = \frac{e^x(2x) - x^2e^x}{e^{2x}} = \frac{x(2 - x)}{e^x}$$

47.
$$y(\ln x) + y^2 = 0$$
$$y\left(\frac{1}{x}\right) + (\ln x)\left(\frac{dy}{dx}\right) + 2y\left(\frac{dy}{dx}\right) = 0$$
$$(2y + \ln x)\frac{dy}{dx} = \frac{-y}{x}$$
$$\frac{dy}{dx} = \frac{-y}{x(2y + \ln x)}$$

49. Let $u = -3x^2$, $du = -6x\,dx$.

$$\int xe^{-3x^2}\,dx = -\frac{1}{6}\int e^{-3x^2}(-6x)\,dx = -\frac{1}{6}e^{-3x^2} + C$$

51.
$$\int \frac{e^{4x} - e^{2x} + 1}{e^x}\,dx = \int (e^{3x} - e^x + e^{-x})\,dx$$
$$= \frac{1}{3}e^{3x} - e^x - e^{-x} + C$$
$$= \frac{e^{4x} - 3e^{2x} - 3}{3e^x} + C$$

53.
$$\int xe^{1-x^2}\,dx = -\frac{1}{2}\int e^{1-x^2}(-2x)\,dx$$
$$= -\frac{1}{2}e^{1-x^2} + C$$

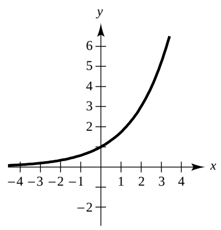
55. Let $u = e^x - 1$, $du = e^x\,dx$.

$$\int \frac{e^x}{e^x - 1}\,dx = \ln|e^x - 1| + C$$

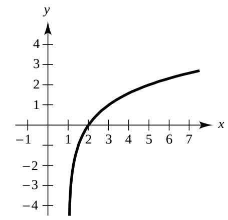
57.
$$y = e^x(a\cos 3x + b\sin 3x)$$
$$y' = e^x(-3a\sin 3x + 3b\cos 3x) + e^x(a\cos 3x + b\sin 3x)$$
$$= e^x[(-3a + b)\sin 3x + (a + 3b)\cos 3x]$$
$$y'' = e^x[3(-3a + b)\cos 3x - 3(a + 3b)\sin 3x] + e^x[(-3a + b)\sin 3x + (a + 3b)\cos 3x]$$
$$= e^x[(-6a - 8b)\sin 3x + (-8a + 6b)\cos 3x]$$
$$y'' - 2y' + 10y = e^x\{[(-6a - 8b) - 2(-3a + b) + 10b]\sin 3x + [(-8a + 6b) - 2(a + 3b) + 10a]\cos 3x\} = 0$$

59. Area $= \int_0^4 xe^{-x^2}\,dx = \left[-\frac{1}{2}e^{-x^2}\right]_0^4 = -\frac{1}{2}(e^{-16} - 1) \approx 0.500$

61. $y = 3^{3/2}$



63. $y = \log_2(x - 1)$



65. $f(x) = 3^{x-1}$

$f'(x) = 3^{x-1}\ln 3$

67. $y = x^{2x+1}$

$\ln y = (2x + 1)\ln x$

$\frac{y'}{y} = \frac{2x + 1}{x} + 2\ln x$

$y' = y\left(\frac{2x + 1}{x} + 2\ln x\right) = x^{2x+1}\left(\frac{2x + 1}{x} + 2\ln x\right)$

