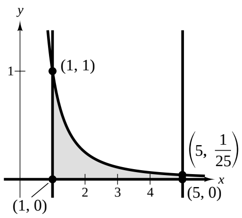
Review Exercises for Chapter 6

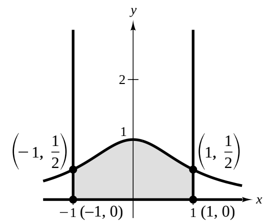
1. $A = \int_1^5 \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^5 = \frac{4}{5}$



3. $A = \int_{-1}^{1} \frac{1}{x^2+1}\,dx$

$= \left[\arctan x\right]_{-1}^{1}$

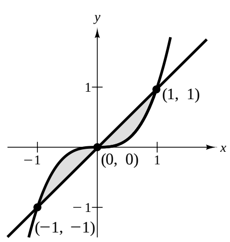
$= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$



5. $A = 2\int_0^1 (x - x^3)\,dx$

$= 2\left[\frac{1}{2}x^2 - \frac{1}{4}x^4\right]_0^1$

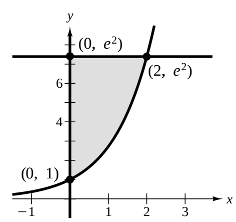
$= \frac{1}{2}$



7. $A = \int_0^2 (e^2 - e^x)\,dx$

$= \left[xe^2 - e^x\right]_0^2$

$= e^2 + 1$



9. $A = \int_{\pi/4}^{5\pi/4} (\sin x - \cos x)\,dx$

$= \left[-\cos x - \sin x\right]_{\pi/4}^{5\pi/4}$

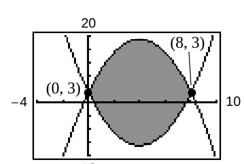
$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)$

$= \frac{4}{\sqrt{2}} = 2\sqrt{2}$

11. $A = \int_0^8 \left[(3 + 8x - x^2) - (x^2 - 8x + 3)\right]dx$

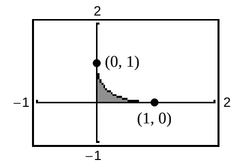
$= \int_0^8 (16x - 2x^2)\,dx$

$= \left[8x^2 - \frac{2}{3}x^3\right]_0^8 = \frac{512}{3} \approx 170.667$



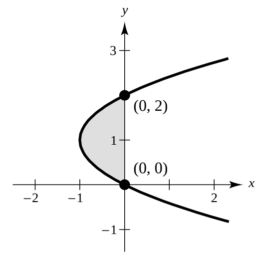
13. $y = (1 - \sqrt{x})^2$

$$A = \int_0^1 (1 - \sqrt{x})^2\,dx$$
$$= \int_0^1 (1 - 2x^{1/2} + x)\,dx$$
$$= \left[x - \frac{4}{3}x^{3/2} + \frac{1}{2}x^2\right]_0^1 = \frac{1}{6} \approx 0.1667$$



15. $x = y^2 - 2y \Rightarrow x + 1 = (y - 1)^2 \Rightarrow y = 1 \pm \sqrt{x + 1}$

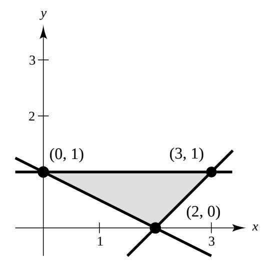
$$A = \int_{-1}^0 \left[\left(1 + \sqrt{x+1}\right) - \left(1 - \sqrt{x+1}\right)\right]dx = \int_{-1}^0 2\sqrt{x+1}\,dx$$
$$A = \int_0^2 [0 - (y^2 - 2y)]\,dy = \int_0^2 (2y - y^2)\,dy = \left[y^2 - \frac{1}{3}y^3\right]_0^2 = \frac{4}{3}$$



