PART I

CHAPTER 6
Applications of Integration

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Section 6.1 Area of a Region Between Two Curves

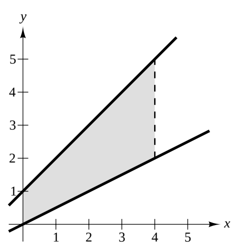
Solutions to Odd-Numbered Exercises

1. $A = \int_0^6 [0 - (x^2 - 6x)]\,dx = -\int_0^6 (x^2 - 6x)\,dx$

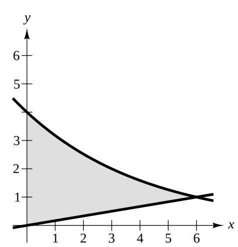
3. $A = \int_0^3 [(-x^2 + 2x + 3) - (x^2 - 4x + 3)]\,dx = \int_0^3 (-2x^2 + 6x)\,dx$

5. $A = 2\int_{-1}^0 3(x^3 - x)\,dx = 6\int_{-1}^0 (x^3 - x)\,dx$ or $-6\int_0^1 (x^3 - x)\,dx$

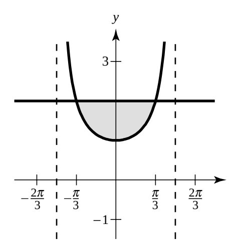
7. $\int_0^4 \left[(x + 1) - \frac{x}{2}\right] dx$



9. $\int_0^6 \left[4(2^{-x/3}) - \frac{x}{6}\right] dx$



11. $\int_{-\pi/3}^{\pi/3} [2 - \sec x]\,dx$

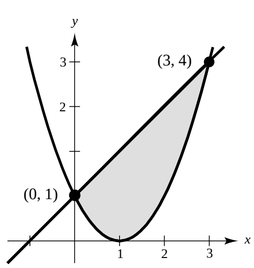


13. $f(x) = x + 1$

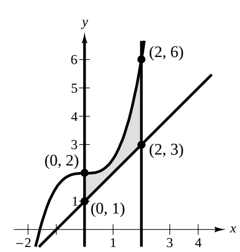
$g(x) = (x - 1)^2$

$A \approx 4$

Matches (d)



15.
$$A = \int_0^2 \left[\left(\frac{1}{2}x^3 + 2\right) - (x + 1)\right] dx$$
$$= \int_0^2 \left(\frac{1}{2}x^3 - x + 1\right) dx$$
$$= \left[\frac{x^4}{8} - \frac{x^2}{2} + x\right]_0^2$$
$$= \left(\frac{16}{8} - \frac{4}{2} + 2\right) - 0 = 2$$



17. The points of intersection are given by:

$$x^2 - 4x = 0$$

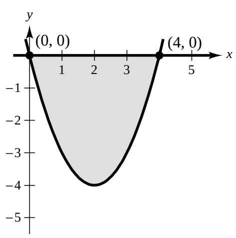
$$x(x - 4) = 0 \quad \text{when} \quad x = 0, 4$$

$$A = \int_0^4 [g(x) - f(x)]\, dx$$

$$= -\int_0^4 (x^2 - 4x)\, dx$$

$$= -\left[\frac{x^3}{3} - 2x^2\right]_0^4$$

$$= \frac{32}{3}$$



19. The points of intersection are given by:

$$x^2 + 2x + 1 = 3x + 3$$

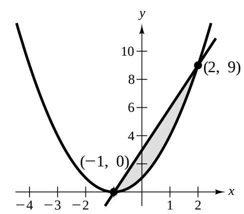
$$(x - 2)(x + 1) = 0 \quad \text{when} \quad x = -1, 2$$

$$A = \int_{-1}^2 [g(x) - f(x)]\, dx$$

$$= \int_{-1}^2 [(3x + 3) - (x^2 + 2x + 1)]\, dx$$

$$= \int_{-1}^2 (2 + x - x^2)\, dx$$

$$= \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^2 = \frac{9}{2}$$



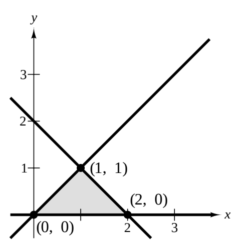
21. The points of intersection are given by:

$$x = 2 - x \quad \text{and} \quad x = 0 \quad \text{and} \quad 2 - x = 0$$

$$x = 1 \qquad\qquad x = 0 \qquad\qquad x = 2$$

$$A = \int_0^1 [(2 - y) - (y)]\, dy = \left[2y - y^2\right]_0^1 = 1$$

Note that if we integrate with respect to x, we need two integrals. Also, note that the region is a triangle.



23. The points of intersection are given by:

$$\sqrt{3x} + 1 = x + 1$$

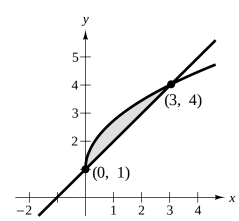
$$\sqrt{3x} = x \quad \text{when} \quad x = 0, 3$$

$$A = \int_0^3 [f(x) - g(x)]\, dx$$

$$= \int_0^3 \left[\left(\sqrt{3x} + 1\right) - (x + 1)\right] dx$$

$$= \int_0^3 \left[(3x)^{1/2} - x\right] dx$$

$$= \left[\frac{2}{9}(3x)^{3/2} - \frac{x^2}{2}\right]_0^3 = \frac{3}{2}$$



25. The points of intersection are given by:

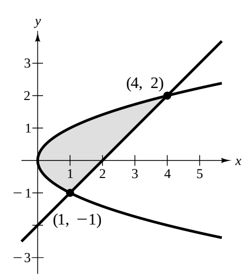
$$y^2 = y + 2$$

$$(y - 2)(y + 1) = 0 \quad \text{when} \quad y = -1, 2$$

$$A = \int_{-1}^2 [g(y) - f(y)]\, dy$$

$$= \int_{-1}^2 [(y + 2) - y^2]\, dy$$

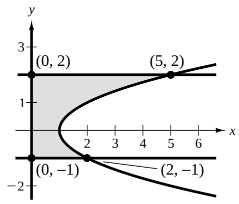
$$= \left[2y + \frac{y^2}{2} - \frac{y^3}{3}\right]_{-1}^2 = \frac{9}{2}$$



27. $A = \int_{-1}^{2} [f(y) - g(y)]\, dy$

$= \int_{-1}^{2} [(y^2 + 1) - 0]\, dy$

$= \left[\frac{y^3}{3} + y\right]_{-1}^{2} = 6$



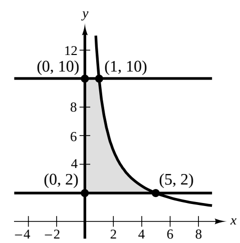
29. $y = \frac{10}{x} \Rightarrow x = \frac{10}{y}$

$A = \int_{2}^{10} \frac{10}{y}\, dy$

$= \Big[10 \ln y\Big]_{2}^{10}$

$= 10(\ln 10 - \ln 2)$

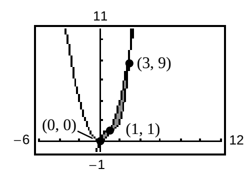
$= 10 \ln 5 \approx 16.0944$



31. The points of intersection are given by:

$x^3 - 3x^2 + 3x = x^2$

$x(x - 1)(x - 3) = 0$ when $x = 0, 1, 3$



$A = \int_0^1 [f(x) - g(x)]\, dx + \int_1^3 [g(x) - f(x)]\, dx$

$= \int_0^1 [(x^3 - 3x^2 + 3x) - x^2]\, dx + \int_1^3 [x^2 - (x^3 - 3x^2 + 3x)]\, dx$

$= \int_0^1 (x^3 - 4x^2 + 3x)\, dx + \int_1^3 (-x^3 + 4x^2 - 3x)\, dx$

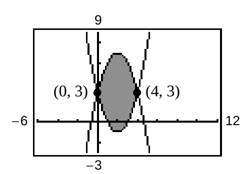
$= \left[\frac{x^4}{4} - \frac{4}{3}x^3 + \frac{3}{2}x^2\right]_0^1 + \left[\frac{-x^4}{4} + \frac{4}{3}x^3 - \frac{3}{2}x^2\right]_1^3 = \frac{37}{12}$

Numerical Approximation: $0.417 + 2.667 \approx 3.083$

33. The points of intersection are given by:

$x^2 - 4x + 3 = 3 + 4x - x^2$

$2x(x - 4) = 0$ when $x = 0, 4$



$A = \int_0^4 [(3 + 4x - x^2) - (x^2 - 4x + 3)]\, dx$

$= \int_0^4 (-2x^2 + 8x)\, dx$

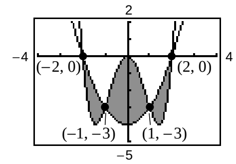
$= \left[-\frac{2x^3}{3} + 4x^2\right]_0^4 = \frac{64}{3}$

Numerical Approximation: 21.333

35. $f(x) = x^4 - 4x^2, \quad g(x) = x^2 - 4$

The points of intersection are given by:

$$x^4 - 4x^2 = x^2 - 4$$
$$x^4 - 5x^2 + 4 = 0$$
$$(x^2 - 4)(x^2 - 1) = 0 \quad \text{when} \quad x = \pm 2, \pm 1$$



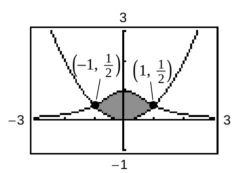
By symmetry,

$$A = 2\int_0^1 [(x^4 - 4x^2) - (x^2 - 4)]\,dx + 2\int_1^2 [(x^2 - 4) - (x^4 - 4x^2)]\,dx$$
$$= 2\int_0^1 (x^4 - 5x^2 + 4)\,dx + 2\int_1^2 (-x^4 + 5x^2 - 4)\,dx$$
$$= 2\left[\frac{x^5}{5} - \frac{5x^3}{3} + 4x\right]_0^1 + 2\left[-\frac{x^5}{5} + \frac{5x^3}{3} - 4x\right]_1^2$$
$$= 2\left[\frac{1}{5} - \frac{5}{3} + 4\right] + 2\left[\left(-\frac{32}{5} + \frac{40}{3} - 8\right) - \left(-\frac{1}{5} + \frac{5}{3} - 4\right)\right] = 8.$$

Numerical Approximation: $5.067 + 2.933 = 8.0$

37. The points of intersection are given by:

$$\frac{1}{1 + x^2} = \frac{x^2}{2}$$
$$x^4 + x^2 - 2 = 0$$
$$(x^2 + 2)(x^2 - 1) = 0$$
$$x = \pm 1$$



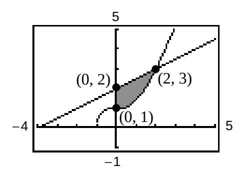
$$A = 2\int_0^1 [f(x) - g(x)]\,dx$$
$$= 2\int_0^1 \left[\frac{1}{1 + x^2} - \frac{x^2}{2}\right]dx$$
$$= 2\left[\arctan x - \frac{x^3}{6}\right]_0^1$$
$$= 2\left(\frac{\pi}{4} - \frac{1}{6}\right) = \frac{\pi}{2} - \frac{1}{3} \approx 1.237$$

Numerical Approximation: 1.237

39. $\sqrt{1 + x^3} \le \frac{1}{2}x + 2$ on $[0, 2]$

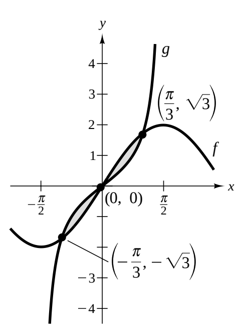
Numerical approximation: 1.759

$$A = \int_0^2 \left[\frac{1}{2}x + 2 - \sqrt{1 + x^3}\right]dx \approx 1.759$$



41. $A = 2\int_0^{\pi/3} [f(x) - g(x)]\,dx$

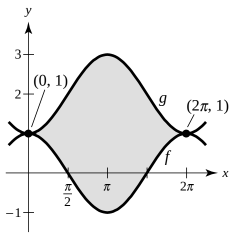
$$= 2\int_0^{\pi/3} (2\sin x - \tan x)\,dx$$
$$= 2\Big[-2\cos x + \ln|\cos x|\Big]_0^{\pi/3}$$
$$= 2(1 - \ln 2) \approx 0.614$$



43. $A = \int_0^{2\pi} [(2 - \cos x) - \cos x]\, dx$

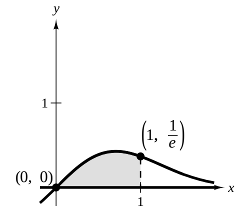
$= 2\int_0^{2\pi} (1 - \cos x)\, dx$

$= 2\Big[x - \sin x\Big]_0^{2\pi} = 4\pi \approx 12.566$



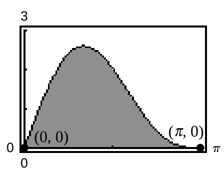
45. $A = \int_0^{1} [xe^{-x^2} - 0]\, dx$

$= \left[-\frac{1}{2}e^{-x^2}\right]_0^1 = \frac{1}{2}\left(1 - \frac{1}{e}\right) \approx 0.316$



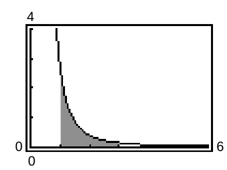
47. $A = \int_0^{\pi} [(2\sin x + \sin 2x) - 0]\, dx$

$= \left[-2\cos x - \frac{1}{2}\cos 2x\right]_0^{\pi} = 4.0$



49. $A = \int_1^{3} \left[\frac{1}{x^2}e^{1/x} - 0\right] dx$

$= \Big[-e^{1/x}\Big]_1^3 = e - e^{1/3} \approx 1.323$

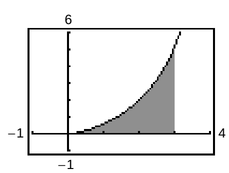


51. (a) $y = \sqrt{\frac{x^3}{4 - x}}, \quad y = 0, \quad x = 3$

(b) $A = \int_0^3 \sqrt{\frac{x^3}{4 - x}}\, dx,$

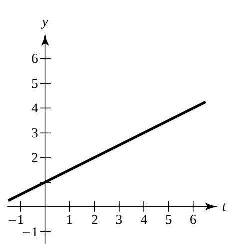
No, it cannot be evaluated by hand.

(c) 4.7721

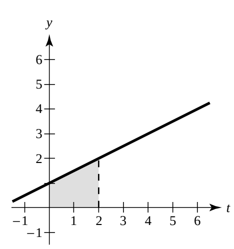


53. $F(x) = \int_0^x \left(\frac{1}{2}t + 1\right) dt = \left[\frac{t^2}{4} + t\right]_0^x = \frac{x^2}{4} + x$

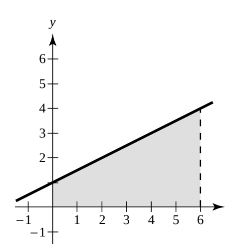
(a) $F(0) = 0$



(b) $F(2) = \frac{2^2}{4} + 2 = 3$

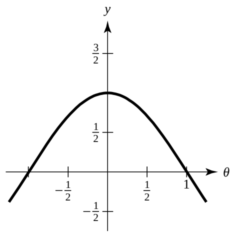


(c) $F(6) = \frac{6^2}{4} + 6 = 15$

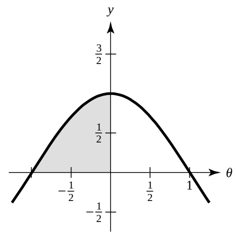


55. $F(\alpha) = \int_{-1}^{\alpha} \cos \frac{\pi\theta}{2}\, d\theta = \left[\frac{2}{\pi} \sin \frac{\pi\theta}{2}\right]_{-1}^{\alpha} = \frac{2}{\pi} \sin \frac{\pi\alpha}{2} + \frac{2}{\pi}$

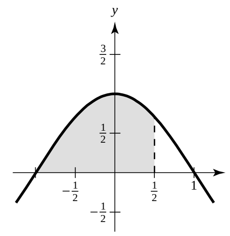
(a) $F(-1) = 0$



(b) $F(0) = \frac{2}{\pi} \approx 0.6366$



(c) $F\left(\frac{1}{2}\right) = \frac{2 + \sqrt{2}}{\pi} \approx 1.0868$

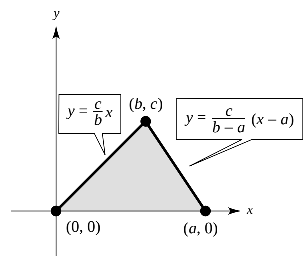


57. $A = \int_0^c \left[\left(\frac{b - a}{c} y + a\right) - \frac{b}{c} y\right] dy$

$= \int_0^c \left(-\frac{a}{c} y + a\right) dy$

$= \left[-\frac{a}{2c} y^2 + ay\right]_0^c$

$= -\frac{ac}{2} + ac = \frac{ac}{2} \quad \left(= \frac{1}{2}\text{(base)(height)}\right)$



59. $f(x) = x^3$

$f'(x) = 3x^2$

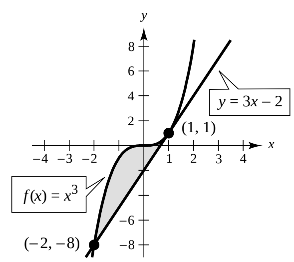
At $(1, 1)$, $f'(1) = 3$.