69. $g(x) = \log_3\sqrt{1 - x} = \frac{1}{2}\log_3(1 - x)$

$g'(x) = \frac{1}{2}\frac{-1}{(1 - x)\ln 3} = \frac{1}{2(x - 1)\ln 3}$

71. $\int (x + 1)5^{(x+1)^2}\,dx = \frac{1}{2}\frac{1}{\ln 5}5^{(x+1)^2} + C$

73. (a) $y = x^a$

$y' = ax^{a-1}$

(b) $y = a^x$

$y' = (\ln a)a^x$

(c) $y = x^x$

$\ln y = x\ln x$

$\frac{1}{y}y' = x \cdot \frac{1}{x} + (1)\ln x$

$y' = y(1 + \ln x)$

$y' = x^x(1 + \ln x)$

(d) $y = a^a$

$y' = 0$

75. $10{,}000 = Pe^{(0.07)(15)}$

$P = \frac{10{,}000}{e^{1.05}} \approx \$3499.38$

77. $P(h) = 30e^{kh}$

$P(18{,}000) = 30e^{18{,}000k} = 15$

$k = \frac{\ln(1/2)}{18{,}000} = \frac{-\ln 2}{18{,}000}$

$P(h) = 30e^{-(h\ln 2)/18{,}000}$

$P(35{,}000) = 30e^{-(35{,}000\ln 2)/18{,}000} \approx 7.79$ inches

79. $P = Ce^{0.015t}$

$2C = Ce^{0.015t}$

$2 = e^{0.015t}$

$\ln 2 = 0.015t$

$t = \frac{\ln 2}{0.015} \approx 46.21$ years

81. $\frac{dy}{dx} = \frac{x^2 + 3}{x}$

$\int dy = \int\left(x + \frac{3}{x}\right)dx$

$y = \frac{x^2}{2} + 3\ln|x| + C$

83. $y' - 2xy = 0$

$$\frac{dy}{dx} = 2xy$$

$$\int \frac{1}{y}\,dy = \int 2x\,dx$$

$$\ln|y| = x^2 + C_1$$

$$e^{x^2 + C_1} = y$$

$$y = Ce^{x^2}$$

85. $\dfrac{dy}{dx} = \dfrac{x^2 + y^2}{2xy}$ (homogeneous differential equation)

$$(x^2 + y^2)\,dx - 2xy\,dy = 0$$

Let $y = vx$, $dy = x\,dv + v\,dx$.

$$(x^2 + v^2x^2)\,dx - 2x(vx)(x\,dv + v\,dx) = 0$$

$$(x^2 + v^2x^2 - 2x^2v^2)\,dx - 2x^3v\,dv = 0$$

$$(x^2 - x^2v^2)\,dx = 2x^3v\,dv$$

$$(1 - v^2)\,dx = 2x\,dv$$

$$\int \frac{dx}{x} = \int \frac{2v}{1 - v^2}\,dv$$

$$\ln|x| = -\ln|1 - v^2| + C_1 = -\ln|1 - v^2| + \ln C$$

$$x = \frac{C}{1 - v^2} = \frac{C}{1 - (y/x)^2} = \frac{Cx^2}{x^2 - y^2}$$

$$1 = \frac{Cx}{x^2 - y^2} \quad \text{or} \quad C_1 = \frac{x}{x^2 - y^2}$$

87. $y = C_1x + C_2x^3$

$$y' = C_1 + 3C_2x^2$$

$$y'' = 6C_2x$$

$$x^2y'' - 3xy' + 3y = x^2(6C_2x) - 3x(C_1 + 3C_2x^2) + (C_1x = C_2x^3)$$

$$= 6C_2x^3 - 3C_1x - 9C_2x^3 + 3C_1x + 3C_2x^3 = 0$$

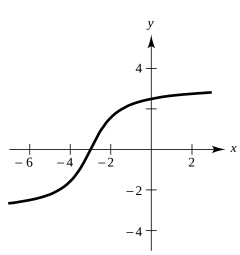
$x = 2, y = 0$: $0 = 2C_1 + 8C_2 \implies C_1 = -4C_2$

$x = 2, y' = 4$: $4 = C_1 + 12C_2$

$$4 = (-4C_2) + 12C_2 = 8C_2 \implies C_2 = \frac{1}{2}, C_1 = -2$$

$$y = -2x + \frac{1}{2}x^3$$

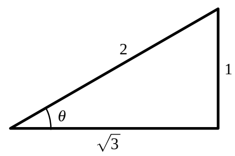
89. $f(x) = 2\arctan(x + 3)$



91. (a) Let $\theta = \arcsin \frac{1}{2}$

$\sin \theta = \frac{1}{2}$

$\sin\left(\arcsin \frac{1}{2}\right) = \sin \theta = \frac{1}{2}.$



(b) Let $\theta = \arcsin \frac{1}{2}$

$\sin \theta = \frac{1}{2}$

$\cos\left(\arcsin \frac{1}{2}\right) = \cos \theta = \frac{\sqrt{3}}{2}.$

93. $y = \tan(\arcsin x) = \dfrac{x}{\sqrt{1 - x^2}}$

$$y' = \frac{(1 - x^2)^{1/2} + x^2(1 - x^2)^{-1/2}}{1 - x^2} = (1 - x^2)^{-3/2}$$

95. $y = x \operatorname{arcsec} x$

$$y' = \frac{x}{|x|\sqrt{x^2 - 1}} + \operatorname{arcsec} x$$

97. $y = x(\arcsin x)^2 - 2x + 2\sqrt{1 - x^2} \arcsin x$

$$y' = \frac{2x \arcsin x}{\sqrt{1 - x^2}} + (\arcsin x)^2 - 2 + \frac{2\sqrt{1 - x^2}}{\sqrt{1 - x^2}} - \frac{2x}{\sqrt{1 - x^2}} \arcsin x = (\arcsin x)^2$$