17. $A = \int_0^2 \left[1 - \left(1 - \frac{x}{2}\right)\right]dx + \int_2^3 [1 - (x - 2)]\,dx$

$$= \int_0^2 \frac{x}{2}\,dx + \int_2^3 (3 - x)\,dx$$

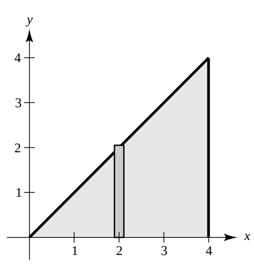
$$y = 1 - \frac{x}{2} \Rightarrow x = 2 - 2y$$
$$y = x - 2 \Rightarrow x = y + 2, y = 1$$
$$A = \int_0^1 [(y + 2) - (2 - 2y)]\,dy$$
$$= \int_0^1 3y\,dy = \left[\frac{3}{2}y^2\right]_0^1 = \frac{3}{2}$$



19. Job 1 is better. The salary for Job 1 is greater than the salary for Job 2 for all the years except the first and 10th years.

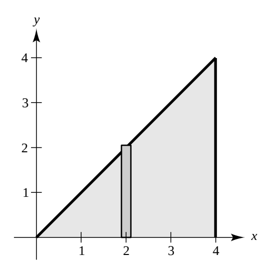
21. (a) **Disk**

$$V = \pi\int_0^4 x^2\,dx = \left[\frac{\pi x^3}{3}\right]_0^4 = \frac{64\pi}{3}$$



(b) **Shell**

$$V = 2\pi\int_0^4 x^2\,dx = \left[\frac{2\pi}{3}x^3\right]_0^4 = \frac{128\pi}{3}$$



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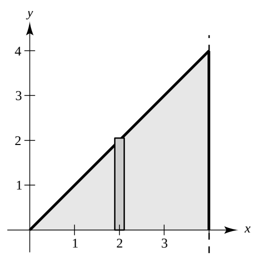
21. —CONTINUED—

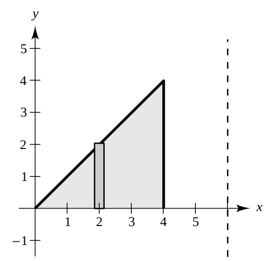
(c) **Shell**

$$V = 2\pi \int_0^4 (4 - x)x\,dx$$
$$= 2\pi \int_0^4 (4x - x^2)\,dx$$
$$= 2\pi \left[2x^2 - \frac{x^3}{3}\right]_0^4 = \frac{64\pi}{3}$$



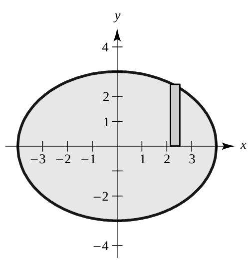
(d) **Shell**

$$V = 2\pi \int_0^4 (6 - x)x\,dx$$
$$= 2\pi \int_0^4 (6x - x^2)\,dx$$
$$= 2\pi \left[3x^2 - \frac{1}{3}x^3\right]_0^4 = \frac{160\pi}{3}$$



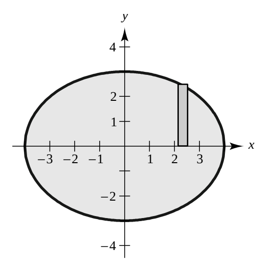
23. (a) **Shell**

$$V = 4\pi \int_0^4 x\left(\frac{3}{4}\right)\sqrt{16 - x^2}\,dx$$
$$= \left[3\pi\left(-\frac{1}{2}\right)\left(\frac{2}{3}\right)(16 - x^2)^{3/2}\right]_0^4 = 64\pi$$



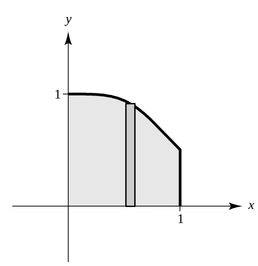
(b) **Disk**

$$V = 2\pi \int_0^4 \left[\frac{3}{4}\sqrt{16 - x^2}\right]^2 dx$$
$$= \frac{9\pi}{8}\left[16x - \frac{x^3}{3}\right]_0^4 = 48\pi$$