Tangent line:

$y - 1 = 3(x - 1)$ or $y = 3x - 2$

The tangent line intersects $f(x) = x^3$ at $x = -2$.

$$A = \int_{-2}^{1} [x^3 - (3x - 2)]\, dx = \left[\frac{x^4}{4} - \frac{3x^2}{2} + 2x\right]_{-2}^{1} = \frac{27}{4}$$



61. The variable is y.

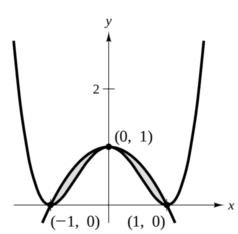
63. $x^4 - 2x^2 + 1 \le 1 - x^2$ on $[-1, 1]$

$A = \int_{-1}^{1} [(1 - x^2) - (x^4 - 2x^2 + 1)]\, dx$

$= \int_{-1}^{1} (x^2 - x^4)\, dx$

$= \left[\frac{x^3}{3} - \frac{x^5}{5}\right]_{-1}^{1} = \frac{4}{15}$

You can use a single integral because $x^4 - 2x^2 + 1 \le 1 - x^2$ on $[-1, 1]$.



65. Offer 2 is better because the accumulated salary (area under the curve) is larger.

67. $$A = \int_{-3}^{3} (9 - x^2)\,dx = 36$$

$$\int_{-\sqrt{9-b}}^{\sqrt{9-b}} [(9 - x^2) - b]\,dx = 18$$

$$\int_{0}^{\sqrt{9-b}} [(9 - b) - x^2]\,dx = 9$$

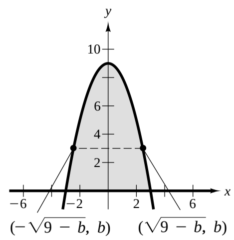
$$\left[(9 - b)x - \frac{x^3}{3}\right]_0^{\sqrt{9-b}} = 9$$

$$\frac{2}{3}(9 - b)^{3/2} = 9$$

$$(9 - b)^{3/2} = \frac{27}{2}$$

$$9 - b = \frac{9}{\sqrt[3]{4}}$$

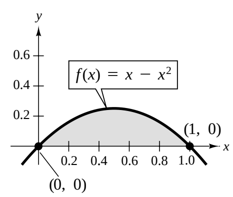
$$b = 9 - \frac{9}{\sqrt[3]{4}} \approx 3.330$$



69. $\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} (x_i - x_i^2)\,\Delta x$

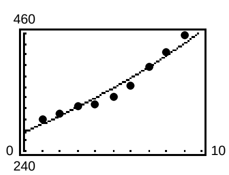
where $x_i = \frac{i}{n}$ and $\Delta x = \frac{1}{n}$ is the same as

$$\int_0^1 (x - x^2)\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{6}.$$

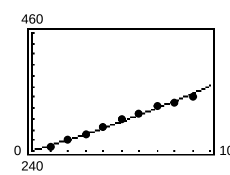


71. $$\int_0^5 [(7.21 + 0.58t) - (7.21 + 0.45t)]\,dt = \int_0^5 0.13t\,dt = \left[\frac{0.13t^2}{2}\right]_0^5 = \$1.625 \text{ billion}$$

73. (a) $y_1 = (275.0675)(1.0537)^t = (275.0675)e^{0.0523t}]$



(b) $y_2 = (239.9407)(1.0417)^t = (239.9407)e^{0.0408t}$

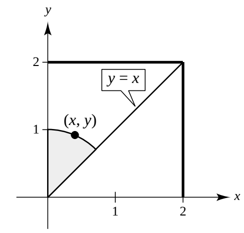


(c) $\int_{10}^{15} (y_1 - y_2)\,dt \approx 649.5$ billion dollars

(d) No, model $y_1 > y_2$ forever because $1.0537 > 1.0417$.

No, these models are not accurate. According to news reports, $E > R$ eventually.

75. The total area is 8 times the area of the shaded region to the right. A point (x, y) is on the upper boundary of the region if



$$\sqrt{x^2 + y^2} = 2 - y$$
$$x^2 + y^2 = 4 - 4y + y^2$$
$$x^2 = 4 - 4y$$
$$4y = 4 - x^2$$
$$y = 1 - \frac{x^2}{4}.$$

We now determine where this curve intersects the line $y = x$.

$$x = 1 - \frac{x^2}{4}$$
$$x^2 + 4x - 4 = 0$$
$$x = \frac{-4 \pm \sqrt{16 + 16}}{2} = -2 \pm 2\sqrt{2} \implies x = -2 + 2\sqrt{2}$$

$$\text{Total area} = 8 \int_0^{-2+2\sqrt{2}} \left(1 - \frac{x^2}{4} - x\right) dx$$

$$= 8\left[x - \frac{x^3}{12} - \frac{x^2}{2}\right]_0^{-2+2\sqrt{2}} = \frac{16}{3}\left(4\sqrt{2} - 5\right) \approx 8(0.4379) = 3.503$$

77. (a) $A = 2\left[\int_0^5 \left(1 - \frac{1}{3}\sqrt{5 - x}\right) dx + \int_5^{5.5} (1 - 0)\, dx\right]$

$$= 2\left(\left[x + \frac{2}{9}(5 - x)^{3/2}\right]_0^5 + \left[x\right]_5^{5.5}\right) = 2\left(5 - \frac{10\sqrt{5}}{9} + 5.5 - 5\right) \approx 6.031 \text{ m}^2$$

(b) $V = 2A \approx 2(6.031) \approx 12.062 \text{ m}^3$

(c) $5000\, V \approx 5000(12.062) = 60{,}310$ pounds

79. True

81. False. Let $f(x) = x$ and $g(x) = 2x - x^2$. f and g intersect at $(1, 1)$, the midpoint of $[0, 2]$. But

$$\int_a^b [f(x) - g(x)]\, dx = \int_0^2 [x - (2x - x^2)]\, dx = \frac{2}{3} \neq 0.$$

Section 6.2 Volume: The Disk Method

1. $V = \pi \int_0^1 (-x + 1)^2\, dx = \pi \int_0^1 (x^2 - 2x + 1)\, dx = \pi\left[\frac{x^3}{3} - x^2 + x\right]_0^1 = \frac{\pi}{3}$

3. $V = \pi \int_1^4 \left(\sqrt{x}\right)^2 dx = \pi \int_1^4 x\, dx = \pi\left[\frac{x^2}{2}\right]_1^4 = \frac{15\pi}{2}$

5. $V = \pi \int_0^1 [(x^2)^2 - (x^3)^2]\, dx = \pi \int_0^1 (x^4 - x^6)\, dx = \pi\left[\frac{x^5}{5} - \frac{x^7}{7}\right]_0^1 = \frac{2\pi}{35}$

7. $y = x^2 \implies x = \sqrt{y}$

$$V = \pi \int_0^4 \left(\sqrt{y}\right)^2 dy = \pi \int_0^4 y\, dy$$
$$= \pi\left[\frac{y^2}{2}\right]_0^4 = 8\pi$$

9. $y = x^{2/3} \implies x = y^{3/2}$

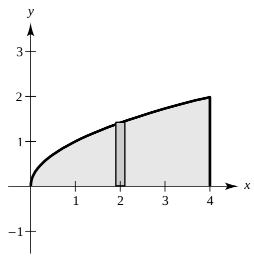
$$V = \pi \int_0^1 (y^{3/2})^2\, dy = \pi \int_0^1 y^3\, dy = \pi\left[\frac{y^4}{4}\right]_0^1 = \frac{\pi}{4}$$

11. $y = \sqrt{x},\ y = 0,\ x = 4$

(a) $R(x) = \sqrt{x},\ r(x) = 0$

$$V = \pi \int_0^4 \left(\sqrt{x}\right)^2 dx$$

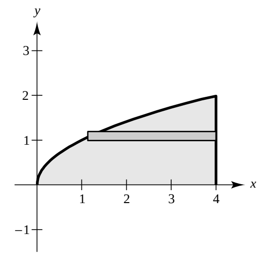
$$= \pi \int_0^4 x\, dx = \left[\frac{\pi}{2}x^2\right]_0^4 = 8\pi$$



(b) $R(y) = 4,\ r(y) = y^2$

$$V = \pi \int_0^2 (16 - y^4)\, dy$$

$$= \pi\left[16y - \frac{1}{5}y^5\right]_0^2 = \frac{128\pi}{5}$$

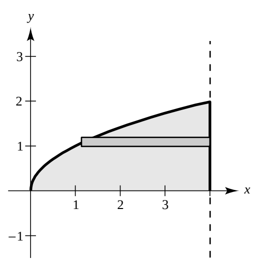


(c) $R(y) = 4 - y^2,\ r(y) = 0$

$$V = \pi \int_0^2 (4 - y^2)^2\, dy$$

$$= \pi \int_0^2 (16 - 8y^2 + y^4)\, dy$$

$$= \pi\left[16y - \frac{8}{3}y^3 + \frac{1}{5}y^5\right]_0^2 = \frac{256\pi}{15}$$

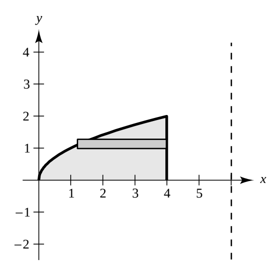


(d) $R(y) = 6 - y^2,\ r(y) = 2$

$$V = \pi \int_0^2 \left[(6 - y^2)^2 - 4\right] dy$$

$$= \pi \int_0^2 (32 - 12y^2 + y^4)\, dy$$

$$= \pi\left[32y - 4y^3 + \frac{1}{5}y^5\right]_0^2 = \frac{192\pi}{5}$$



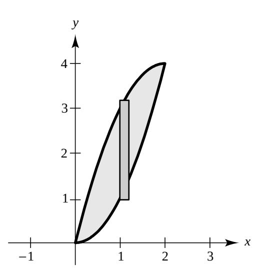
13. $y = x^2,\ y = 4x - x^2$ intersect at $(0, 0)$ and $(2, 4)$.

(a) $R(x) = 4x - x^2\ \ r(x) = x^2$

$$V = \pi \int_0^2 \left[(4x - x^2)^2 - x^4\right] dx$$

$$= \pi \int_0^2 (16x^2 - 8x^3)\, dx$$

$$= \pi\left[\frac{16}{3}x^3 - 2x^4\right]_0^2 = \frac{32\pi}{3}$$

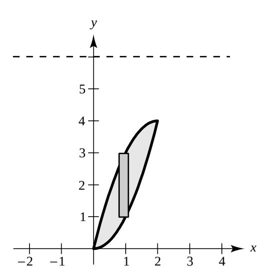


(b) $R(x) = 6 - x^2,\ r(x) = 6 - (4x - x^2)$

$$V = \pi \int_0^2 \left[(6 - x^2)^2 - (6 - 4x + x^2)^2\right] dx$$

$$= 8\pi \int_0^2 (x^3 - 5x^2 + 6x)\, dx$$

$$= 8\pi\left[\frac{x^4}{4} - \frac{5}{3}x^3 + 3x^2\right]_0^2 = \frac{64\pi}{3}$$

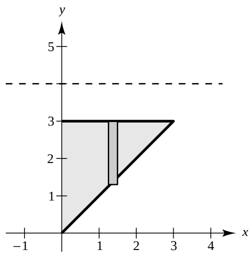


15. $R(x) = 4 - x,\ r(x) = 1$

$$V = \pi \int_0^3 \left[(4 - x)^2 - (1)^2\right] dx$$

$$= \pi \int_0^3 (x^2 - 8x + 15)\, dx$$

$$= \pi \left[\frac{x^3}{3} - 4x^2 + 15x\right]_0^3 = 18\pi$$



17. $R(x) = 4,\ r(x) = 4 - \dfrac{1}{1 + x}$

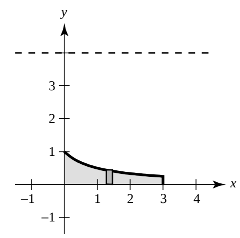
$$V = \pi \int_0^3 \left[4^2 - \left(4 - \frac{1}{1 + x}\right)^2\right] dx$$

$$= \pi \int_0^3 \left[\frac{8}{1 + x} - \frac{1}{(1 + x)^2}\right] dx$$

$$= \pi \left[8 \ln(1 + x) + \frac{1}{1 + x}\right]_0^3$$

$$= \pi \left[8 \ln 4 + \frac{1}{4} - 1\right]$$

$$= \left(8 \ln 4 - \frac{3}{4}\right)\pi \approx 32.485$$



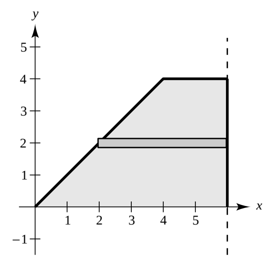
19. $R(y) = 6 - y,\ r(y) = 0$

$$V = \pi \int_0^4 (6 - y)^2\, dy$$

$$= \pi \int_0^4 (y^2 - 12y + 36)\, dy$$

$$= \pi \left[\frac{y^3}{3} - 6y^2 + 36y\right]_0^4$$

$$= \frac{208\pi}{3}$$



21. $R(y) = 6 - y^2,\ r(y) = 2$

$$V = \pi \int_{-2}^2 \left[(6 - y^2)^2 - (2)^2\right] dy$$

$$= 2\pi \int_0^2 (y^4 - 12y^2 + 32)\, dy$$

$$= 2\pi \left[\frac{y^5}{5} - 4y^3 + 32y\right]_0^2$$

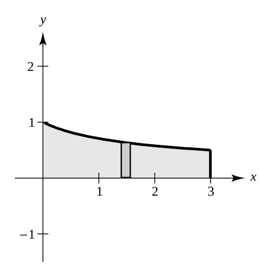
$$= \frac{384\pi}{5}$$

23. $R(x) = \dfrac{1}{\sqrt{x + 1}},\ r(x) = 0$

$$V = \pi \int_0^3 \left(\frac{1}{\sqrt{x + 1}}\right)^2 dx$$

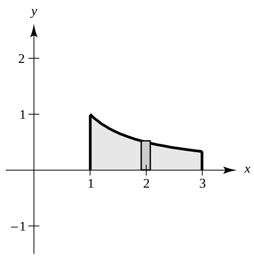
$$= \pi \int_0^3 \frac{1}{x + 1}\, dx$$

$$= \left[\pi \ln|x + 1|\right]_0^3 = \pi \ln 4$$



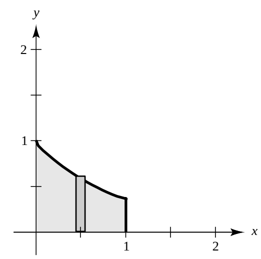
25. $R(x) = \frac{1}{x},\ r(x) = 0$

$$V = \pi\int_1^4 \left(\frac{1}{x}\right)^2 dx = \pi\left[-\frac{1}{x}\right]_1^4 = \frac{3\pi}{4}$$