99. Let $u = e^{2x}$, $du = 2e^{2x}\, dx$.

$$\int \frac{1}{e^{2x} + e^{-2x}}\, dx = \int \frac{e^{2x}}{1 + e^{4x}}\, dx = \frac{1}{2}\int \frac{1}{1 + (e^{2x})^2}(2e^{2x})\, dx = \frac{1}{2}\arctan(e^{2x}) + C$$

101. Let $u = x^2$, $du = 2x\, dx$.

$$\int \frac{x}{\sqrt{1 - x^4}}\, dx = \frac{1}{2}\int \frac{1}{\sqrt{1 - (x^2)^2}}(2x)\, dx = \frac{1}{2}\arcsin x^2 + C$$

103. Let $u = 16 + x^2$, $du = 2x\, dx$.

$$\int \frac{x}{16 + x^2}\, dx = \frac{1}{2}\int \frac{1}{16 + x^2}(2x)\, dx = \frac{1}{2}\ln(16 + x^2) + C$$

105. Let $u = \arctan\left(\frac{x}{2}\right)$, $du = \dfrac{2}{4 + x^2}\, dx$.

$$\int \frac{\arctan(x/2)}{4 + x^2}\, dx = \frac{1}{2}\int \left(\arctan \frac{x}{2}\right)\left(\frac{2}{4 + x^2}\right) dx = \frac{1}{4}\left(\arctan \frac{x}{2}\right)^2 + C$$

107. $\displaystyle\int \frac{dy}{\sqrt{A^2 - y^2}} = \int \sqrt{\frac{k}{m}}\, dt$

$\arcsin\left(\dfrac{y}{A}\right) = \sqrt{\dfrac{k}{m}}\, t + C$

Since $y = 0$ when $t = 0$, you have $C = 0$. Thus,

$\sin\left(\sqrt{\dfrac{k}{m}}\, t\right) = \dfrac{y}{A}$

$y = A \sin\left(\sqrt{\dfrac{k}{m}}\, t\right)$

109. $y = 2x - \cosh\sqrt{x}$

$$y' = 2 - \frac{1}{2\sqrt{x}}(\sinh \sqrt{x}) = 2 - \frac{\sinh\sqrt{x}}{2\sqrt{x}}$$

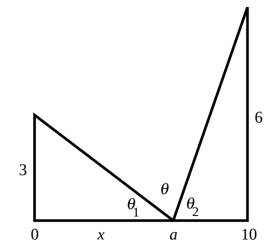
111. Let $u = x^2$, $du = 2x\, dx$.

$$\int \frac{x}{\sqrt{x^4 - 1}}\, dx = \frac{1}{2}\int \frac{1}{\sqrt{(x^2)^2 - 1}}(2x)\, dx = \frac{1}{2}\ln\left(x^2 + \sqrt{x^4 - 1}\right) + C$$

Problem Solving for Chapter 5

1. $\tan\theta_1 = \dfrac{3}{x}$

$\tan\theta_2 = \dfrac{6}{10 - x}$



Minimize $\theta_1 + \theta_2$:

$$f(x) = \theta_1 + \theta_2 = \arctan\left(\frac{3}{x}\right) + \arctan\left(\frac{6}{10 - x}\right)$$

$$f'(x) = \frac{1}{1 + \frac{9}{x^2}}\left(\frac{-3}{x^2}\right) + \frac{1}{1 + \frac{36}{(10 - x)^2}}\left(\frac{6}{(10 - x)^2}\right) = 0$$

$$\frac{3}{x^2 + 9} = \frac{6}{(10 - x)^2 + 36}$$

$$(10 - x)^2 + 36 = 2(x^2 + 9)$$

$$100 - 20x + x^2 + 36 = 2x^2 + 18$$

$$x^2 + 20x - 118 = 0$$

$$x = \frac{-20 \pm \sqrt{20^2 - 4(-118)}}{2} = -10 \pm \sqrt{218}$$

$a = -10 + \sqrt{218} \approx 4.7648 \qquad f(a) \approx 1.4153$

$\theta = \pi - (\theta_1 + \theta_2) \approx 1.7263$ or $98.9°$

Endpoints: $a = 0$: $\theta \approx 1.0304$

$a = 10$: $\theta \approx 1.2793$

Maximum is 1.7263 at $a = -10 + \sqrt{218} \approx 4.7648$.

