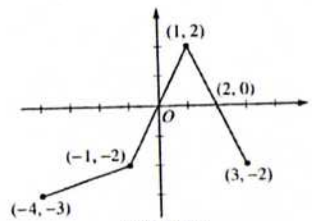
Graphical Analysis / FTC
SA Answers

AP® CALCULUS BC
2005 SCORING GUIDELINES (Form B)

Question 4



Graph of f

The graph of the function f above consists of three line segments.

(a) Let g be the function given by $g(x) = \int_{-4}^{x} f(t)\,dt$.
For each of $g(-1)$, $g'(-1)$, and $g''(-1)$, find the value or state that it does not exist.

(b) For the function g defined in part (a), find the x-coordinate of each point of inflection of the graph of g on the open interval $-4 < x < 3$. Explain your reasoning.

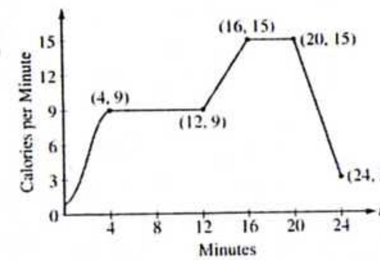
(c) Let h be the function given by $h(x) = \int_{x}^{3} f(t)\,dt$. Find all values of x in the closed interval $-4 \le x \le 3$ for which $h(x) = 0$.

(d) For the function h defined in part (c), find all intervals on which h is decreasing. Explain your reasoning.

(a) $g(-1) = \int_{-4}^{-1} f(t)\,dt = -\frac{1}{2}(3)(5) = -\frac{15}{2}$ $g'(-1) = f(-1) = -2$ $g''(-1)$ does not exist because f is not differentiable at $x = -1$.	$3: \begin{cases} 1: g(-1) \\ 1: g'(-1) \\ 1: g''(-1) \end{cases}$
(b) $x = 1$ $g' = f$ changes from increasing to decreasing at $x = 1$.	$2: \begin{cases} 1: x = 1 \text{ (only)} \\ 1: \text{reason} \end{cases}$
(c) $x = -1, 1, 3$	2 : correct values $\langle -1 \rangle$ each missing or extra value
(d) h is decreasing on $[0, 2]$ $h' = -f < 0$ when $f > 0$	$2: \begin{cases} 1: \text{interval} \\ 1: \text{reason} \end{cases}$

AP® CALCULUS BC
2006 SCORING GUIDELINES (Form B)

Question 4



The rate, in calories per minute, at which a person using an exercise machine burns calories is modeled by the function f. In the figure above, $f(t) = -\frac{1}{4}t^3 + \frac{3}{2}t^2 + 1$ for $0 \le t \le 4$ and f is piecewise linear for $4 \le t \le 24$.

(a) Find $f'(22)$. Indicate units of measure.

(b) For the time interval $0 \le t \le 24$, at what time t is f increasing at its greatest rate? Show the reasoning that supports your answer.

(c) Find the total number of calories burned over the time interval $6 \le t \le 18$ minutes.

(d) The setting on the machine is now changed so that the person burns $f(t) + c$ calories per minute. For this setting, find c so that an average of 15 calories per minute is burned during the time interval $6 \le t \le 18$.

(a) $f'(22) = \frac{15-3}{20-24} = -3$ calories/min/min	1 : $f'(22)$ and units
(b) f is increasing on $[0, 4]$ and on $[12, 16]$. On $(12, 16)$, $f'(t) = \frac{15-9}{16-12} = \frac{3}{2}$ since f has constant slope on this interval. On $(0, 4)$, $f'(t) = -\frac{3}{4}t^2 + 3t$ and $f''(t) = -\frac{3}{2}t + 3 = 0$ when $t = 2$. This is where f' has a maximum on $[0, 4]$ since $f'' > 0$ on $(0, 2)$ and $f'' < 0$ on $(2, 4)$. On $[0, 24]$, f is increasing at its greatest rate when $t = 2$ because $f'(2) = 3 > \frac{3}{2}$.	$4: \begin{cases} 1: f' \text{ on } (0, 4) \\ 1: \text{shows } f' \text{ has a max at } t = 2 \text{ on } (0, 4) \\ 1: \text{shows for } 12 < t < 16,\ f'(t) < f'(2) \\ 1: \text{answer} \end{cases}$
(c) $\int_{6}^{18} f(t)\,dt = 6(9) + \frac{1}{2}(4)(9 + 15) + 2(15)$ $= 132$ calories	$2: \begin{cases} 1: \text{method} \\ 1: \text{answer} \end{cases}$
(d) We want $\frac{1}{12}\int_{6}^{18} (f(t) + c)\,dt = 15$. This means $132 + 12c = 15(12)$. So, $c = 4$. OR Currently, the average is $\frac{132}{12} = 11$ calories/min. Adding c to $f(t)$ will shift the average by c. So $c = 4$ to get an average of 15 calories/min.	$2: \begin{cases} 1: \text{setup} \\ 1: \text{value of } c \end{cases}$