27. $R(x) = e^{-x},\ r(x) = 0$

$$V = \pi\int_0^1 (e^{-x})^2\, dx = \pi\int_0^1 e^{-2x}\, dx = \left[-\frac{\pi}{2}e^{-2x}\right]_0^1 = \frac{\pi}{2}(1 - e^{-2}) \approx 1.358$$

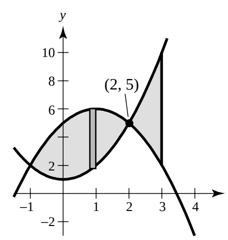


29. $V = \pi\int_0^2 [(5 + 2x - x^2)^2 - (x^2 + 1)^2)]\, dx + \pi\int_2^3 [(x^2 + 1)^2 - (5 + 2x - x^2)^2]\, dx$

$$= \pi\int_0^2 (-4x^3 - 8x^2 + 20x + 24)\, dx + \pi\int_2^3 (4x^3 + 8x^2 - 20x - 24)\, dx$$

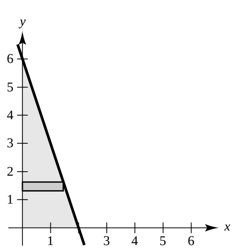
$$= \pi\left[-x^4 - \frac{8}{3}x^3 + 10x^2 + 24x\right]_0^2 + \pi\left[x^3 + \frac{8}{3}x^3 - 10x^2 - 24x\right]_2^3$$

$$= \pi\frac{152}{3} + \pi\frac{125}{3} = \frac{277\pi}{3}$$

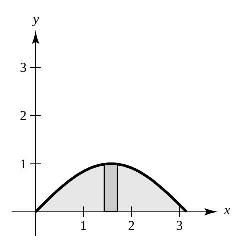


31. $y = 6 - 3x \implies x = \frac{1}{3}(6 - y)$

$$V = \pi\int_0^6 \left[\frac{1}{3}(6 - y)\right]^2 dy = \frac{\pi}{9}\int_0^6 [36 - 12y + y^2]\, dy = \frac{\pi}{9}\left[36y - 6y^2 + \frac{y^3}{3}\right]_0^6 = \frac{\pi}{9}\left[216 - 216 + \frac{216}{3}\right] = 8\pi$$



33. $V = \pi\int_0^\pi [\sin x]^2\, dx \approx 4.9348$

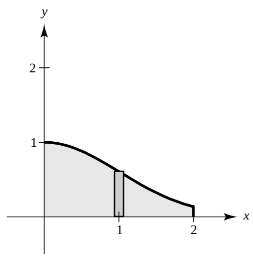


35. $V = \pi \int_0^2 [e^{-x^2}]^2\,dx \approx 1.9686$

37. $V = \pi \int_{-1}^2 [e^{x/2} + e^{-x/2}]^2\,dx \approx 49.0218$

39. $A \approx 3$

Matches (a)



41. Disk Method:

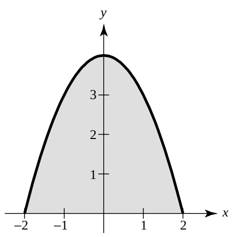
$$V = \pi \int_a^b [R(x)]^2\,dx \quad \text{or} \quad V = \pi \int_c^d [R(y)]^2\,dy$$

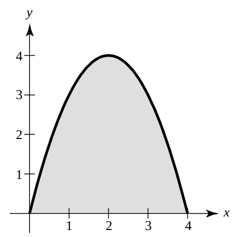
Washer Method:

$$V = \pi \int_a^b \left([R(x)]^2 - [r(x)]^2\right)dx \quad \text{or}$$

$$V = \pi \int_c^d \left([R(y)]^2 - [r(y)]^2\right)dy$$

43.





The volumes are the same because the solid has been translated horizontally.

45. $R(x) = \frac{1}{2}x,\ r(x) = 0$

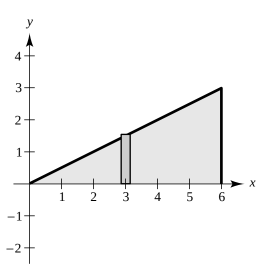
$$V = \pi \int_0^6 \frac{1}{4}x^2\,dx$$

$$= \left[\frac{\pi}{12}x^3\right]_0^6 = 18\pi$$

Note: $V = \frac{1}{3}\pi r^2 h$

$$= \frac{1}{3}\pi(3^2)6$$

$$= 18\pi$$



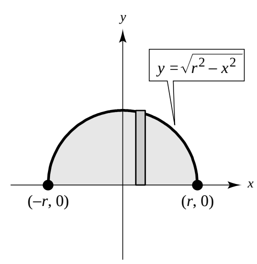
47. $R(x) = \sqrt{r^2 - x^2},\ r(x) = 0$

$$V = \pi \int_{-r}^r (r^2 - x^2)\,dx$$

$$= 2\pi \int_0^r (r^2 - x^2)\,dx$$

$$= 2\pi\left[r^2x - \frac{1}{3}x^3\right]_0^r$$

$$= 2\pi\left(r^3 - \frac{1}{3}r^3\right) = \frac{4}{3}\pi r^3$$



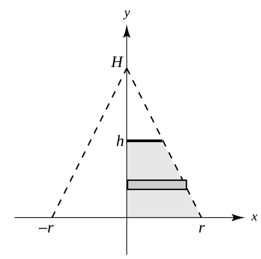
49. $x = r - \frac{r}{H}y = r\left(1 - \frac{y}{H}\right),\ R(y) = r\left(1 - \frac{y}{H}\right),\ r(y) = 0$

$$V = \pi\int_0^h \left[r\left(1 - \frac{y}{H}\right)\right]^2 dy = \pi r^2 \int_0^h \left(1 - \frac{2}{H}y + \frac{1}{H^2}y^2\right) dy$$

$$= \pi r^2\left[y - \frac{1}{H}y^2 + \frac{1}{3H^2}y^3\right]_0^h$$

$$= \pi r^2\left(h - \frac{h^2}{H} + \frac{h^3}{3H^2}\right)$$

$$= \pi r^2 h\left(1 - \frac{h}{H} + \frac{h^2}{3H^2}\right)$$



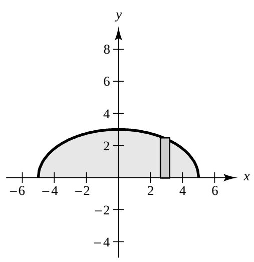
51. $V = \pi\int_0^2 \left(\frac{1}{8}x^2\sqrt{2 - x}\right)^2 dx = \frac{\pi}{64}\int_0^2 x^4(2 - x)\,dx = \frac{\pi}{64}\left[\frac{2x^5}{5} - \frac{x^6}{6}\right]_0^2 = \frac{\pi}{30}$

53. (a) $R(x) = \frac{3}{5}\sqrt{25 - x^2},\ r(x) = 0$

$$V = \frac{9\pi}{25}\int_{-5}^5 (25 - x^2)\,dx$$

$$= \frac{18\pi}{25}\int_0^5 (25 - x^2)\,dx$$

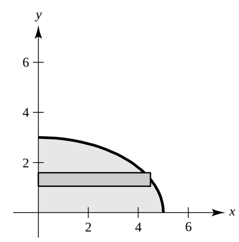
$$= \frac{18\pi}{25}\left[25x - \frac{x^3}{3}\right]_0^5 = 60\pi$$



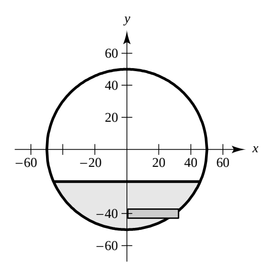
(b) $R(y) = \frac{5}{3}\sqrt{9 - y^2},\ r(y) = 0,\ x \geq 0$

$$V = \frac{25\pi}{9}\int_0^3 (9 - y^2)\,dy$$

$$= \frac{25\pi}{9}\left[9y - \frac{y^3}{3}\right]_0^3 = 50\pi$$



55. Total volume: $V = \frac{4\pi(50)^3}{3} = \frac{500{,}000}{3}$ ft^3



Volume of water in the tank:

$$\pi\int_{-50}^{y_0} \left(\sqrt{2500 - y^2}\right)^2 dy = \pi\int_{-50}^{y_0} (2500 - y^2)\,dy$$

$$= \pi\left[2500y - \frac{y^3}{3}\right]_{-50}^{y_0}$$

$$= \pi\left(2500y_0 - \frac{y_0^3}{3} + \frac{250{,}000}{3}\right)$$

When the tank is one-fourth of its capacity:

$$\frac{1}{4}\left(\frac{500{,}000\pi}{3}\right) = \pi\left(2500y_0 - \frac{y_0^3}{3} + \frac{250{,}00}{3}\right)$$

$$125{,}000 = 7500y_0 - y_0^3 + 250{,}000$$

$$y_0^3 - 7500y_0 - 125{,}000 = 0$$

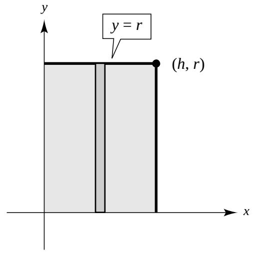
$$y_0 \approx -17.36$$

Depth: $-17.36 - (-50) = 32.64$ feet

When the tank is three-fourths of its capacity the depth is $100 - 32.64 = 67.36$ feet.

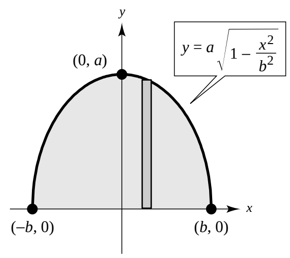
57. (a) $\pi\int_0^h r^2\,dx$ (ii)

is the volume of a right circular cylinder with radius r and height h.



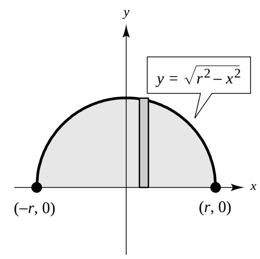
(b) $\pi\int_{-b}^{b}\left(a\sqrt{1-\dfrac{x^2}{b^2}}\right)^2 dx$ (iv)

is the volume of an ellipsoid with axes $2a$ and $2b$.



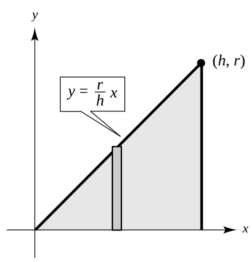
(c) $\pi\int_{-r}^{r}\left(\sqrt{r^2-x^2}\right)^2 dx$ (iii)

is the volume of a sphere with radius r.



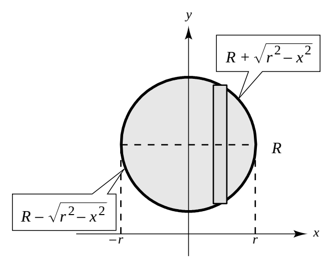
(d) $\pi\int_0^h \left(\dfrac{rx}{h}\right)^2 dx$ (i)

is the volume of a right circular cone with the radius of the base as r and height h.

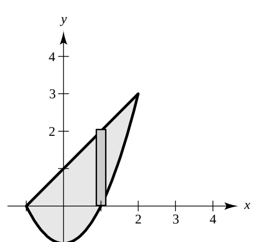


(e) $\pi\int_{-r}^{r}\left[\left(R+\sqrt{r^2-x^2}\right)^2-\left(R-\sqrt{r^2-x^2}\right)^2\right] dx$ (v)

is the volume of a torus with the radius of its circular cross section as r and the distance from the axis of the torus to the center of its cross section as R.



59.



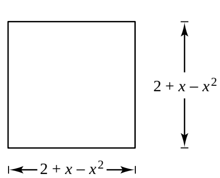
Base of Cross Section $= (x+1)-(x^2-1) = 2+x-x^2$

(a) $A(x) = b^2 = (2+x-x^2)^2$

$= 4+4x-3x^2-2x^3+x^4$

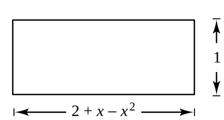
$$V = \int_{-1}^{2}(4+4x-3x^2-2x^3+x^4)\,dx$$

$$= \left[4x+2x^3-x^3-\frac{1}{2}x^4+\frac{1}{5}x^5\right]_{-1}^{2} = \frac{81}{10}$$

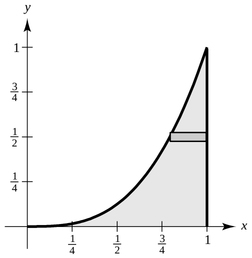


(b) $A(x) = bh = (2+x-x^2)1$

$$V = \int_{-1}^{2}(2+x-x^2)\,dx = \left[2x+\frac{x^2}{2}-\frac{x^3}{3}\right]_{-1}^{2} = \frac{9}{2}$$



61.



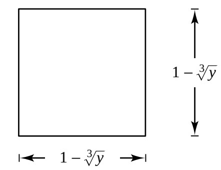
Base of Cross Section $= 1 - \sqrt[3]{y}$

(a) $A(y) = b^2 = \left(1 - \sqrt[3]{y}\right)^2$

$$V = \int_0^1 \left(1 - \sqrt[3]{y}\right)^2 \, dy$$

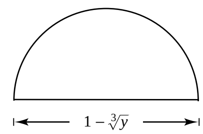
$$= \int_0^1 (1 - 2y^{1/3} + y^{2/3}) \, dy$$

$$= \left[y - \frac{3}{2}y^{4/3} + \frac{3}{5}y^{5/3} \right]_0^1 = \frac{1}{10}$$



(b) $A(y) = \dfrac{1}{2}\pi r^2 = \dfrac{1}{2}\pi\left(\dfrac{1 - \sqrt[3]{y}}{2}\right)^2 = \dfrac{1}{8}\pi\left(1 - \sqrt[3]{y}\right)^2$

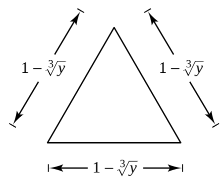
$$V = \frac{1}{8}\pi \int_0^1 \left(1 - \sqrt[3]{y}\right)^2 \, dy = \frac{\pi}{8}\left(\frac{1}{10}\right) = \frac{\pi}{80}$$



(c) $A(y) = \dfrac{1}{2}bh = \dfrac{1}{2}\left(1 - \sqrt[3]{y}\right)\left(\dfrac{\sqrt{3}}{2}\right)\left(1 - \sqrt[3]{y}\right)$

$$= \frac{\sqrt{3}}{4}\left(1 - \sqrt[3]{y}\right)^2$$

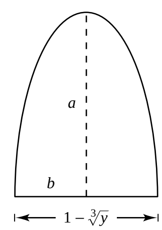
$$V = \frac{\sqrt{3}}{4}\int_0^1 \left(1 - \sqrt[3]{y}\right)^2 \, dy = \frac{\sqrt{3}}{4}\left(\frac{1}{10}\right) = \frac{\sqrt{3}}{40}$$



(d) $A(y) = \dfrac{1}{2}\pi ab = \dfrac{\pi}{2}(2)\left(1 - \sqrt[3]{y}\right)\dfrac{1 - \sqrt[3]{y}}{2}$

$$= \frac{\pi}{2}\left(1 - \sqrt[3]{y}\right)^2$$

$$V = \frac{\pi}{2}\int_0^1 \left(1 - \sqrt[3]{y}\right)^2 \, dy = \frac{\pi}{2}\left(\frac{1}{10}\right) = \frac{\pi}{20}$$



63. Let $A_1(x)$ and $A_2(x)$ equal the areas of the cross sections of the two solids for $a \le x \le b$. Since $A_1(x) = A_2(x)$, we have

$$V_1 = \int_a^b A_1(x) \, dx = \int_a^b A_2(x) \, dx = V_2$$

Thus, the volumes are the same.