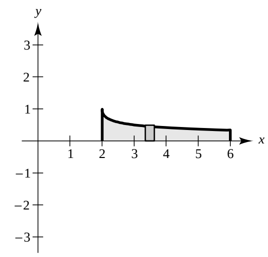


25. **Shell**

$$V = 2\pi \int_0^1 \frac{x}{x^4 + 1}\,dx$$
$$= \pi \int_0^1 \frac{(2x)}{(x^2)^2 + 1}\,dx$$
$$= \left[\pi \arctan(x^2)\right]_0^1$$
$$= \pi\left[\frac{\pi}{4} - 0\right] = \frac{\pi^2}{4}$$



27. Shell



$$u = \sqrt{x-2}$$

$$x = u^2 + 2$$

$$dx = 2u\,du$$

$$V = 2\pi \int_2^6 \frac{x}{1+\sqrt{x-2}}\,dx = 4\pi \int_0^2 \frac{(u^2+2)u}{1+u}\,du$$

$$= 4\pi \int_0^2 \frac{u^3+2u}{1+u}\,du = 4\pi \int_0^2 \left(u^2 - u + 3 - \frac{3}{1+u}\right) du$$

$$= 4\pi \left[\frac{1}{3}u^3 - \frac{1}{2}u^2 + 3u - 3\ln(1+u)\right]_0^2 = \frac{4\pi}{3}(20 - 9\ln 3) \approx 42.359$$

29. Since $y \le 0$, $A = -\int_{-1}^{0} x\sqrt{x+1}\,dx$.

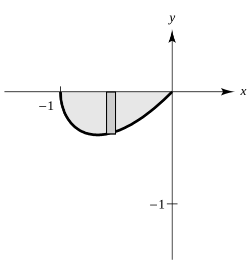
$$u = x + 1$$

$$x = u - 1$$

$$dx = du$$

$$A = -\int_0^1 (u-1)\sqrt{u}\,du = -\int_0^1 (u^{3/2} - u^{1/2})\,du$$

$$= -\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_0^1 = \frac{4}{15}$$



31. From Exercise 23(a) we have: $V = 64\pi \text{ ft}^3$

$$\frac{1}{4}V = 16\pi$$

Disk: $$\pi \int_{-3}^{y_0} \frac{16}{9}(9 - y^2)\,dy = 16\pi$$

$$\frac{1}{9}\int_{-3}^{y_0} (9 - y^2)\,dy = 1$$

$$\left[9y - \frac{1}{3}y^3\right]_{-3}^{y_0} = 9$$

$$\left(9y_0 - \frac{1}{3}y_0^3\right) - (-27 + 9) = 9$$

$$y_0^3 - 27y_0 - 27 = 0$$

By Newton's Method, $y_0 \approx -1.042$ and the depth of the gasoline is $3 - 1.042 = 1.958$ ft.

33. $$f(x) = \frac{4}{5}x^{5/4}$$

$$f'(x) = x^{1/4}$$

$$1 + [f'(x)]^2 = 1 + \sqrt{x}$$

$$u = 1 + \sqrt{x}$$

$$x = (u-1)^2$$

$$dx = 2(u-1)\,du$$

$$s = \int_0^4 \sqrt{1+\sqrt{x}}\,dx = 2\int_1^3 \sqrt{u}(u-1)\,du$$

$$= 2\int_1^3 (u^{3/2} - u^{1/2})\,du$$

$$= 2\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_1^3 = \frac{4}{15}\left[u^{3/2}(3u-5)\right]_1^3$$

$$= \frac{8}{15}\left(1 + 6\sqrt{3}\right) \approx 6.076$$

35. $y = 300\cosh\left(\frac{x}{2000}\right) - 280$, $-2000 \le x \le 2000$

$$y' = \frac{3}{20}\sinh\left(\frac{x}{2000}\right)$$

$$s = \int_{-2000}^{2000} \sqrt{1 + \left[\frac{3}{20}\sinh\left(\frac{x}{2000}\right)\right]^2}\,dx$$

$$= \frac{1}{20}\int_{-2000}^{2000} \sqrt{400 + 9\sinh^2\left(\frac{x}{2000}\right)}\,dx$$

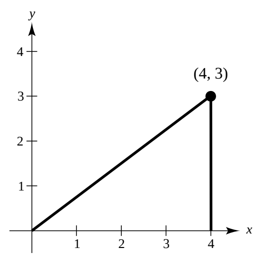
≈ 4018.2 ft (by Simpson's Rule or graphing utility)