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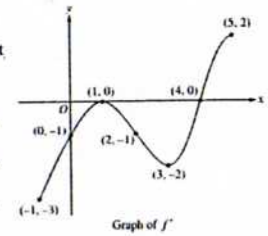
2004 SCORING GUIDELINES (Form B)

Question 4



Graph of f'

The figure above shows the graph of f', the derivative of the function f, on the closed interval $-1 \le x \le 5$. The graph of f' has horizontal tangent lines at $x = 1$ and $x = 3$. The function f is twice differentiable with $f(2) = 6$.

(a) Find the x-coordinate of each of the points of inflection of the graph of f. Give a reason for your answer.

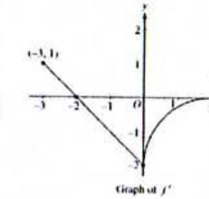
(b) At what value of x does f attain its absolute minimum value on the closed interval $-1 \le x \le 5$? At what value of x does f attain its absolute maximum value on the closed interval $-1 \le x \le 5$? Show the analysis that leads to your answers.

(c) Let g be the function defined by $g(x) = xf(x)$. Find an equation for the line tangent to the graph of g at $x = 2$.

(a) $x = 1$ and $x = 3$ because the graph of f' changes from increasing to decreasing at $x = 1$, and changes from decreasing to increasing at $x = 3$.	$2: \begin{cases} 1: x = 1, x = 3 \\ 1: \text{reason} \end{cases}$
(b) The function f decreases from $x = -1$ to $x = 4$, then increases from $x = 4$ to $x = 5$. Therefore, the absolute minimum value for f is at $x = 4$. The absolute maximum value must occur at $x = -1$ or at $x = 5$. $f(5) - f(-1) = \int_{-1}^{5} f'(t)\,dt < 0$ Since $f(5) < f(-1)$, the absolute maximum value occurs at $x = -1$.	$4: \begin{cases} 1: \text{indicates } f \text{ decreases then increases} \\ 1: \text{eliminates } x = 5 \text{ for maximum} \\ 1: \text{absolute minimum at } x = 4 \\ 1: \text{absolute maximum at } x = -1 \end{cases}$
(c) $g'(x) = f(x) + xf'(x)$ $g'(2) = f(2) + 2f'(2) = 6 + 2(-1) = 4$ $g(2) = 2f(2) = 12$ Tangent line is $y = 4(x-2) + 12$	$3: \begin{cases} 2: g'(x) \\ 1: \text{tangent line} \end{cases}$

AP® CALCULUS BC
2003 SCORING GUIDELINES

Question 4



Graph of f'

Let f be a function defined on the closed interval $-3 \le x \le 4$ with $f(0) = 3$. The graph of f', the derivative of f, consists of one line segment and a semicircle, as shown above.

(a) On what intervals, if any, is f increasing? Justify your answer.

(b) Find the x-coordinate of each point of inflection of the graph of f on the open interval $-3 < x < 4$. Justify your answer.

(c) Find an equation for the line tangent to the graph of f at the point $(0, 3)$.

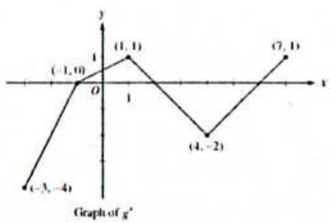
(d) Find $f(-3)$ and $f(4)$. Show the work that leads to your answers.

(a) The function f is increasing on $[-3, -2]$ since $f' > 0$ for $-3 \le x < -2$.	$2: \begin{cases} 1: \text{interval} \\ 1: \text{reason} \end{cases}$
(b) $x = 0$ and $x = 2$ f' changes from decreasing to increasing at $x = 0$ and from increasing to decreasing at $x = 2$	$2: \begin{cases} 1: x = 0 \text{ and } x = 2 \text{ only} \\ 1: \text{justification} \end{cases}$
(c) $f'(0) = -2$ Tangent line is $y = -2x + 3$.	1 : equation
(d) $f(0) - f(-3) = \int_{-3}^{0} f'(t)\,dt = \frac{1}{2}(1)(1) - \frac{1}{2}(2)(2) = -\frac{3}{2}$ $f(-3) = f(0) + \frac{3}{2} = \frac{9}{2}$ $f(4) - f(0) = \int_{0}^{4} f'(t)\,dt = -\left(8 - \frac{1}{2}(2)^2\pi\right) = -8 + 2\pi$ $f(4) = f(0) - 8 + 2\pi = -5 + 2\pi$	4 : 1 : $\pm\left(\frac{1}{2} - 2\right)$ (difference of areas of triangles) 1 : answer for $f(-3)$ using FTC 1 : $\pm\left(8 - \frac{1}{2}(2)^2\pi\right)$ (area of rectangle − area of semicircle) 1 : answer for $f(4)$ using FTC

AP® CALCULUS BC
2008 SCORING GUIDELINES (Form B)

Question 5

t g be a continuous function with $g(2) = 5$. The graph the piecewise-linear function g', the derivative of g, is own above for $-3 \le x \le 7$.