65. $\dfrac{4}{3}\pi(25 - r^2)^{3/2} = \dfrac{1}{2}\left(\dfrac{4}{3}\right)\pi(125)$

$$(25 - r^2)^{3/2} = \frac{125}{2}$$

$$25 - r^2 = \left(\frac{125}{2}\right)^{2/3}$$

$$25 - \frac{25}{(2^{2/3})} = r^2$$

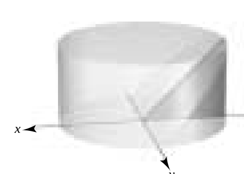
$$25(1 - 2^{-2/3}) = r^2$$

$$r = 5\sqrt{1 - 2^{-2/3}} \approx 3.0415$$

67. (a) Since the cross sections are isosceles right triangles:

$$A(x) = \frac{1}{2}bh = \frac{1}{2}\left(\sqrt{r^2 - y^2}\right)\left(\sqrt{r^2 - y^2}\right) = \frac{1}{2}(r^2 - y^2)$$

$$V = \frac{1}{2}\int_{-r}^{r} (r^2 - y^2) \, dy = \int_0^r (r^2 - y^2) \, dy = \left[r^2 y - \frac{y^3}{3} \right]_0^r = \frac{2}{3}r^3$$



(b) $A(x) = \dfrac{1}{2}bh = \dfrac{1}{2}\sqrt{r^2 - y^2}\left(\sqrt{r^2 - y^2}\tan\theta\right) = \dfrac{\tan\theta}{2}(r^2 - y^2)$

$$V = \frac{\tan\theta}{2}\int_{-r}^{r} (r^2 - y^2) \, dy = \tan\theta \int_0^r (r^2 - y^2) \, dy = \tan\theta \left[r^2 y - \frac{y^3}{3} \right]_0^r = \frac{2}{3}r^3 \tan\theta$$

As $\theta \to 90^\circ$, $V \to \infty$.

Section 6.3 Volume: The Shell Method

1. $p(x) = x$

$h(x) = x$

$$V = 2\pi\int_0^2 x(x)\,dx = \left[\frac{2\pi x^3}{3}\right]_0^2 = \frac{16\pi}{3}$$

$$= \left[\frac{2\pi x^3}{3}\right]_0^2 = \frac{16\pi}{3}$$

3. $p(x) = x$

$h(x) = \sqrt{x}$

$$V = 2\pi\int_0^4 x\sqrt{x}\,dx$$

$$= 2\pi\int_0^4 x^{3/2}\,dx$$

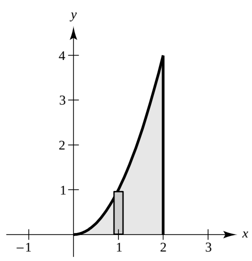
$$= \left[\frac{4\pi}{5}x^{5/2}\right]_0^4 = \frac{128\pi}{5}$$

5. $p(x) = x$

$h(x) = x^2$

$$V = 2\pi\int_0^2 x^3\,dx$$

$$= \left[\frac{\pi}{2}x^4\right]_0^2 = 8\pi$$



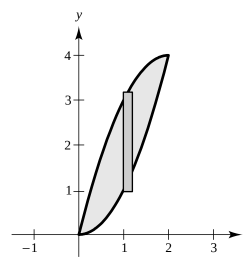
7. $p(x) = x$

$h(x) = (4x - x^2) - x^2 = 4x - 2x^2$

$$V = 2\pi\int_0^2 x(4x - 2x^2)\,dx$$

$$= 4\pi\int_0^2 (2x^2 - x^3)\,dx$$

$$= 4\pi\left[\frac{2}{3}x^3 - \frac{1}{4}x^4\right]_0^2 = \frac{16\pi}{3}$$

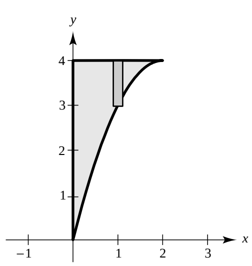


9. $p(x) = x$

$h(x) = 4 - (4x - x^2) = x^2 - 4x + 4$

$$V = 2\pi\int_0^2 (x^3 - 4x^2 + 4x)\,dx$$

$$= 2\pi\left[\frac{x^4}{4} - \frac{4}{3}x^3 + 2x^2\right]_0^2 = \frac{8\pi}{3}$$



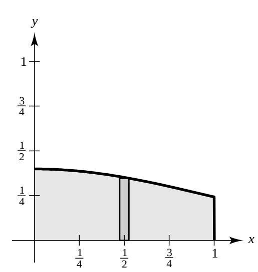
11. $p(x) = x$

$$h(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$$

$$V = 2\pi\int_0^1 x\left(\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\right)dx$$

$$= \sqrt{2\pi}\int_0^1 e^{-x^2/2}x\,dx$$

$$= \left[-\sqrt{2\pi}\,e^{-x^2/2}\right]_0^1 = \sqrt{2\pi}\left(1 - \frac{1}{\sqrt{e}}\right) \approx 0.986$$



13. $p(y) = y$

$h(y) = 2 - y$

$$V = 2\pi\int_0^2 y(2 - y)\,dy$$

$$= 2\pi\int_0^2 (2y - y^2)\,dy$$

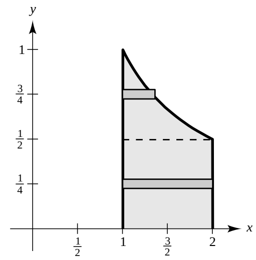
$$= 2\pi\left[y^2 - \frac{y^3}{3}\right]_0^2 = \frac{8\pi}{3}$$

15. $p(y) = y$ and $h(y) = 1$ if $0 \le y < \frac{1}{2}$.

$p(y) = y$ and $h(y) = \frac{1}{y} - 1$ if $\frac{1}{2} \le y \le 1$.

$$V = 2\pi \int_0^{1/2} y\,dy + 2\pi \int_{1/2}^{1} (1 - y)\,dy$$

$$= 2\pi\left[\frac{y^2}{2}\right]_0^{1/2} + 2\pi\left[y - \frac{y^2}{2}\right]_{1/2}^{1} = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$



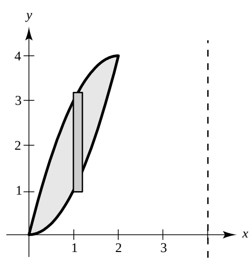
17. $p(x) = 4 - x$

$h(x) = 4x - x^2 - x^2 = 4x - 2x^2$

$$V = 2\pi \int_0^2 (4 - x)(4x - 2x^2)\,dx$$

$$= 2\pi(2) \int_0^2 (x^3 - 6x^2 + 8x)\,dx$$

$$= 4\pi\left[\frac{x^4}{4} - 2x^3 + 4x^2\right]_0^2 = 16\pi$$



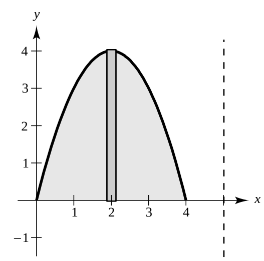
19. $p(x) = 5 - x$

$h(x) = 4x - x^2$

$$V = 2\pi \int_0^4 (5 - x)(4x - x^2)\,dx$$

$$= 2\pi \int_0^4 (x^3 - 9x^2 + 20x)\,dx$$

$$= 2\pi\left[\frac{x^4}{4} - 3x^3 + 10x^2\right]_0^4 = 64\pi$$

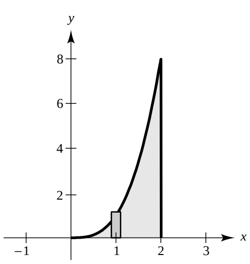


21. (a) **Disk**

$R(x) = x^3$

$r(x) = 0$

$$V = \pi \int_0^2 x^6\,dx = \pi\left[\frac{x^7}{7}\right]_0^2 = \frac{128\pi}{7}$$

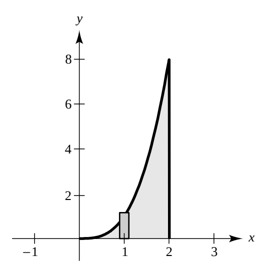


(b) **Shell**

$p(x) = x$

$h(x) = x^3$

$$V = 2\pi \int_0^2 x^4\,dx = 2\pi\left[\frac{x^5}{5}\right]_0^2 = \frac{64\pi}{5}$$



(c) **Shell**

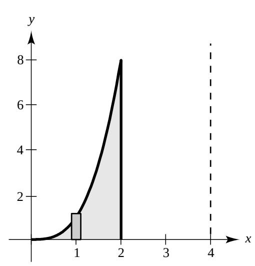
$p(x) = 4 - x$

$h(x) = x^3$

$$V = 2\pi \int_0^2 (4 - x)x^3\,dx$$

$$= 2\pi \int_0^2 (4x^3 - x^4)\,dx$$

$$= 2\pi\left[x^4 - \frac{1}{5}x^5\right]_0^2 = \frac{96\pi}{5}$$



23. (a) **Shell**

$$p(y) = y$$

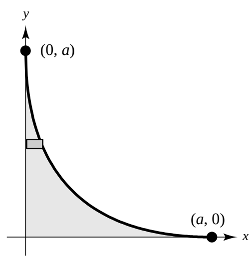
$$h(y) = (a^{1/2} - y^{1/2})^2$$

$$V = 2\pi \int_0^a y(a - 2a^{1/2}y^{1/2} + y)\, dy$$

$$= 2\pi \int_0^a (ay - 2a^{1/2}y^{3/2} + y^2)\, dy$$

$$= 2\pi \left[\frac{a}{2}y^2 - \frac{4a^{1/2}}{5}y^{5/2} + \frac{y^3}{3} \right]_0^a$$

$$= 2\pi \left[\frac{a^3}{2} - \frac{4a^3}{5} + \frac{a^3}{3} \right] = \frac{\pi a^3}{15}$$



(b) Same as part (a) by symmetry

(c) **Shell**

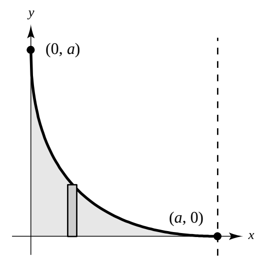
$$p(x) = a - x$$

$$h(x) = (a^{1/2} - x^{1/2})^2$$

$$V = 2\pi \int_0^a (a - x)(a^{1/2} - x^{1/2})^2\, dx$$

$$= 2\pi \int_0^a (a^2 - 2a^{3/2}x^{1/2} + 2a^{1/2}x^{3/2} - x^2)\, dx$$

$$= 2\pi \left[a^2x - \frac{4}{3}a^{3/2}x^{3/2} + \frac{4}{5}a^{1/2}x^{5/2} - \frac{1}{3}x^3 \right]_0^a = \frac{4\pi a^3}{15}$$



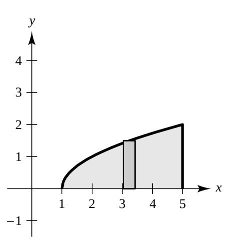
25. $V = 2\pi \int_x^d p(y)h(y)\, dy$ or $V = 2\pi \int_a^b p(x)h(x)\, dx$

27. $\pi \int_1^5 (x - 1)\, dx = \pi \int_1^5 \left(\sqrt{x - 1}\right)^2 dx$

This integral represents the volume of the solid generated by revolving the region bounded by $y = \sqrt{x - 1}$, $y = 0$, and $x = 5$ about the x-axis by using the Disk Method.

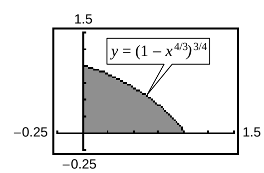
$$2\pi \int_0^2 y[5 - (y^2 + 1)]\, dy$$

represents this same volume by using the Shell Method.



Disk Method

29. (a)

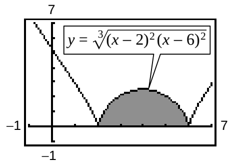


(b) $x^{4/3} + y^{4/3} = 1, x = 0, y = 0$

$$y = (1 - x^{4/3})^{3/4}$$

$$V = 2\pi \int_0^1 x(1 - x^{4/3})^{3/4}\, dx \approx 1.5056$$

31. (a)

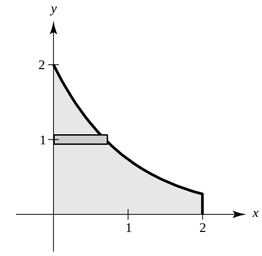


(b) $V = 2\pi \int_2^6 x\sqrt[3]{(x-2)^2(x-6)^2}\,dx \approx 187.249$

33. $y = 2e^{-x}, y = 0, x = 0, x = 2$

Volume ≈ 7.5

Matches (d)



35. $p(x) = x$

$h(x) = 2 - \frac{1}{2}x^2$

$V = 2\pi \int_0^2 x\left(2 - \frac{1}{2}x^2\right) dx = 2\pi \int_0^2 \left(2x - \frac{1}{2}x^3\right) dx = 2\pi\left[x^2 - \frac{1}{8}x^4\right]_0^2 = 4\pi$ (total volume)

Now find x_0 such that

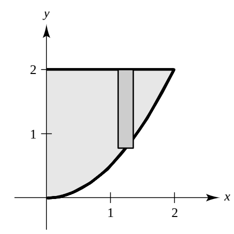
$$\pi = 2\pi \int_0^{x_0} \left(2x - \frac{1}{2}x^3\right) dx$$

$$1 = 2\left[x^2 - \frac{1}{8}x^4\right]_0^{x_0}$$

$$1 = 2x_0^2 - \frac{1}{4}x_0^4$$

$$x_0^4 - 8x_0^2 + 4 = 0$$

$$x_0^2 = 4 \pm 2\sqrt{3} \quad \text{(Quadratic Formula)}$$



Take $x_0 = \sqrt{4 - 2\sqrt{3}}$ since the other root is too large.

Diameter: $2\sqrt{4 - 2\sqrt{3}} \approx 1.464$

37. $V = 4\pi \int_{-1}^1 (2 - x)\sqrt{1 - x^2}\,dx$

$= 8\pi \int_{-1}^1 \sqrt{1 - x^2}\,dx - 4\pi \int_{-1}^1 x\sqrt{1 - x^2}\,dx$

$= 8\pi\left(\frac{\pi}{2}\right) + 2\pi \int_{-1}^1 x(1 - x^2)^{1/2}(-2)\,dx$

$= 4\pi^2 + \left[2\pi\left(\frac{2}{3}\right)(1 - x^2)^{3/2}\right]_{-1}^1 = 4\pi^2$

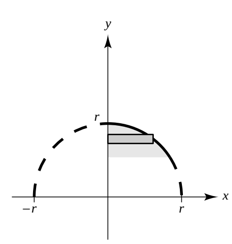
39. Disk Method

$R(y) = \sqrt{r^2 - y^2}$

$r(y) = 0$

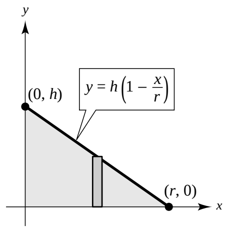
$V = \pi \int_{r-h}^r (r^2 - y^2)\,dy$

$= \pi\left[r^2y - \frac{y^3}{3}\right]_{r-h}^r = \frac{1}{3}\pi h^2(3r - h)$



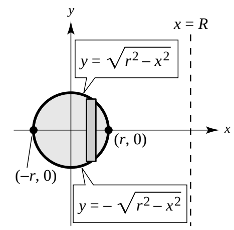
41. (a) $2\pi \int_0^r hx\left(1 - \frac{x}{r}\right) dx$ (ii)

is the volume of a right circular cone with the radius of the base as r and height h.

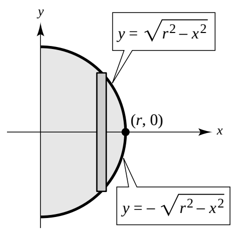


(b) $2\pi \int_{-r}^r (R - x)\left(2\sqrt{r^2 - x^2}\right) dx$ (v)

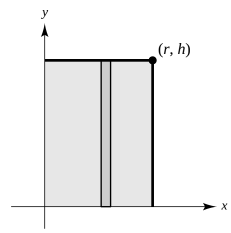
is the volume of a torus with the radius of its circular cross section as r and the distance from the axis of the torus to the center of its cross section as R.



(c) $2\pi \int_0^r 2x\sqrt{r^2 - x^2}\, dx$ (iii) is the volume of a sphere with radius r.