37. $y = \frac{3}{4}x$

$$y' = \frac{3}{4}$$

$$1 + (y')^2 = \frac{25}{16}$$

$$S = 2\pi \int_0^4 \left(\frac{3}{4}x\right)\sqrt{\frac{25}{16}}\,dx = \left[\left(\frac{15\pi}{8}\right)\frac{x^2}{2}\right]_0^4 = 15\pi$$



39. $F = kx$

$$4 = k(1)$$

$$F = 4x$$

$$W = \int_0^5 4x\,dx = \left[2x^2\right]_0^5$$

$$= 50 \text{ in} \cdot \text{lb} \approx 4.167 \text{ ft} \cdot \text{lb}$$

41. Volume of disk: $\pi\left(\frac{1}{3}\right)^2 \Delta y$

Weight of disk: $62.4\pi\left(\frac{1}{3}\right)^2 \Delta y$

Distance: $175 - y$

$$W = \frac{62.4\pi}{9}\int_0^{150} (175 - y)\,dy = \frac{62.4\pi}{9}\left[175y - \frac{y^2}{2}\right]_0^{150}$$

$$= 104{,}000\pi \text{ ft} \cdot \text{lb} \approx 163.4 \text{ ft} \cdot \text{ton}$$

43. Weight of section of chain: $5\,\Delta x$

Distance moved: $10 - x$

$$W = 5\int_0^{10} (10 - x)\,dx = \left[-\frac{5}{2}(10 - x)^2\right]_0^{10}$$

$$= 250 \text{ ft} \cdot \text{lb}$$

45. $W = \int_a^b F(x)\,dx$

$$80 = \int_0^4 ax^2 dx = \frac{ax^3}{3}\Big]_0^4 = \frac{64}{3}a$$

$$a = \frac{3(80)}{64} = \frac{15}{4} = 3.75$$

47. $A = \int_0^a \left(\sqrt{a} - \sqrt{x}\right)^2 dx = \int_0^a \left(a - 2\sqrt{a}\,x^{1/2} + x\right) dx = \left[ax - \frac{4}{3}\sqrt{a}\,x^{3/2} + \frac{1}{2}x^2\right]_0^a = \frac{a^2}{6}$

$$\frac{1}{A} = \frac{6}{a^2}$$

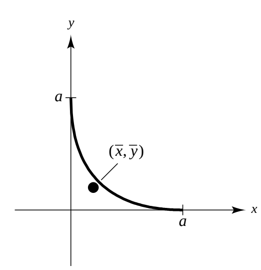
$$\bar{x} = \frac{6}{a^2}\int_0^a x\left(\sqrt{a} - \sqrt{x}\right)^2 dx = \frac{6}{a^2}\int_0^a \left(ax - 2\sqrt{a}\,x^{3/2} + x^2\right) dx$$

$$\bar{y} = \left(\frac{6}{a^2}\right)\frac{1}{2}\int_0^a \left(\sqrt{a} - \sqrt{x}\right)^4 dx$$

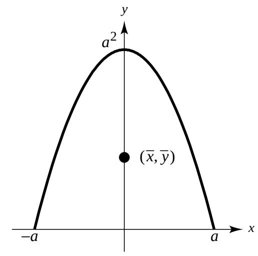
$$= \frac{3}{a^2}\int_0^a \left(a^2 - 4a^{3/2}x^{1/2} + 6ax - 4a^{1/2}x^{3/2} + x^2\right) dx$$

$$= \frac{3}{a^2}\left[a^2x - \frac{8}{3}a^{3/2}x^{3/2} + 3ax^2 - \frac{8}{5}a^{1/2}x^{5/2} + \frac{1}{3}x^3\right]_0^a = \frac{a}{5}$$

$$(\bar{x}, \bar{y}) = \left(\frac{a}{5}, \frac{a}{5}\right)$$



49. By symmetry, $x = 0$.



$$A = 2\int_0^1 (a^2 - x^2)\,dx = 2\left[a^2x - \frac{x^3}{3}\right]_0^a = \frac{4a^3}{3}$$

$$\frac{1}{A} = \frac{3}{4a^3}$$

$$\bar{y} = \left(\frac{3}{4a^3}\right)\frac{1}{2}\int_{-a}^{a} (a^2 - x^2)^2\,dx$$