3. $f(x) = \sin(\ln x)$

(a) Domain: $x > 0$ or $(0, \infty)$

(b) $f(x) = 1 = \sin(\ln x) \implies \ln x = \dfrac{\pi}{2} + 2k\pi.$

Two values are $x = e^{\pi/2}, e^{(\pi/2)+2\pi}$.

(c) $f(x) = -1 = \sin(\ln x) \implies \ln x = \dfrac{3\pi}{2} + 2k\pi.$

Two values are $x = e^{-\pi/2}, e^{3\pi/2}$.

(d) Since the range of the sine function is $[-1, 1]$, parts (b) and (c) show that the range of f is $[-1, 1]$.

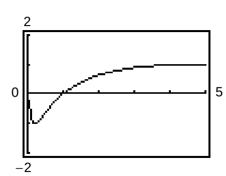
(e) $f'(x) = \dfrac{1}{x}\cos(\ln x)$

$f'(x) = 0 \implies \cos(\ln x) = 0 \implies \ln x = \dfrac{\pi}{2} + k\pi \implies$

$x = e^{\pi/2}$ on $[1, 10]$

$$\left.\begin{aligned} f(e^{\pi/2}) &= 1 \\ f(1) &= 0 \\ f(10) &\approx 0.7440 \end{aligned}\right\} \text{Maximum is 1 at } x = e^{\pi/2} \approx 4.8105$$

(f)



$\lim\limits_{x\to 0^+} f(x)$ seems to be $-\dfrac{1}{2}$. (This in incorrect.)

(g) For the points $x = e^{\pi/2}, e^{-3\pi/2}, e^{-7\pi/2}, \ldots$ we have $f(x) = 1$.

For the points $x = e^{-\pi/2}, e^{-5\pi/2}, e^{-9\pi/2}, \ldots$ we have $f(x) = -1$.

That is, as $x \to 0^+$, there is an infinite number of points where $f(x) = 1$, and an infinite number where $f(x) = -1$. Thus $\lim\limits_{x\to 0+} \sin(\ln x)$ does not exist.

You can verifiy this by graphing $f(x)$ on small intervals close to the origin.

5. (a) $\dfrac{\text{Area sector}}{\text{Area circle}} = \dfrac{t}{2\pi} \implies \text{Area sector} = \dfrac{t}{2\pi}(\pi) = \dfrac{t}{2}$

(b) $\text{Area } AOP = \dfrac{1}{2}(\text{base})(\text{height}) - \displaystyle\int_1^{\cosh t} \sqrt{x^2 - 1}\, dx$

$$A(t) = \frac{1}{2}\cosh t \cdot \sinh t - \int_1^{\cosh t} \sqrt{x^2 - 1}\, dx$$

$$A'(t) = \frac{1}{2}[\cosh^2 t + \sinh^2 t] - \sqrt{\cosh^2 t - 1}\sinh t$$

$$= \frac{1}{2}[\cosh^2 t + \sinh^2 t] - \sinh^2 t$$

$$= \frac{1}{2}[\cosh^2 t - \sinh^2 t] = \frac{1}{2}$$

$A(t) = \dfrac{1}{2}t + C$. But, $A(0) = C = 0 \implies C = 0$

Thus, $A(t) = \dfrac{1}{2}t$ or $t = 2A(t)$.

7.

$$y = \ln x$$

$$y' = \frac{1}{x}$$

$$y - b = \frac{1}{a}(x - a)$$

$$y = \frac{1}{a}x + b - 1 \quad \text{Tangent line}$$

If $x = 0$, $c = b - 1$. Thus, $b - c = b - (b - 1) = 1$.

9. Let $u = 1 + \sqrt{x}$, $\sqrt{x} = u - 1$, $x = u^2 - 2u + 1$,

$dx = (2u - 2)du$.

$$\text{Area} = \int_1^4 \frac{1}{\sqrt{x} + x}\, dx = \int_2^3 \frac{2u - 2}{(u - 1) + (u^2 - 2u + 1)}\, du$$

$$= \int_2^3 \frac{2(u - 1)}{u^2 - u}\, du$$

$$= \int_2^3 \frac{2}{u}\, du$$

$$= \Big[2 \ln u\Big]_2^3$$

$$= 2\ln 3 - 2\ln 2 = 2\ln\left(\frac{3}{2}\right)$$

$$\approx 0.8109$$

11. (a)

$$\frac{dy}{dt} = y^{1.01}$$

$$\int y^{-1.01}\, dy = \int dt$$

$$\frac{y^{-0.01}}{-0.01} = t + C_1$$

$$\frac{1}{y^{0.01}} = -0.01t + C$$

$$y^{0.01} = \frac{1}{C - 0.01t}$$

$$y = \frac{1}{(C - 0.01t)^{100}}$$

$y(0) = 1$: $1 = \dfrac{1}{C^{100}} \implies C = 1$

Hence, $y = \dfrac{1}{(1 - 0.01t)^{100}}$.