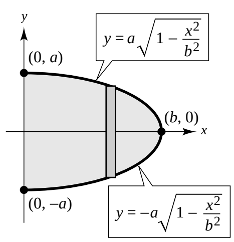


(d) $2\pi \int_0^r hx\, dx$ (i) is the volume of a right circular cylinder with a radius of r and a height of h.



(e) $2\pi \int_0^b 2ax\sqrt{1 - (x^2/b^2)}\, dx$ (iv)

is the volume of an ellipsoid with axes $2a$ and $2b$.

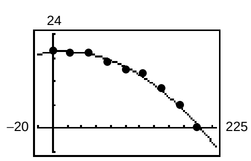


43. (a) $V = 2\pi \int_0^{200} x f(x)\, dx$

$$\approx \frac{2\pi(200)}{3(8)}[0 + 4(25)(19) + 2(50)(19) + 4(75)(17) + 2(100)15 + 4(125)(14) + 2(150)(10) + 4(175)(6) + 0]$$

$\approx$ 1,366,593 cubic feet

(b) $d = -0.000561x^2 + 0.0189x + 19.39$



(c) $V \approx 2\pi \int_0^{200} xd(x)\, dx \approx 2\pi(213{,}800) = 1{,}343{,}345$ cubic feet

(d) Number gallons $\approx V(7.48) = 10{,}048{,}221$ gallons

Section 6.4 Arc Length and Surfaces of Revolution

1. (0, 0), (5, 12)

(a) $d = \sqrt{(5-0)^2 + (12-0)^2} = 13$

(b) $y = \frac{12}{5}x$

$y' = \frac{12}{5}$

$s = \int_0^5 \sqrt{1 + \left(\frac{12}{5}\right)^2}\,dx = \left[\frac{13}{5}x\right]_0^5 = 13$

3. $y = \frac{2}{3}x^{3/2} + 1$

$y' = x^{1/2}, [0, 1]$

$s = \int_0^1 \sqrt{1+x}\,dx$

$= \left[\frac{2}{3}(1+x)^{3/2}\right]_0^1$

$= \frac{2}{3}\left(\sqrt{8} - 1\right) \approx 1.219$

5. $y = \frac{3}{2}x^{2/3}$

$y' = \frac{1}{x^{1/3}}, [1, 8]$

$s = \int_1^8 \sqrt{1 + \left(\frac{1}{x^{1/3}}\right)^2}\,dx$

$= \int_1^8 \sqrt{\frac{x^{2/3}+1}{x^{2/3}}}\,dx$

$= \frac{3}{2}\int_1^8 \sqrt{x^{2/3}+1}\left(\frac{2}{3x^{1/3}}\right)dx$

$= \frac{3}{2}\left[\frac{2}{3}(x^{2/3}+1)^{3/2}\right]_1^8$

$= 5\sqrt{5} - 2\sqrt{2} \approx 8.352$

7. $y = \frac{x^4}{8} + \frac{1}{4x^2}$

$y' = \frac{1}{2}x^3 - \frac{1}{2x^3}, [1, 2]$

$1 + (y')^2 = \left(\frac{1}{2}x^3 + \frac{1}{2x^3}\right)^2, [1, 2]$

$s = \int_a^b \sqrt{1 + (y')^2}\,dx$

$= \int_1^2 \left(\frac{1}{2}x^3 + \frac{1}{2x^3}\right)dx$

$= \left[\frac{1}{8}x^4 - \frac{1}{4x^2}\right]_1^2 = \frac{33}{16} \approx 2.063$

9. $y = \ln(\sin x), \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$

$y' = \frac{1}{\sin x}\cos x = \cot x$

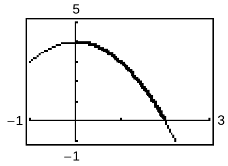
$1 + (y')^2 = 1 + \cot^2 x = \csc^2 x$

$s = \int_{\pi/4}^{3\pi/4} \csc x\,dx$

$= \left[\ln|\csc x - \cot x|\right]_{\pi/4}^{3\pi/4}$

$= \ln\left(\sqrt{2}+1\right) - \ln\left(\sqrt{2}-1\right) \approx 1.763$

11. (a) $y = 4 - x^2, 0 \le x \le 2$



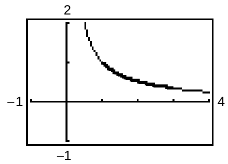
(b) $y' = -2x$

$1 + (y')^2 = 1 + 4x^2$

$L = \int_0^2 \sqrt{1 + 4x^2}\,dx$

(c) $L \approx 4.647$

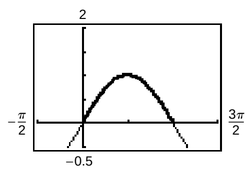
13. (a) $y = \frac{1}{x}, 1 \le x \le 3$



(b) $$y' = -\frac{1}{x^2}$$
$$1 + (y')^2 = 1 + \frac{1}{x^4}$$
$$L = \int_1^3 \sqrt{1 + \frac{1}{x^4}}\, dx$$

(c) $L \approx 2.147$

15. (a) $y = \sin x, 0 \le x \le \pi$



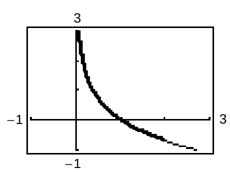
(b) $$y' = \cos x$$
$$1 + (y')^2 = 1 + \cos^2 x$$
$$L = \int_0^{\pi} \sqrt{1 + \cos^2 x}\, dx$$

(c) $L \approx 3.820$

17. (a) $x = e^{-y}, 0 \le y \le 2$

$y = -\ln x$

$1 \ge x \ge e^{-2} \approx 0.135$



(b) $$y' = -\frac{1}{x}$$
$$1 + (y')^2 = 1 + \frac{1}{x^2}$$
$$L = \int_{e^{-2}}^{1} \sqrt{1 + \frac{1}{x^2}}\, dx$$

(c) $L \approx 2.221$

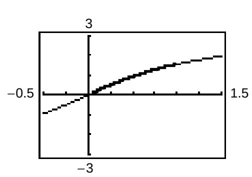
Alternatively, you can do all the computations with respect to y.

(a) $x = e^{-y} \quad 0 \le y \le 2$

(b) $$\frac{dx}{dy} = -e^{-y}$$
$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + e^{-2y}$$
$$L = \int_0^2 \sqrt{1 + e^{-2y}}\, dy$$

(c) $L \approx 2.221$

19. (a) $y = 2 \arctan x, 0 \le x \le 1$



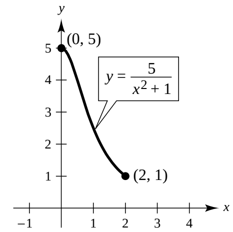
(b) $$y' = \frac{2}{1 + x^2}$$
$$L = \int_0^1 \sqrt{1 + \frac{4}{(1 + x^2)^2}}\, dx$$

(c) $L \approx 1.871$

21. $\int_0^2 \sqrt{1 + \left[\frac{d}{dx}\left(\frac{5}{x^2+1}\right)\right]^2}\, dx$

$s \approx 5$

Matches (b)



23. $y = x^3, [0, 4]$

(a) $d = \sqrt{(4-0)^2 + (64-0)^2} \approx 64.125$

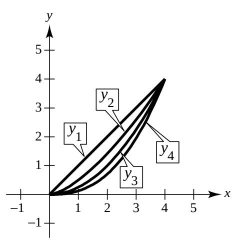
(b) $d = \sqrt{(1-0)^2 + (1-0)^2} + \sqrt{(2-1)^2 + (8-1)^2} + \sqrt{(3-2)^2 + (27-8)^2} + \sqrt{(4-3)^2 + (64-27)^2}$

≈ 64.525

(c) $s = \int_0^4 \sqrt{1 + (3x^2)^2}\, dx = \int_0^4 \sqrt{1 + 9x^4}\, dx \approx 64.666$

(d) 64.672

25. (a)



(b) y_1, y_2, y_3, y_4

(c) $y_1' = 1,\ L_1 = \int_0^4 \sqrt{2}\, dx \approx 5.657$

$y_2' = \frac{3}{4}x^{1/2},\ L_2 = \int_0^4 \sqrt{1 + \frac{9x}{16}}\, dx \approx 5.759$

$y_3' = \frac{1}{2}x,\ L_3 = \int_0^4 \sqrt{1 + \frac{x^2}{4}}\, dx \approx 5.916$

$y_4' = \frac{5}{16}x^{3/2},\ L_4 = \int_0^4 \sqrt{1 + \frac{25}{256}x^3}\, dx \approx 6.063$

27. $y = \frac{1}{3}[x^{3/2} - 3x^{1/2} + 2]$

When $x = 0$, $y = \frac{2}{3}$. Thus, the fleeting object has traveled $\frac{2}{3}$ units when it is caught.

$$y' = \frac{1}{3}\left[\frac{3}{2}x^{1/2} - \frac{3}{2}x^{-1/2}\right] = \left(\frac{1}{2}\right)\frac{x-1}{x^{1/2}}$$

$$1 + (y')^2 = 1 + \frac{(x-1)^2}{4x} = \frac{(x+1)^2}{4x}$$

$$s = \int_0^1 \frac{x+1}{2x^{1/2}}\, dx = \frac{1}{2}\int_0^1 (x^{1/2} + x^{-1/2})\, dx = \frac{1}{2}\left[\frac{2}{3}x^{3/2} + 2x^{1/2}\right]_0^1 = \frac{4}{3} = 2\left(\frac{2}{3}\right)$$

The pursuer has traveled twice the distance that the fleeing object has traveled when it is caught.

29. $$y = 20\cosh\frac{x}{20},\ -20 \le x \le 20$$

$$y' = \sinh\frac{x}{20}$$

$$1 + (y')^2 = 1 + \sinh^2\frac{x}{20} = \cosh^2\frac{x}{20}$$

$$L = \int_{-20}^{20} \cosh\frac{x}{20}\, dx = 2\int_0^{20} \cosh\frac{x}{20}\, dx = 2(20)\sinh\frac{x}{20}\Big]_0^{20}$$

$$= 40\sinh(1) \approx 47.008 \text{ m.}$$

31.
$$y = \sqrt{9 - x^2}$$
$$y' = \frac{-x}{\sqrt{9 - x^2}}$$
$$1 + (y')^2 = \frac{9}{9 - x^2}$$
$$s = \int_0^2 \sqrt{\frac{9}{9 - x^2}}\,dx$$
$$= \int_0^2 \frac{3}{\sqrt{9 - x^2}}\,dx$$
$$= \left[3 \arcsin \frac{x}{3}\right]_0^2$$
$$= 3\left(\arcsin \frac{2}{3} - \arcsin 0\right)$$
$$= 3 \arcsin \frac{2}{3} \approx 2.1892$$

33. $y = \dfrac{x^3}{3}$
$$y' = x^2, [0, 3]$$
$$S = 2\pi \int_0^3 \frac{x^3}{3}\sqrt{1 + x^4}\,dx$$
$$= \frac{\pi}{6}\int_0^3 (1 + x^4)^{1/2}(4x^3)\,dx$$
$$= \left[\frac{\pi}{9}(1 + x^4)^{3/2}\right]_0^3$$
$$= \frac{\pi}{9}\left(82\sqrt{82} - 1\right) \approx 258.85$$

35.
$$y = \frac{x^3}{6} + \frac{1}{2x}$$
$$y' = \frac{x^2}{2} - \frac{1}{2x^2}$$
$$1 + (y')^2 = \left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2, [1, 2]$$
$$S = 2\pi \int_1^2 \left(\frac{x^3}{6} + \frac{1}{2x}\right)\left(\frac{x^2}{2} + \frac{1}{2x^2}\right) dx$$
$$= 2\pi \int_1^2 \left(\frac{x^5}{12} + \frac{x}{3} + \frac{1}{4x^3}\right) dx$$
$$= 2\pi\left[\frac{x^6}{72} + \frac{x^2}{6} - \frac{1}{8x^2}\right]_1^2 = \frac{47\pi}{16}$$

37. $y = \sqrt[3]{x} + 2$
$$y' = \frac{1}{3x^{2/3}}, [1, 8]$$
$$S = 2\pi \int_1^8 x\sqrt{1 + \frac{1}{9x^{4/3}}}\,dx$$
$$= \frac{2\pi}{3}\int_1^8 x^{1/3}\sqrt{9x^{4/3} + 1}\,dx$$
$$= \frac{\pi}{18}\int_1^8 (9x^{4/3} + 1)^{1/2}(12x^{1/3})\,dx$$
$$= \left[\frac{\pi}{27}(9x^{4/3} + 1)^{3/2}\right]_1^8$$
$$= \frac{\pi}{27}\left(145\sqrt{145} - 10\sqrt{10}\right) \approx 199.48$$

39. $y = \sin x$
$$y' = \cos x, [0, \pi]$$
$$S = 2\pi \int_0^{\pi} \sin x\sqrt{1 + \cos^2 x}\,dx$$
$$\approx 14.4236$$

41. A rectifiable curve is one that has a finite arc length.

43. The precalculus formula is the surface area formula for the lateral surface of the frustum of a right circular cone. The representative element is
$$2\pi f(d_i)\sqrt{\Delta x_i^2 + \Delta y_i^2} = 2\pi f(di)\sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\,\Delta x_i.$$

45.

$$y = \frac{hx}{r}$$

$$y' = \frac{h}{r}$$

$$1 + (y')^2 = \frac{r^2 + h^2}{r^2}$$

$$S = 2\pi \int_0^r x\sqrt{\frac{r^2 + h^2}{r^2}}\,dx$$

$$= \left[\frac{2\pi\sqrt{r^2 + h^2}}{r}\left(\frac{x^2}{2}\right)\right]_0^r = \pi r\sqrt{r^2 + h^2}$$

47.

$$y = \sqrt{9 - x^2}$$

$$y' = \frac{-x}{\sqrt{9 - x^2}}$$

$$\sqrt{1 + (y')^2} = \frac{3}{\sqrt{9 - x^2}}$$

$$S = 2\pi \int_0^2 \frac{3x}{\sqrt{9 - x^2}}\,dx$$

$$= -3\pi \int_0^2 \frac{-2x}{\sqrt{9 - x^2}}\,dx$$

$$= \left[-6\pi\sqrt{9 - x^2}\right]_0^2$$

$$= 6\pi\left(3 - \sqrt{5}\right) \approx 14.40$$

See figure in Exercise 48.

49.

$$y = \frac{1}{3}x^{1/2} - x^{3/2}$$

$$y' = \frac{1}{6}x^{-1/2} - \frac{3}{2}x^{1/2} = \frac{1}{6}(x^{-1/2} - 9x^{1/2})$$

$$1 + (y')^2 = 1 + \frac{1}{36}(x^{-1} - 18 + 81x) = \frac{1}{36}(x^{-1/2} + 9x^{1/2})^2$$

$$S = 2\pi \int_0^{1/3} \left(\frac{1}{3}x^{1/2} - x^{3/2}\right)\sqrt{\frac{1}{36}(x^{-1/2} + 9^{1/2})^2}\,dx = \frac{2\pi}{6}\int_0^{1/3}\left(\frac{1}{3}x^{1/2} - x^{3/2}\right)(x^{-1/2} + 9x^{1/2})\,dx$$

$$= \frac{\pi}{3}\int_0^{1/3}\left(\frac{1}{3} + 2x - 9x^2\right)dx = \frac{\pi}{3}\left[\frac{1}{3}x + x^2 - 3x^3\right]_0^{1/3} = \frac{\pi}{27}\text{ ft}^2 \approx 0.1164\text{ ft}^2 \approx 16.8\text{ in}^2$$

Amount of glass needed: $V = \dfrac{\pi}{27}\left(\dfrac{0.015}{12}\right) \approx 0.00015\text{ ft}^3 \approx 0.25\text{ in}^3$

51. (a) $y = f(x) = 0.0000001953x^4 - 0.0001804x^3 + 0.0496x^2 - 4.8323x + 536.9270$

(b) Area $= \displaystyle\int_0^{400} f(x)\,dx \approx 131{,}734.5$ square feet

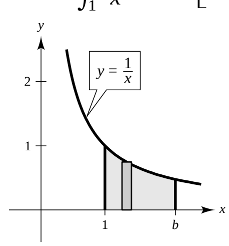
≈ 3.0 acres

(Answers will vary.)