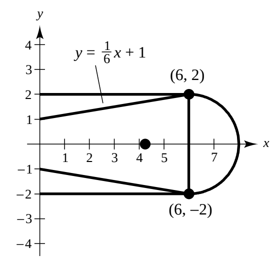
$$= \frac{6}{8a^3}\int_0^a (a^4 - 2a^2x^2 + x^4)\,dx$$

$$= \frac{6}{8a^3}\left[a^4x - \frac{2a^2}{3}x^3 + \frac{1}{5}x^5\right]_0^a$$

$$= \frac{6}{8a^3}\left(a^5 - \frac{2}{3}a^5 + \frac{1}{5}a^5\right) = \frac{2a^2}{5}$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{2a^2}{5}\right)$$

51. $\bar{y} = 0$ by symmetry



For the trapezoid:

$$m = [(4)(6) - (1)(6)]\rho = 18\rho$$

$$M_y = \rho\int_0^6 x\left[\left(\frac{1}{6}x + 1\right) - \left(-\frac{1}{6}x - 1\right)\right]dx$$

$$= \rho\int_0^6 \left(\frac{1}{3}x^2 + 2x\right)dx = \rho\left[\frac{x^3}{9} + x^2\right]_0^6 = 60\rho$$

For the semicircle:

$$m = \left(\frac{1}{2}\right)(\pi)(2)^2\rho = 2\pi\rho$$

$$M_y = \rho\int_6^8 x\left[\sqrt{4 - (x - 6)^2} - \left(-\sqrt{4 - (x - 6)^2}\right)\right]dx = 2\rho\int_6^8 x\sqrt{4 - (x - 6)^2}\,dx$$

Let $u = x - 6$, then $x = u + 6$ and $dx = du$. When $x = 6$, $u = 0$. When $x = 8$, $u = 2$.

$$M_y = 2\rho\int_0^2 (u + 6)\sqrt{4 - u^2}\,du = 2\rho\int_0^2 u\sqrt{4 - u^2}\,du + 12\rho\int_0^2 \sqrt{4 - u^2}\,du$$

$$= 2\rho\left[\left(-\frac{1}{2}\right)\left(\frac{2}{3}\right)(4 - u^2)^{3/2}\right]_0^2 + 12\rho\left[\frac{\pi(2)^2}{4}\right] = \frac{16\rho}{3} + 12\pi\rho = \frac{4\rho(4 + 9\pi)}{3}$$

Thus, we have:

$$\bar{x}(18\rho + 2\pi\rho) = 60\rho + \frac{4\rho(4 + 9\pi)}{3}$$

$$\bar{x} = \frac{180\rho + 4\rho(4 + 9\pi)}{3}\cdot\frac{1}{2\rho(9 + \pi)} = \frac{2(9\pi + 49)}{3(\pi + 9)}$$

The centroid of the blade is $\left(\dfrac{2(9\pi + 49)}{3(\pi + 9)}, 0\right)$.

53. Let D = surface of liquid; ρ = weight per cubic volume.

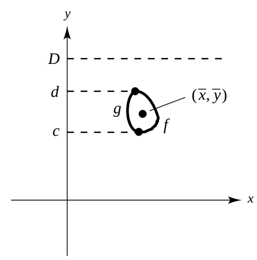
$$F = \rho \int_c^d (D - y)[f(y) - g(y)]\,dy$$

$$= \rho\left[\int_c^d D[f(y) - g(y)]\,dy - \int_c^d y[f(y) - g(y)]\,dy\right]$$

$$= \rho\left[\int_c^d [f(y) - g(y)]\,dy\right]\left[D - \frac{\int_c^d y[f(y) - g(y)]\,dy}{\int_c^d [f(y) - g(y)]\,dy}\right]$$

$$= \rho(\text{Area})(D - \bar{y})$$

$$= \rho(\text{Area})(\text{depth of centroid})$$



Problem Solving for Chapter 6

1. $T = \frac{1}{2}c(c^2) = \frac{1}{2}c^3$

$$R = \int_0^c (cx - x^2)\,dx = \left[\frac{cx^2}{2} - \frac{x^3}{3}\right]_0^c = \frac{c^3}{2} - \frac{c^3}{3} = \frac{c^3}{6}$$

$$\lim_{c\to 0^+} \frac{T}{R} = \lim_{c\to 0^+} \frac{\frac{1}{2}c^3}{\frac{1}{6}c^3} = 3$$