For $T = 100$, $\displaystyle\lim_{t\to T^-} y = \infty$.

(b)

$$\int y^{-(1+\varepsilon)}\, dy = \int k\, dt$$

$$\frac{y^{-\varepsilon}}{-\varepsilon} = kt + C_1$$

$$y^{-\varepsilon} = -\varepsilon kt + C$$

$$y = \frac{1}{(C - \varepsilon kt)^{1/\varepsilon}}$$

$y(0) = y_0 = \dfrac{1}{C^{1/\varepsilon}} \implies C^{1/\varepsilon} = \dfrac{1}{y_0} \implies C = \left(\dfrac{1}{y_0}\right)^{\varepsilon}$

Hence, $y = \dfrac{1}{\left(\dfrac{1}{y_0^{\varepsilon}} - \varepsilon kt\right)^{1/\varepsilon}}$.

For $t \to \dfrac{1}{y_0^{\varepsilon}\varepsilon k}$, $y \to \infty$.

13. Since $\dfrac{dy}{dt} = k(y - 20)$,

$$\int \frac{1}{y - 20}\,dy = \int k\,dt$$

$$\ln(y - 20) = kt + C$$

$$y = Ce^{kt} + 20.$$

When $t = 0$, $y = 72$. Therefore, $C = 52$.

When $t = 1$, $y = 48$. Therefore, $48 = 52e^k + 20$, $e^k = (28/52) = (7/13)$, and $k = \ln(7/13)$.
Thus, $y = 52e^{[\ln(7/13)]t} + 20$.

When $t = 5$, $y = 52e^{5\ln(7/13)} + 20 \approx 22.35°$.

15. (a) $\dfrac{dS}{dt} = k_1 S(L - S)$

$S = \dfrac{L}{1 + Ce^{-kt}}$ is a solution because

$$\frac{dS}{dt} = -L(1 + Ce^{-kt})^{-2}(-C\,ke^{-kt})$$

$$= \frac{LC\,ke^{-kt}}{(1 + Ce^{-kt})^2}$$

$$= \left(\frac{k}{L}\right)\frac{L}{1 + Ce^{-kt}} \cdot \frac{C\,Le^{-kt}}{1 + Ce^{-kt}}$$

$$= \left(\frac{k}{L}\right)\frac{L}{1 + Ce^{-kt}} \cdot \left(L - \frac{L}{1 + Ce^{-kt}}\right)$$

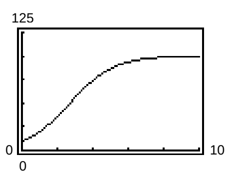
$$= k_1 S(L - S), \text{ where } k_1 = \frac{k}{L}.$$

$L = 100$. Also, $S = 10$ when $t = 0 \implies C = 9$. And,
$S = 20$ when $t = 1 \implies k = -\ln(4/9)$.

Particular Solution. $S = \dfrac{100}{1 + 9e^{\ln(4/9)t}}$

$$= \frac{100}{1 + 9e^{-0.8109t}}$$

(c)



(b) $\dfrac{dS}{dt} = \ln\left(\dfrac{4}{9}\right) S(100 - S)$

$$\frac{d^2S}{dt^2} = \ln\left(\frac{4}{9}\right)\left[S\left(-\frac{dS}{dt}\right) + (100 - S)\frac{dS}{dt}\right]$$

$$= \ln\left(\frac{4}{9}\right)(100 - 2S)\frac{dS}{dt}$$

$$= 0 \text{ when } S = 50 \text{ or } \frac{dS}{dt} = 0.$$

Choosing $S = 50$, we have:

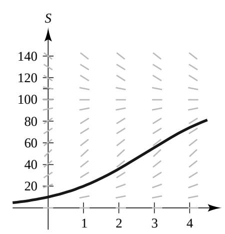
$$50 = \frac{100}{1 + 9e^{\ln(4/9)t}}$$

$$2 = 1 + 9e^{\ln(4/9)t}$$

$$\frac{\ln(1/9)}{\ln(4/9)} = t$$

$$t \approx 2.7 \text{ months}$$

(d)



(e) Sales will decrease toward the line $S = L$.