(c) $L = \displaystyle\int_0^{400} \sqrt{1 + f'(x)^2}\,dx \approx 794.9$ feet

(Answers will vary.)

53. (a) $V = \pi\displaystyle\int_1^b \frac{1}{x^2}\,dx = \left[-\frac{\pi}{x}\right]_1^b = \pi\left(1 - \frac{1}{b}\right)$



(b) $S = 2\pi\displaystyle\int_1^b \frac{1}{x}\sqrt{1 + \left(-\frac{1}{x^2}\right)^2}\,dx$

$$= 2\pi\int_1^b \frac{1}{x}\sqrt{1 + \frac{1}{x^4}}\,dx$$

$$= 2\pi\int_1^b \frac{\sqrt{x^4 + 1}}{x^3}\,dx$$

—CONTINUED—

53. —CONTINUED—

(c) $\lim_{b\to\infty} V = \lim_{b\to\infty} \pi\left(1 - \frac{1}{b}\right) = \pi$

(d) Since

$$\frac{\sqrt{x^4+1}}{x^3} > \frac{\sqrt{x^4}}{x^3} = \frac{1}{x} > 0 \text{ on } [1, b]$$

we have

$$\int_1^b \frac{\sqrt{x^4+1}}{x^3}\,dx > \int_1^b \frac{1}{x}\,dx = \Big[\ln x\Big]_1^b = \ln b$$

and $\lim_{b\to\infty} \ln b \to \infty$. Thus,

$$\lim_{b\to\infty} 2\pi \int_1^b \frac{\sqrt{x^4+1}}{x^3}\,dx = \infty.$$

55. (a) Area of circle with radius L: $A = \pi L^2$

Area of sector with central angle θ (in radians)

$$S = \frac{\theta}{2\pi}A = \frac{\theta}{2\pi}(\pi L^2) = \frac{1}{2}L^2\theta$$

(b) Let s be the arc length of the sector, which is the circumference of the base of the cone. Here, $s = L\theta = 2\pi r$, and you have

$$S = \frac{1}{2}L^2\theta = \frac{1}{2}L^2\left(\frac{s}{L}\right) = \frac{1}{2}Ls = \frac{1}{2}L(2\pi r) = \pi rL$$

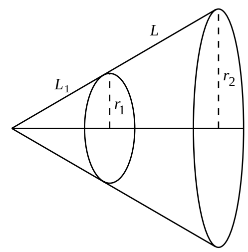
(c) The lateral surface area of the frustum is the difference of the large cone and the small one.

$$S = \pi r_2(L + L_1) - \pi r_1 L_1$$
$$= \pi r_2 L + \pi L_1(r_2 - r_1)$$

By similar triangles, $\frac{L + L_1}{r_2} = \frac{L_1}{r_1} \implies Lr_1 = L_1(r_2 - r_1)$

Hence,

$$S = \pi r_2 L + \pi L_1(r_2 - r_1) = \pi r_2 L + \pi L r_1$$
$$= \pi L(r_1 + r_2).$$



Section 6.5 Work

1. $W = Fd = (100)(10) = 1000 \text{ ft} \cdot \text{lb}$

3. $W = Fd = (112)(4) = 448$ joules (newton-meters)

5. Work equals force times distance, $W = FD$.

7. Since the work equals the area under the force function, you have $(c) < (d) < (a) < (b)$.

9. $F(x) = kx$

$5 = k(4)$

$k = \frac{5}{4}$

$$W = \int_0^7 \frac{5}{4}x\,dx = \left[\frac{5}{8}x^2\right]_0^7$$
$$= \frac{245}{8} \text{ in} \cdot \text{lb}$$
$$= 30.625 \text{ in} \cdot \text{lb} \approx 2.55 \text{ ft} \cdot \text{lb}$$

11. $F(x) = kx$

$250 = k(30) \implies k = \frac{25}{3}$

$$W = \int_{20}^{50} F(x)\,dx = \int_{20}^{50} \frac{25}{3}x\,dx = \left.\frac{25x^2}{6}\right]_{20}^{50}$$
$$= 8750 \text{ n} \cdot \text{cm} = 87.5 \text{ joules or Nm}$$

13. $F(x) = kx$

$20 = k(9)$

$k = \frac{20}{9}$

$W = \int_0^{12} \frac{20}{9}x\,dx = \left[\frac{10}{9}x^2\right]_0^{12} = \frac{40}{3}$ ft · lb

15. $W = 18 = \int_0^{1/3} kx\,dx = \frac{kx^2}{2}\Big]_0^{1/3} = \frac{k}{18} \Rightarrow k = 324$

$W = \int_{1/3}^{7/12} 324x\,dx = 162x^2\Big]_{1/3}^{7/12} = 37.125$ ft · lbs

[**Note:** 4 inches $= \frac{1}{3}$ foot]

17. Assume that Earth has a radius of 4000 miles.

$F(x) = \frac{k}{x^2}$

$s = \frac{k}{(4000)^2}$

$k = 80{,}000{,}000$

$F(x) = \frac{80{,}000{,}000}{x^2}$

(a) $W = \int_{4000}^{4100} \frac{80{,}000{,}000}{x^2}\,dx = \left[\frac{-80{,}000{,}000}{x}\right]_{4000}^{4100} \approx 487.8$ mi · tons

$\approx 5.15 \times 10^9$ ft · lb

(b) $W = \int_{4000}^{4300} \frac{80{,}000{,}000}{x^2}\,dx \approx 1395.3$ mi · ton

$\approx 1.47 \times 10^{10}$ ft · lb

19. Assume that the earth has a radius of 4000 miles.

$F(x) = \frac{k}{x^2}$

$10 = \frac{k}{(4000)^2}$

$k = 160{,}000{,}000$

$F(x) = \frac{160{,}000{,}000}{x^2}$

(a) $W = \int_{4000}^{15{,}000} \frac{160{,}000{,}000}{x^2}\,dx = \left[-\frac{160{,}000{,}000}{x}\right]_{4000}^{15{,}000} \approx -10{,}666.667 + 40{,}000$

$= 29{,}333.333$ mi · ton

$\approx 2.93 \times 10^4$ mi · ton

$\approx 3.10 \times 10^{11}$ ft · lb

(b) $W = \int_{4000}^{26{,}000} \frac{160{,}000{,}000}{x^2}\,dx = \left[-\frac{160{,}000{,}000}{x}\right]_{4000}^{26{,}000} \approx -6{,}153.846 + 40{,}000$

$= 33{,}846.154$ mi · ton

$\approx 3.38 \times 10^4$ mi · ton

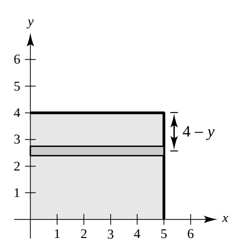
$\approx 3.57 \times 10^{11}$ ft · lb

21. Weight of each layer: $62.4(20)\,\Delta y$

Distance: $4 - y$

(a) $W = \int_2^4 62.4(20)(4-y)\,dy = \left[4992y - 624y^2\right]_2^4 = 2496$ ft · lb

(b) $W = \int_0^4 62.4(20)(4-y)\,dy = \left[4992y - 624y^2\right]_0^4 = 9984$ ft · lb



23. Volume of disk: $\pi(2)^2\,\Delta y = 4\pi\,\Delta y$

Weight of disk of water: $9800(4\pi)\,\Delta y$

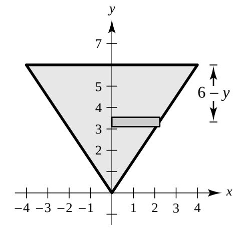
Distance the disk of water is moved: $5 - y$

$$W = \int_0^4 (5-y)(9800)4\pi\,dy = 39{,}200\pi\int_0^4 (5-y)\,dy$$

$$= 39{,}200\pi\left[5y - \frac{y^2}{2}\right]_0^4$$

$$= 39{,}200\pi(12) = 470{,}400\pi \text{ newton–meters}$$

25. Volume of disk: $\pi\left(\frac{2}{3}y\right)^2 \Delta y$

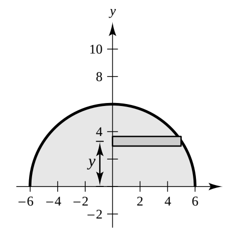


Weight of disk: $62.4\pi\left(\frac{2}{3}y\right)^2 \Delta y$

Distance: $6 - y$

$$W = \frac{4(62.4)\pi}{9}\int_0^6 (6-y)y^2\,dy = \frac{4}{9}(62.4)\pi\left[2y^3 - \frac{1}{4}y^4\right]_0^6 = 2995.2\pi \text{ ft} \cdot \text{lb}$$

27. Volume of disk: $\pi\left(\sqrt{36 - y^2}\right)^2 \Delta y$



Weight of disk: $62.4\pi(36 - y^2)\,\Delta y$

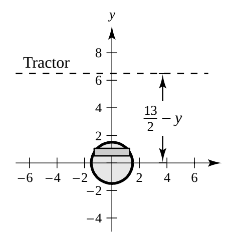
Distance: y

$$W = 62.4\pi\int_0^6 y(36 - y^2)\,dy$$

$$= 62.4\pi\int_0^6 (36y - y^3)\,dy = 62.4\pi\left[18y^2 - \frac{1}{4}y^4\right]_0^6$$

$$= 20{,}217.6\pi \text{ ft} \cdot \text{lb}$$

29. Volume of layer: $V = lwh = 4(2)\sqrt{(9/4) - y^2}\,\Delta y$



Weight of layer: $W = 42(8)\sqrt{(9/4) - y^2}\,\Delta y$

Distance: $\frac{13}{2} - y$

$$W = \int_{-1.5}^{1.5} 42(8)\sqrt{(9/4) - y^2}\left(\frac{13}{2} - y\right)dy$$

$$= 336\left[\frac{13}{2}\int_{-1.5}^{1.5}\sqrt{(9/4) - y^2}\,dy - \int_{-1.5}^{1.5}\sqrt{(9/4) - y^2}\,y\,dy\right]$$

The second integral is zero since the integrand is odd and the limits of integration are symmetric to the origin. The first integral represents the area of a semicircle of radius $\frac{3}{2}$. Thus, the work is

$$W = 336\left(\frac{13}{2}\right)\pi\left(\frac{3}{2}\right)^2\left(\frac{1}{2}\right) = 2457\pi \text{ ft} \cdot \text{lb}$$

31. Weight of section of chain: $3\,\Delta y$

Distance: $15 - y$

$$W = 3\int_0^{15}(15 - y)\,dy$$

$$= \left[-\frac{3}{2}(15 - y)^2\right]_0^{15}$$

$$= 337.5 \text{ ft} \cdot \text{lb}$$

33. The lower 5 feet of chain are raised 10 feet with a constant force.

$$W_1 = 3(5)(10) = 150 \text{ ft} \cdot \text{lb}$$

The top 10 feet of chain are raised with a variable force.

Weight per section: $3\,\Delta y$

Distance: $10 - y$

$$W_2 = 3\int_0^{10}(10 - y)\,dy = \left[-\frac{3}{2}(10 - y)^2\right]_0^{10}$$

$$= 150 \text{ ft} \cdot \text{lb}$$

$$W = W_1 + W_2 = 300 \text{ ft} \cdot \text{lb}$$

35. Weight of section of chain: $3\,\Delta y$

Distance: $15 - 2y$

$$W = 3\int_0^{7.5}(15 - 2y)\,dy = \left[-\frac{3}{4}(15 - 2y)^2\right]_0^{7.5}$$
$$= \frac{3}{4}(15)^2 = 168.75 \text{ ft} \cdot \text{lb}$$

37. Work to pull up the ball: $W_1 = 500(15) = 7500$ ft · lb

Work to wind up the top 15 feet of cable: force is variable

Weight per section: $1\,\Delta y$

Distance: $15 - x$

$$W_2 = \int_0^{15}(15 - x)\,dx = \left[-\frac{1}{2}(15 - x)^2\right]_0^{15}$$
$$= 112.5 \text{ ft} \cdot \text{lb}$$

Work to lift the lower 25 feet of cable with a constant force:

$$W_3 = (1)(25)(15) = 375 \text{ ft} \cdot \text{lb}$$
$$W = W_1 + W_2 + W_3 = 7500 + 112.5 + 375$$
$$= 7987.5 \text{ ft} \cdot \text{lb}$$

39.
$$p = \frac{k}{V}$$
$$1000 = \frac{k}{2}$$
$$k = 2000$$
$$W = \int_2^3 \frac{2000}{V}\,dV = \Big[2000 \ln|V|\Big]_2^3$$
$$= 2000 \ln\left(\frac{3}{2}\right) \approx 810.93 \text{ ft} \cdot \text{lb}$$

41. $F(x) = \dfrac{k}{(2 - x)^2}$

$$W = \int_{-2}^{1}\frac{k}{(2 - x)^2}\,dx = \left[\frac{k}{2 - x}\right]_{-2}^{1} = k\left(1 - \frac{1}{4}\right)$$
$$= \frac{3k}{4}\text{(units of work)}$$

43. $W = \displaystyle\int_0^5 1000[1.8 - \ln(x + 1)]\,dx \approx 3249.44$ ft · lb

45. $W = \displaystyle\int_0^5 100x\sqrt{125 - x^3}\,dx \approx 10{,}330.3$ ft · lb

Section 6.6 Moments, Centers of Mass, and Centroids

1. $\bar{x} = \dfrac{6(-5) + 3(1) + 5(3)}{6 + 3 + 5} = -\dfrac{6}{7}$

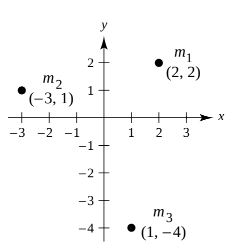
3. $\bar{x} = \dfrac{1(7) + 1(8) + 1(12) + 1(15) + 1(18)}{1 + 1 + 1 + 1 + 1} = 12$

5. (a) $\bar{x} = \dfrac{(7 + 5) + (8 + 5) + (12 + 5) + (15 + 5) + (18 + 5)}{5} = 17 = 12 + 5$

(b) $\bar{x} = \dfrac{12(-6 - 3) + 1(-4 - 3) + 6(-2 - 3) + 3(0 - 3) + 11(8 - 3)}{12 + 1 + 6 + 3 + 11} = \dfrac{-99}{33} = -3$

7.
$$50x = 75(L - x) = 75(10 - x)$$
$$50x = 750 - 75x$$
$$125x = 750$$
$$x = 6 \text{ feet}$$

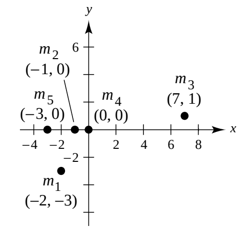
9.
$$\bar{x} = \frac{5(2) + 1(-3) + 3(1)}{5 + 1 + 3} = \frac{10}{9}$$
$$\bar{y} = \frac{5(2) + 1(1) + 3(-4)}{5 + 1 + 3} = -\frac{1}{9}$$
$$(\bar{x}, \bar{y}) = \left(\frac{10}{9}, -\frac{1}{9}\right)$$



11. $\bar{x} = \dfrac{3(-2) + 4(-1) + 2(7) + 1(0) + 6(-3)}{3 + 4 + 2 + 1 + 6} = -\dfrac{7}{8}$

$\bar{y} = \dfrac{3(-3) + 4(0) + 2(1) + 1(0) + 6(0)}{3 + 4 + 2 + 1 + 6} = -\dfrac{7}{16}$

$(\bar{x}, \bar{y}) = \left(-\dfrac{7}{8}, -\dfrac{7}{16}\right)$



13. $m = \rho \displaystyle\int_0^4 \sqrt{x}\, dx = \left[\frac{2\rho}{3}x^{3/2}\right]_0^4 = \frac{16\rho}{3}$