3. (a) $\frac{1}{2}V = \int_0^1 \left[\pi\left(2 + \sqrt{1 - y^2}\right)^2 - \pi\left(2 - \sqrt{1 - y^2}\right)^2\right] dy$

$$= \pi\int_0^1 \left[\left(4 + 4\sqrt{1 - y^2} + (1 - y^2)\right) - \left(4 - 4\sqrt{1 - y^2} + (1 - y^2)\right)\right] dy$$

$$= 8\pi\int_0^1 \sqrt{1 - y^2}\,dy \quad \text{(Integral represents 1/4 (area of circle))}$$

$$= 8\pi\left(\frac{\pi}{4}\right) = 2\pi^2 \implies V = 4\pi^2$$

(b) $(x - R)^2 + y^2 = r^2 \implies x = R \pm \sqrt{r^2 - y^2}$

$$\frac{1}{2}V = \int_0^r \left[\pi\left(R + \sqrt{r^2 - y^2}\right)^2 - \pi\left(R - \sqrt{r^2 - y^2}\right)^2\right] dy$$

$$= \pi\int_0^r 4R\sqrt{r^2 - y^2}\,dy$$

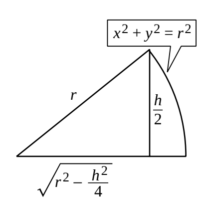
$$= \pi(4R)\frac{1}{4}\pi r^2 - \pi^2 r^2 R$$

$$V = 2\pi^2 r^2 R$$

5. $V = 2(2\pi)\int_{\sqrt{r^2 - (h^2/4)}}^r x\sqrt{r^2 - x^2}\,dx$

$$= -2\pi\left[\frac{2}{3}(r^2 - x^2)^{3/2}\right]_{\sqrt{r^2 - (h^2/4)}}^r$$

$$= \frac{-4\pi}{3}\left[-\frac{h^3}{8}\right] = \frac{\pi h^3}{6} \text{ which does not depend on } r!$$



7. (a) Tangent at A: $y = x^3, y' = 3x^2$

$$y - 1 = 3(x - 1)$$
$$y = 3x - 2$$

To find point B: $x^3 = 3x - 2$

$$x^3 - 3x + 2 = 0$$
$$(x-1)^2(x+2) = 0 \implies B = (-2, -8)$$

Tangent at B: $y = x^3, y' = 3x^2$

$$y + 8 = 12(x + 2)$$
$$y = 12x + 16$$

To find point C: $x^3 = 12x + 16$

$$x^3 - 12x - 16 = 0$$
$$(x+2)^2(x-4) = 0 \implies C = (4, 64)$$

Area of $R = \int_{-2}^{1} (x^3 - 3x + 2)\,dx = \frac{27}{4}$

Area of $S = \int_{-2}^{4} (12x + 16 - x^3)\,dx = 108$

Area of $S = 16$(area of R) $\left[\frac{\text{area } S}{\text{area } R} = 16\right]$

(b) Tangent at $A(a, a^3)$: $y - a^3 = 3a^2(x - a)$

$$y = 3a^2x - 2a^3$$

To find point B: $x^3 - 3a^2x + 2a^3 = 0$

$$(x-a)^2(x+2a) = 0 \implies$$
$$B = (-2a, -8a^3)$$

Tangent at B: $y + 8a^3 = 12a^2(x + 2a)$

$$y = 12a^2x + 16a^3$$

To find point C: $x^3 - 12a^2x - 16a^3 = 0$

$$(x+2a)^2(x-4a) = 0 \implies$$
$$C = (4a, 64a^3)$$

Area of $R = \int_{-2a}^{a} \left[x^3 - 3a^2x + 2a^3\right] dx = \frac{27}{4}a^4$