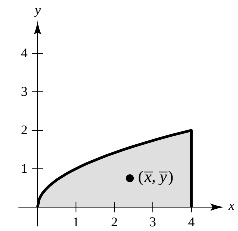
$M_x = \rho \displaystyle\int_0^4 \frac{\sqrt{x}}{2}\left(\sqrt{x}\right) dx = \left[\rho\frac{x^2}{4}\right]_0^4 = 4\rho$

$\bar{y} = \dfrac{M_x}{m} = 4\rho\left(\dfrac{3}{16\rho}\right) = \dfrac{3}{4}$

$M_y = \rho \displaystyle\int_0^4 x\sqrt{x}\, dx = \left[\rho\frac{2}{5}x^{5/2}\right]_0^4 = \frac{64\rho}{5}$

$\bar{x} = \dfrac{M_y}{m} = \dfrac{64\rho}{5}\left(\dfrac{3}{16\rho}\right) = \dfrac{12}{5}$

$(\bar{x}, \bar{y}) = \left(\dfrac{12}{5}, \dfrac{3}{4}\right)$



15. $m = \rho \displaystyle\int_0^1 (x^2 - x^3)\, dx = \rho\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{\rho}{12}$

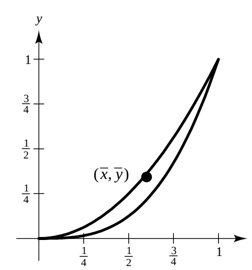
$M_x = \rho \displaystyle\int_0^1 \frac{(x^2 + x^3)}{2}(x^2 - x^3)\, dx = \frac{\rho}{2}\int_0^1 (x^4 - x^6)\, dx = \frac{\rho}{2}\left[\frac{x^5}{5} - \frac{x^7}{7}\right]_0^1 = \frac{\rho}{35}$

$\bar{y} = \dfrac{M_x}{m} = \dfrac{\rho}{35}\left(\dfrac{12}{\rho}\right) = \dfrac{12}{35}$

$M_y = \rho \displaystyle\int_0^1 x(x^2 - x^3)\, dx = \rho\int_0^1 (x^3 - x^4)\, dx = \rho\left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1 = \frac{\rho}{20}$

$\bar{x} = \dfrac{M_y}{m} = \dfrac{\rho}{20}\left(\dfrac{12}{\rho}\right) = \dfrac{3}{5}$

$(\bar{x}, \bar{y}) = \left(\dfrac{3}{5}, \dfrac{12}{35}\right)$



17. $m = \rho \displaystyle\int_0^3 [(-x^2 + 4x + 2) - (x + 2)]\, dx = -\rho\left[\frac{x^3}{3} + \frac{3x^2}{2}\right]_0^3 = \frac{9\rho}{2}$

$M_x = \rho \displaystyle\int_0^3 \left[\frac{(-x^2 + 4x + 2) + (x + 2)}{2}\right][(-x^2 + 4x + 2) - (x + 2)]\, dx$

$= \dfrac{\rho}{2}\displaystyle\int_0^3 (-x^2 + 5x + 4)(-x^2 + 3x)\, dx = \frac{\rho}{2}\int_0^3 (x^4 - 8x^3 + 11x^2 + 12x)\, dx$

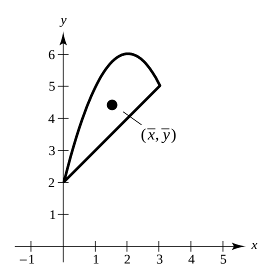
$= \dfrac{\rho}{2}\left[\dfrac{x^5}{5} - 2x^4 + \dfrac{11x^3}{3} + 6x^2\right]_0^3 = \dfrac{99\rho}{5}$

$\bar{y} = \dfrac{M_x}{m} = \dfrac{99\rho}{5}\left(\dfrac{2}{9\rho}\right) = \dfrac{22}{5}$

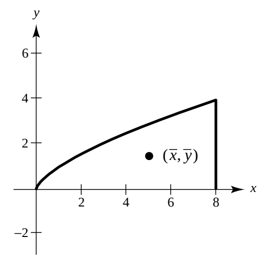
$M_y = \rho \displaystyle\int_0^3 x[(-x^2 + 4x - 2) - (x + 2)]\, dx = \rho\int_0^3 (-x^3 + 3x^2)\, dx = \rho\left[-\frac{x^4}{4} + x^3\right]_0^3 = \frac{27\rho}{4}$

$\bar{x} = \dfrac{M_y}{m} = \dfrac{27\rho}{4}\left(\dfrac{2}{9\rho}\right) = \dfrac{3}{2}$

$(\bar{x}, \bar{y}) = \left(\dfrac{3}{2}, \dfrac{22}{5}\right)$



19. $m = \rho \int_0^8 x^{2/3}\,dx = \rho\left[\frac{3}{5}x^{5/3}\right]_0^8 = \frac{96\rho}{5}$



$M_x = \rho \int_0^8 \frac{x^{2/3}}{2}(x^{2/3})\,dx = \frac{\rho}{2}\left[\frac{3}{7}x^{7/3}\right]_0^8 = \frac{192\rho}{7}$

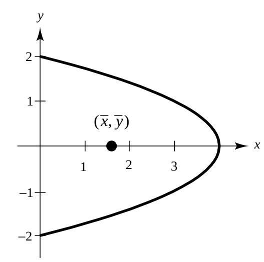
$\bar{y} = \frac{M_x}{m} = \frac{192\rho}{7}\left(\frac{5}{96\rho}\right) = \frac{10}{7}$

$M_y = \rho \int_0^8 x(x^{2/3})\,dx = \rho\left[\frac{3}{8}x^{8/3}\right]_0^8 = 96\rho$

$\bar{x} = \frac{M_y}{m} = 96\rho\left(\frac{5}{96\rho}\right) = 5$

$(\bar{x}, \bar{y}) = \left(5, \frac{10}{7}\right)$

21. $m = 2\rho \int_0^2 (4 - y^2)\,dy = 2\rho\left[4y - \frac{y^3}{3}\right]_0^2 = \frac{32\rho}{3}$



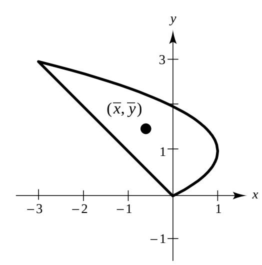
$M_y = 2\rho \int_0^2 \left(\frac{4 - y^2}{2}\right)(4 - y^2)\,dy = \rho\left[16y - \frac{8}{3}y^3 + \frac{y^5}{5}\right]_0^2 = \frac{256\rho}{15}$

$\bar{x} = \frac{M_y}{m} = \frac{256\rho}{15}\left(\frac{3}{32\rho}\right) = \frac{8}{5}$

By symmetry, M_x and $\bar{y} = 0$.

$(\bar{x}, \bar{y}) = \left(\frac{8}{5}, 0\right)$

23. $m = \rho \int_0^3 [(2y - y^2) - (-y)]\,dy = \rho\left[\frac{3y^2}{2} - \frac{y^3}{3}\right]_0^3 = \frac{9\rho}{2}$



$M_y = \rho \int_0^3 \frac{[(2y - y^2) + (-y)]}{2}[(2y - y^2) - (-y)]\,dy = \frac{\rho}{2}\int_0^3 (y - y^2)(3y - y^2)\,dy$

$= \frac{\rho}{2}\int_0^3 (y^4 - 4y^3 + 3y^2)\,dy = \frac{\rho}{2}\left[\frac{y^5}{5} - y^4 + y^3\right]_0^3 = -\frac{27\rho}{10}$

$\bar{x} = \frac{M_y}{m} = -\frac{27\rho}{10}\left(\frac{2}{9\rho}\right) = -\frac{3}{5}$

$M_x = \rho \int_0^3 y[(2y - y^2) - (-y)]\,dy = \rho \int_0^3 (3y^2 - y^3)\,dy = \rho\left[y^3 - \frac{y^4}{4}\right]_0^3 = \frac{27\rho}{4}$

$\bar{y} = \frac{M_x}{m} = \frac{27\rho}{4}\left(\frac{2}{9\rho}\right) = \frac{3}{2}$

$(\bar{x}, \bar{y}) = \left(-\frac{3}{5}, \frac{3}{2}\right)$

25. $A = \int_0^1 (x - x^2)\,dx = \left[\frac{1}{2}x^2 - \frac{x^3}{3}\right]_0^1 = \frac{1}{6}$

$M_x = \frac{1}{2}\int_0^1 (x^2 - x^4)\,dx = \frac{1}{2}\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{1}{15}$

$M_y = \int_0^1 (x^2 - x^3)\,dx = \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{12}$

27. $A = \int_0^3 (2x + 4)\,dx = \Big[x^2 + 4x\Big]_0^3 = 9 + 12 = 21$

$M_x = \frac{1}{2}\int_0^3 (2x + 4)^2\,dx = \int_0^3 (2x^2 + 8x + 8)\,dx = \left[\frac{2x^3}{3} + 4x^2 + 8x\right]_0^3 = 18 + 36 + 24 = 78$

$M_y = \int_0^3 (2x^2 + 4x)\,dx = \left[\frac{2x^3}{3} + 2x^2\right]_0^3 = 18 + 18 = 36$

29. $m = \rho\int_0^5 10x\sqrt{125 - x^3}\,dx \approx 1033.0\rho$

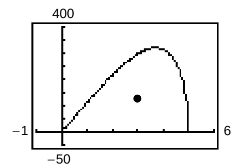
$M_x = \rho\int_0^5 \left(\frac{10x\sqrt{125 - x^3}}{2}\right)\left(10x\sqrt{125 - x^3}\right)dx = 50\rho\int_0^5 x^2(125 - x^3)\,dx = \frac{3{,}124{,}375\rho}{24} \approx 130{,}208\rho$

$M_y = \rho\int_0^5 10x^2\sqrt{125 - x^3}\,dx = -\frac{10\rho}{3}\int_0^5 \sqrt{125 - x^3}(-3x^2)\,dx = \frac{12{,}500\sqrt{5}\rho}{9} \approx 3105.6\rho$

$\bar{x} = \frac{M_y}{m} \approx 3.0$

$\bar{y} = \frac{M_x}{m} \approx 126.0$

Therefore, the centroid is $(3.0, 126.0)$.



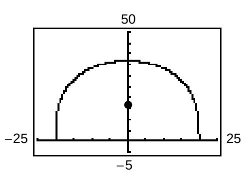
31. $m = \rho\int_{-20}^{20} 5\sqrt[3]{400 - x^2}\,dx \approx 1239.76\rho$

$M_x = \rho\int_{-20}^{20} \frac{5\sqrt[3]{400 - x^2}}{2}\left(5\sqrt[3]{400 - x^2}\right)dx$

$= \frac{25\rho}{2}\int_{-20}^{20} (400 - x^2)^{2/3}\,dx \approx 20064.27$

$\bar{y} = \frac{M_x}{m} \approx 16.18$

$\bar{x} = 0$ by symmetry. Therefore, the centroid is $(0, 16.2)$.



33. $A = \frac{1}{2}(2a)c = ac$

$\frac{1}{A} = \frac{1}{ac}$

$\bar{x} = \left(\frac{1}{ac}\right)\frac{1}{2}\int_0^c \left[\left(\frac{b - a}{c}y + a\right)^2 - \left(\frac{b + a}{c}y - a\right)^2\right]dy$

$= \frac{1}{2ac}\int_0^c \left[\frac{4ab}{c}y - \frac{4ab}{c^2}y^2\right]dy$

$= \frac{1}{2ac}\left[\frac{2ab}{c}y^2 - \frac{4ab}{3c^2}y^3\right]_0^c = \frac{1}{2ac}\left(\frac{2}{3}abc\right) = \frac{b}{3}$

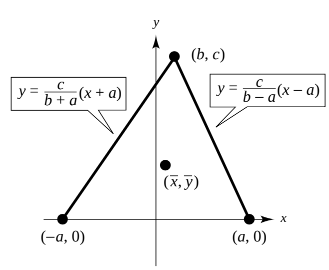
$\bar{y} = \frac{1}{ac}\int_0^c y\left[\left(\frac{b - a}{c}y + a\right) - \left(\frac{b + a}{c}y - a\right)\right]dy$

$= \frac{1}{ac}\int_0^c y\left(-\frac{2a}{c}y + 2a\right)dy = \frac{2}{c}\int_0^c \left(y - \frac{y^2}{c}\right)dy$

$= \frac{2}{c}\left[\frac{y^2}{2} - \frac{y^3}{3c}\right]_0^c = \frac{c}{3}$

$(\bar{x}, \bar{y}) = \left(\frac{b}{3}, \frac{c}{3}\right)$

In Exercise 566 of Section P.2, you found that $(b/3, c/3)$ is the point of intersection of the medians.



35. $A = \frac{c}{2}(a + b)$

$$\frac{1}{A} = \frac{2}{c(a+b)}$$

$$\bar{x} = \frac{2}{c(a+b)}\int_0^c x\left(\frac{b-a}{c}x + a\right)dx = \frac{2}{c(a+b)}\int_0^c \left(\frac{b-a}{c}x^2 + ax\right)dx = \frac{2}{c(a+b)}\left[\frac{b-a}{c}\frac{x^3}{3} + \frac{ax^2}{2}\right]_0^c$$

$$= \frac{2}{c(a+b)}\left[\frac{(b-a)c^2}{3} + \frac{ac^2}{2}\right] = \frac{2}{c(a+b)}\left[\frac{2bc^2 - 2ac^2 + 3ac^2}{6}\right] = \frac{c(2b+a)}{3(a+b)} = \frac{(a+2b)c}{3(a+b)}$$

$$\bar{y} = \frac{2}{c(a+b)}\frac{1}{2}\int_0^c \left(\frac{b-a}{c}x + a\right)^2 dx = \frac{1}{c(a+b)}\int_0^c \left[\left(\frac{b-a}{c}\right)^2 x^2 + \frac{2a(b-a)}{c}x + a^2\right]dx$$

$$= \frac{1}{c(a+b)}\left[\left(\frac{b-a}{c}\right)^2\frac{x^3}{3} + \frac{2a(b-a)}{c}\frac{x^2}{2} + a^2x\right]_0^c = \frac{1}{c(a+b)}\left[\frac{(b-a)^2c}{3} + ac(b-a) + a^2c\right]$$

$$= \frac{1}{3c(a+b)}[(b^2 - 2ab + a^2)c + 3ac(b-a) + 3a^2c]$$

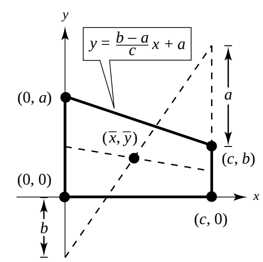
$$= \frac{1}{3(a+b)}[b^2 - 2ab + a^2 + 3ab - 3a^2 + 3a^2] = \frac{a^2 + ab + b^2}{3(a+b)}$$

Thus, $(\bar{x}, \bar{y}) = \left(\frac{(a+2b)c}{3(a+b)}, \frac{a^2+ab+b^2}{3(a+b)}\right)$.

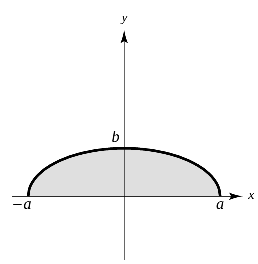
The one line passes through $(0, a/2)$ and $(c, b/2)$. It's equation is $y = \frac{b-a}{2c}x + \frac{a}{2}$.

The other line passes through $(0, -b)$ and $(c, a+b)$. It's equation is $y = \frac{a+2b}{c}x - b$.

$(\bar{x}, \bar{y})$ is the point of intersection of these two lines.



37. $\bar{x} = 0$ by symmetry



$$A = \frac{1}{2}\pi ab$$

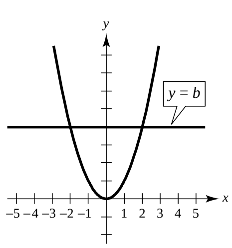
$$\frac{1}{A} = \frac{2}{\pi ab}$$

$$\bar{y} = \frac{2}{\pi ab}\frac{1}{2}\int_{-a}^{a}\left(\frac{b}{a}\sqrt{a^2 - x^2}\right)^2 dx$$

$$= \frac{1}{\pi ab}\left(\frac{b^2}{a^2}\right)\left[a^2x - \frac{x^3}{3}\right]_{-a}^{a} = \frac{b}{\pi a^3}\left[\frac{4a^3}{3}\right] = \frac{4b}{3\pi}$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{4b}{3\pi}\right)$$

39. (a)



(b) $\bar{x} = 0$ by symmetry

(c) $M_y = \int_{-\sqrt{b}}^{\sqrt{b}} x(b - x^2)\,dx = 0$ because $bx - x^3$ is odd

(d) $\bar{y} > \frac{b}{2}$ since there is more area above $y = \frac{b}{2}$ than below

(e) $M_x = \int_{-\sqrt{b}}^{\sqrt{b}} \frac{(b+x^2)(b-x^2)}{2}\,dx$

$$= \int_{-\sqrt{b}}^{\sqrt{b}} \frac{b^2 - x^4}{2}\,dx = \frac{1}{2}\left[b^2x - \frac{x^5}{5}\right]_{-\sqrt{b}}^{\sqrt{b}}$$

$$= b^2\sqrt{b} - \frac{b^2\sqrt{b}}{5} = \frac{4b^2\sqrt{b}}{5}$$

$$A = \int_{-\sqrt{b}}^{\sqrt{b}} (b - x^2)\,dx = \left[bx - \frac{x^3}{3}\right]_{-\sqrt{b}}^{\sqrt{b}}$$

$$= \left(b\sqrt{b} - \frac{b\sqrt{b}}{3}\right)2 = 4\frac{b\sqrt{b}}{3}$$

$$\bar{y} = \frac{M_x}{A} = \frac{4b^2\sqrt{b}/5}{4b\sqrt{b}/3} = \frac{3}{5}b.$$

41. (a) $\bar{x} = 0$ by symmetry

$$A = 2\int_0^{40} f(x)\,dx = \frac{2(40)}{3(4)}[30 + 4(29) + 2(26) + 4(20) + 0] = \frac{20}{3}(278) = \frac{5560}{3}$$

$$M_x = \int_{-40}^{40} \frac{f(x)^2}{2}\,dx = \frac{40}{3(4)}[30^2 + 4(29)^2 + 2(26)^2 + 4(20)^2 + 0] = \frac{10}{3}(7216) = \frac{72160}{3}$$

$$\bar{y} = \frac{M_x}{A} = \frac{72160/3}{5560/3} = \frac{72160}{5560} \approx 12.98$$

$(\bar{x}, \bar{y}) = (0, 12.98)$

(b) $y = (-1.02 \times 10^{-5})x^4 - 0.0019x^2 + 29.28$

(c) $\bar{y} = \dfrac{M_x}{A} \approx \dfrac{23697.68}{1843.54} \approx 12.85$

$(\bar{x}, \bar{y}) = (0, 12.85)$

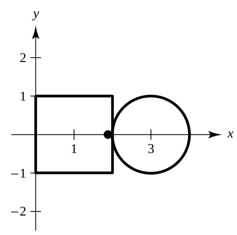
43. Centroids of the given regions: (1, 0) and (3, 0)

Area: $A = 4 + \pi$

$$\bar{x} = \frac{4(1) + \pi(3)}{4 + \pi} = \frac{4 + 3\pi}{4 + \pi}$$

$$\bar{y} = \frac{4(0) + \pi(0)}{4 + \pi} = 0$$

$$(\bar{x}, \bar{y}) = \left(\frac{4 + 3\pi}{4 + \pi}, 0\right) \approx (1.88, 0)$$



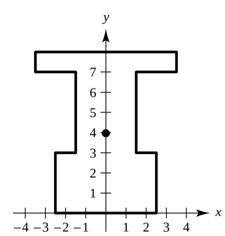
45. Centroids of the given regions: $\left(0, \frac{3}{2}\right)$, (0, 5), and $\left(0, \frac{15}{2}\right)$

Area: $A = 15 + 12 + 7 = 34$

$$\bar{x} = \frac{15(0) + 12(0) + 7(0)}{34} = 0$$

$$\bar{y} = \frac{15(3/2) + 12(5) + 7(15/2)}{34} = \frac{135}{34}$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{135}{34}\right)$$



47. Centroids of the given regions: (1, 0) and (3, 0)

Mass: $4 + 2\pi$

$$\bar{x} = \frac{4(1) + 2\pi(3)}{4 + 2\pi} = \frac{2 + 3\pi}{2 + \pi}$$

$$\bar{y} = 0$$

$$(\bar{x}, \bar{y}) = \left(\frac{2 + 3\pi}{2 + \pi}, 0\right) \approx (2.22, 0)$$

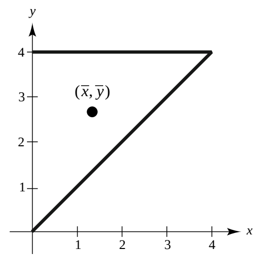
49. $V = 2\pi rA = 2\pi(5)(16\pi) = 160\pi^2 \approx 1579.14$

51. $A = \frac{1}{2}(4)(4) = 8$

$\bar{y} = \left(\frac{1}{8}\right)\frac{1}{2}\int_0^4 (4 + x)(4 - x)\,dx = \frac{1}{16}\left[16x - \frac{x^3}{3}\right]_0^4 = \frac{8}{3}$

$r = \bar{y} = \frac{8}{3}$

$V = 2\pi rA = 2\pi\left(\frac{8}{3}\right)(8) = \frac{128\pi}{3} \approx 134.04$



53. $m = m_1 + \cdots + m_n$

$M_y = m_1x_1 + \cdots + m_nx_n$

$M_x = m_1y_1 + \cdots + m_ny_n$

$\bar{x} = \frac{M_y}{m}, \bar{y} = \frac{M_x}{m}$

55. (a) Yes. $(\bar{x}, \bar{y}) = \left(\frac{5}{6}, \frac{5}{18} + 2\right) = \left(\frac{5}{6}, \frac{41}{18}\right)$

(b) Yes. $(\bar{x}, \bar{y}) = \left(\frac{5}{6} + 2, \frac{5}{18}\right) = \left(\frac{17}{6}, \frac{5}{18}\right)$

(c) Yes. $(\bar{x}, \bar{y}) = \left(\frac{5}{6}, -\frac{5}{18}\right)$

(d) No.

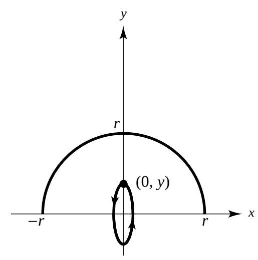
57. The surface area of the sphere is $S = 4\pi r^2$. The arc length of C is $s = \pi r$. The distance traveled by the centroid is

$$d = \frac{S}{s} = \frac{4\pi r^2}{\pi r} = 4r.$$

This distance is also the circumference of the circle of radius y.

$$d = 2\pi y$$

Thus, $2\pi y = 4r$ and we have $y = 2r/\pi$. Therefore, the centroid of the semicircle $y = \sqrt{r^2 - x^2}$ is $(0, 2r/\pi)$.



59. $A = \int_0^1 x^n\,dx = \left[\frac{x^{n+1}}{n+1}\right]_0^1 = \frac{1}{n+1}$

$m = \rho A = \frac{\rho}{n+1}$

$M_x = \frac{\rho}{2}\int_0^1 (x^n)^2\,dx = \left[\frac{\rho}{2}\cdot\frac{x^{2n+1}}{2n+1}\right]_0^1 = \frac{\rho}{2(2n+1)}$

$M_y = \rho\int_0^1 x(x^n)\,dx = \left[\rho\cdot\frac{x^{n+2}}{n+2}\right]_0^1 = \frac{\rho}{n+2}$

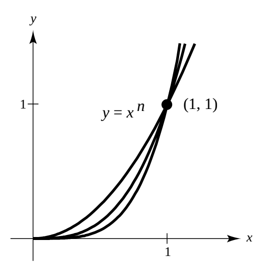
$\bar{x} = \frac{M_y}{m} = \frac{n+1}{n+2}$

$\bar{y} = \frac{M_x}{m} = \frac{n+1}{2(2n+1)} = \frac{n+1}{4n+2}$

Centroid: $\left(\frac{n+1}{n+2}, \frac{n+1}{4n+2}\right)$

As $n \to \infty$, $(\bar{x}, \bar{y}) \to \left(1, \frac{1}{4}\right)$.

The graph approaches the x-axis and the line $x = 1$ as $n \to \infty$.



Section 6.7 Fluid Pressure and Fluid Force

1. $F = PA = [62.4(5)](3) = 936 \text{ lb}$

3. $F = 62.4(h + 2)(6) - (62.4)(h)(6)$

$= 62.4(2)(6) = 748.8 \text{ lb}$

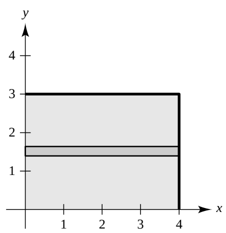
5. $h(y) = 3 - y$

$L(y) = 4$

$F = 62.4 \int_0^3 (3 - y)(4)\, dy$

$= 249.6 \int_0^3 (3 - y)\, dy$

$= 249.6\left[3y - \frac{y^2}{2}\right]_0^3 = 1123.2 \text{ lb}$



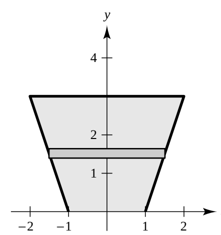
7. $h(y) = 3 - y$

$L(y) = 2\left(\frac{y}{3} + 1\right)$

$F = 2(62.4) \int_0^3 (3 - y)\left(\frac{y}{3} + 1\right) dy$

$= 124.8 \int_0^3 \left(3 - \frac{y^2}{3}\right) dy$

$= 124.8\left[3y - \frac{y^3}{9}\right]_0^3 = 748.8 \text{ lb}$



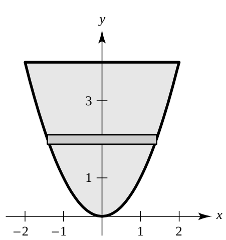
9. $h(y) = 4 - y$

$L(y) = 2\sqrt{y}$

$F = 2(62.4) \int_0^4 (4 - y)\sqrt{y}\, dy$

$= 124.8 \int_0^4 (4y^{1/2} - y^{3/2})\, dy$

$= 124.8\left[\frac{8y^{3/2}}{3} - \frac{2y^{5/2}}{5}\right]_0^4 = 1064.96 \text{ lb}$

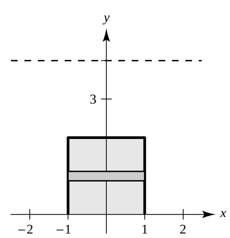


11. $h(y) = 4 - y$

$L(y) = 2$

$F = 9800 \int_0^2 2(4 - y)\, dy$

$= 9800\left[8y - y^2\right]_0^2 = 117{,}600 \text{ Newtons}$

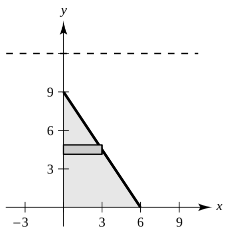


13. $h(y) = 12 - y$

$L(y) = 6 - \frac{2y}{3}$

$F = 9800 \int_0^9 (12 - y)\left(6 - \frac{2y}{3}\right) dy$

$= 9800\left[72y - 7y^2 + \frac{2y^3}{9}\right]_0^9 = 2{,}381{,}400$ Newtons



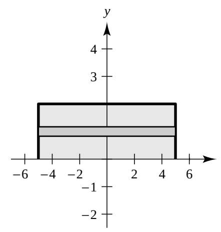
15. $h(y) = 2 - y$

$L(y) = 10$

$F = 140.7 \int_0^2 (2 - y)(10)\, dy$

$= 1407 \int_0^2 (2 - y)\, dy$

$= 1407\left[2y - \frac{y^2}{2}\right]_0^2 = 2814$ lb



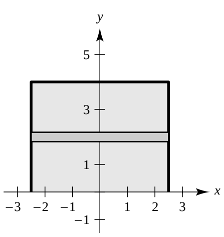
17. $h(y) = 4 - y$

$L(y) = 6$

$F = 140.7 \int_0^4 (4 - y)(6)\, dy$

$= 844.2 \int_0^4 (4 - y)\, dy$

$= 844.2\left[4y - \frac{y^2}{2}\right]_0^4 = 6753.6$ lb



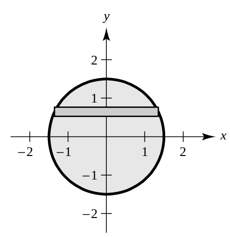
19. $h(y) = -y$

$L(y) = 2\left(\frac{1}{2}\right)\sqrt{9 - 4y^2}$

$F = 42 \int_{-3/2}^{0} (-y)\sqrt{9 - 4y^2}\, dy$

$= \frac{42}{8} \int_{-3/2}^{0} (9 - 4y^2)^{1/2}(-8y)\, dy$

$= \left[\left(\frac{21}{4}\right)\left(\frac{2}{3}\right)(9 - 4y^2)^{3/2}\right]_{-3/2}^{0} = 94.5$ lb

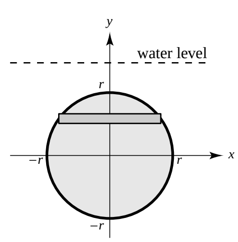


21. $h(y) = k - y$

$L(y) = 2\sqrt{r^2 - y^2}$

$F = w \int_{-r}^{r} (k - y)\sqrt{r^2 - y^2}(2)\, dy$

$= w\left[2k \int_{-r}^{r} \sqrt{r^2 - y^2}\, dy + \int_{-r}^{r} \sqrt{r^2 - y^2}(-2y)\, dy\right]$



The second integral is zero since its integrand is odd and the limits of integration are symmetric to the origin. The first integral is the area of a semicircle with radius r.

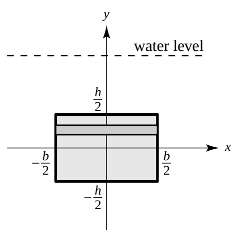
$F = w\left[(2k)\frac{\pi r^2}{2} + 0\right] = wk\pi r^2$

23. $h(y) = k - y$

$L(y) = b$

$$F = w\int_{-h/2}^{h/2} (k - y)b\,dy$$

$$= wb\left[ky - \frac{y^2}{2}\right]_{-h/2}^{h/2} = wb(hk) = wkhb$$



25. From Exercise 22:

$F = 64(15)(1)(1) = 960$ lb

27. $h(y) = 4 - y$

$$F = 62.4\int_0^4 (4 - y)L(y)\,dy$$

Using Simpson's Rule with $n = 8$ we have:

$$F \approx 62.4\left(\frac{4-0}{3(8)}\right)[0 + 4(3.5)(3) + 2(3)(5) + 4(2.5)(8) + 2(2)(9) + 4(1.5)(10) + 2(1)(10.25) + 4(0.5)(10.5) + 0]$$

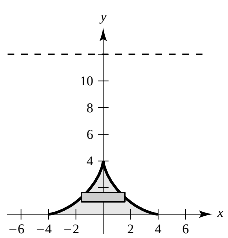
$= 3010.8$ lb

29. $h(y) = 12 - y$

$L(y) = 2(4^{2/3} - y^{2/3})^{3/2}$

$$F = 62.4\int_0^4 2(12 - y)(4^{2/3} - y^{2/3})^{3/2}\,dy$$

≈ 6448.73 lb



31. (a) If the fluid force is one half of 1123.2 lb, and the height of the water is b, then

$h(y) = b - y$

$L(y) = 4$

$$F = 62.4\int_0^b (b - y)(4)\,dy = \frac{1}{2}(1123.2)$$

$$\int_0^b (b - y)\,dy = 2.25$$

$$\left[by - \frac{y^2}{2}\right]_0^b = 2.25$$

$$b^2 - \frac{b^2}{2} = 2.25$$

$$b^2 = 4.5 \implies b \approx 2.12 \text{ ft.}$$

(b) The pressure increases with increasing depth.

33. $F = Fw = w\int_c^d h(y)L(y)\,dy$, see page 471.