Area of $S = \int_{-2a}^{4a} \left[12a^2x + 16a^3 - x^3\right] dx = 108a^4$

Area of $S = 16$(area of R)

9. $s(x) = \int_{\alpha}^{x} \sqrt{1 + f'(t)^2}\,dt$

(a) $s'(x) = \frac{ds}{dx} = \sqrt{1 + f'(x)^2}$

(b) $ds = \sqrt{1 + f'(x)^2}\,dx$

$$(ds)^2 = [1 + f'(x)^2](dx)^2 = \left[1 + \left(\frac{dy}{dx}\right)^2\right](dx)^2 = (dx)^2 + (dy)^2$$

(c) $s(x) = \int_1^x \sqrt{1 + \left(\frac{3}{2}t^{1/2}\right)^2}\,dt = \int_1^x \sqrt{1 + \frac{9}{4}t}\,dt$

(d) $s(2) = \int_1^2 \sqrt{1 + \frac{9}{4}t}\,dt = \left[\frac{8}{27}\left(1 + \frac{9}{4}t\right)^{3/2}\right]_1^2 = \frac{22}{27}\sqrt{22} - \frac{13}{27}\sqrt{13} \approx 2.0858$

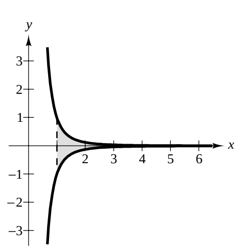
This is the length of the curve $y = x^{3/2}$ from $x = 1$ to $x = 2$.

11. (a) $\bar{y} = 0$ by symmetry

$$M_y = \int_1^6 x\left(\frac{1}{x^3} - \left(-\frac{1}{x^3}\right)\right) dx = \int_1^6 \frac{2}{x^2}\,dx = \left[-2\frac{1}{x}\right]_1^6 = \frac{5}{3}$$

$$m = 2\int_1^6 \frac{1}{x^3}\,dx = \left[-\frac{1}{x^2}\right]_1^6 = \frac{35}{36}$$

$$\bar{x} = \frac{5/3}{35/36} = \frac{12}{7} \qquad (\bar{x}, \bar{y}) = \left(\frac{12}{7}, 0\right)$$



(b) $m = 2\int_1^b \frac{1}{x^3}\,dx = \frac{b^2 - 1}{b^2}$

$$M_y = 2\int_1^6 \frac{1}{x^2}\,dx = \frac{2(b-1)}{b}$$

$$\bar{x} = \frac{2(b-1)/b}{(b^2-1)/b^2} = \frac{2b}{b+1} \qquad (\bar{x}, \bar{y}) = \left(\frac{2b}{b+1}, 0\right)$$

(c) $\lim_{b\to\infty} \bar{x} = \lim_{b\to\infty} \frac{2b}{b+1} = 2 \qquad (\bar{x}, \bar{y}) = (2, 0)$

13. (a) $W = \text{area} = 2 + 4 + 6 = 12$

(b) $W = \text{area} = 3 + (1 + 1) + 2 + \frac{1}{2} = 7\frac{1}{2}$

15. Point of equilibrium: $50 - 0.5x = 0.125x$

$$x = 80,\ p = 10$$

$(P_0, x_0) = (10, 80)$

$$\text{Consumer surplus} = \int_0^{80} [(50 - 0.5x) - 10]\, dx = 1600$$

$$\text{Producer surplus} = \int_0^{80} [10 - 0.125x]\, dx = 400$$

17. (a) Wall at shallow end

From Exercise 22: $F = 62.4(2)(4)(20) = 9984$ lb

(b) Wall at deep end

From Exercise 22: $F = 62.4(4)(8)(20) = 39{,}936$ lb

(c) Side wall

From Exercise 22: $F_1 = 62.4(2)(4)(40) = 19{,}968$ lb

$$F_2 = 62.4 \int_0^4 (8 - y)(10y)\, dy$$

$$= 624 \int_0^4 (8y - y^2)\, dy = 624\left[4y^2 - \frac{y^3}{3}\right]_0^4$$

$$= 26{,}624 \text{ lb}$$

Total force: $F_1 + F_2 = 46{,}592$ lb