Graph of g'

Find the x-coordinate of all points of inflection of the graph of $y = g(x)$ for $-3 < x < 7$. Justify your answer.

Find the absolute maximum value of g on the interval $-3 \le x \le 7$. Justify your answer.

Find the average rate of change of $g(x)$ on the interval $-3 \le x \le 7$.

Find the average rate of change of $g'(x)$ on the interval $-3 \le x \le 7$. Does the Mean Value Theorem applied on the interval $-3 \le x \le 7$ guarantee a value of c, for $-3 < c < 7$, such that $g''(c)$ is equal to this average rate of change? Why or why not?

g' changes from increasing to decreasing at $x = 1$; g' changes from decreasing to increasing at $x = 4$. Points of inflection for the graph of $y = g(x)$ occur at $x = 1$ and $x = 4$.	2 : { 1 : x-values 1 : justification
The only sign change of g' from positive to negative in the interval is at $x = 2$. $g(-3) = 5 + \int_2^{-3} g'(x)\,dx = 5 + \left(-\frac{3}{2}\right) + 4 = \frac{15}{2}$ $g(2) = 5$ $g(7) = 5 + \int_2^7 g'(x)\,dx = 5 + (-4) + \frac{1}{2} = \frac{3}{2}$ The maximum value of g for $-3 \le x \le 7$ is $\frac{15}{2}$.	3 : { 1 : identifies $x = 2$ as a candidate 1 : considers endpoints 1 : maximum value and justification
$\frac{g(7) - g(-3)}{7 - (-3)} = \frac{\frac{3}{2} - \frac{15}{2}}{10} = -\frac{3}{5}$	2 : { 1 : difference quotient 1 : answer
$\frac{g'(7) - g'(-3)}{7 - (-3)} = \frac{1 - (-4)}{10} = \frac{1}{2}$ No, the MVT does *not* guarantee the existence of a value c with the stated properties because g' is not differentiable for at least one point in $-3 < x < 7$.	2 : { 1 : average value of $g'(x)$ 1 : answer "No" with reason

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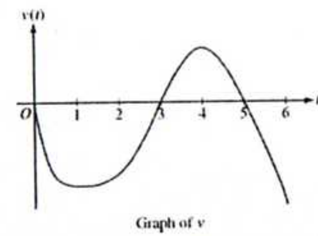
2008 SCORING GUIDELINES

Question 4



Graph of v

A particle moves along the x-axis so that its velocity at time t, for $0 \le t \le 6$, is given by a differentiable function v whose graph is shown above. The velocity is 0 at $t = 0$, $t = 3$, and $t = 5$, and the graph has horizontal tangents at $t = 1$ and $t = 4$. The areas of the regions bounded by the t-axis and the graph of v on the intervals $[0, 3]$, $[3, 5]$, and $[5, 6]$ are 8, 3, and 2, respectively. At time $t = 0$, the particle is at $x = -2$.

(a) For $0 \le t \le 6$, find both the time and the position of the particle when the particle is farthest to the left. Justify your answer.

(b) For how many values of t, where $0 \le t \le 6$, is the particle at $x = -8$? Explain your reasoning.

(c) On the interval $2 < t < 3$, is the speed of the particle increasing or decreasing? Give a reason for your answer.

(d) During what time intervals, if any, is the acceleration of the particle negative? Justify your answer.

(a) Since $v(t) < 0$ for $0 < t < 3$ and $5 < t < 6$, and $v(t) > 0$ for $3 < t < 5$, we consider $t = 3$ and $t = 6$. $x(3) = -2 + \int_0^3 v(t)\,dt = -2 - 8 = -10$ $x(6) = -2 + \int_0^6 v(t)\,dt = -2 - 8 + 3 - 2 = -9$ Therefore, the particle is farthest left at time $t = 3$ when its position is $x(3) = -10$.	3 : { 1 : identifies $t = 3$ as a candidate 1 : considers $\int_0^6 v(t)\,dt$ 1 : conclusion
(b) The particle moves continuously and monotonically from $x(0) = -2$ to $x(3) = -10$. Similarly, the particle moves continuously and monotonically from $x(3) = -10$ to $x(5) = -7$ and also from $x(5) = -7$ to $x(6) = -9$. By the Intermediate Value Theorem, there are three values of t for which the particle is at $x(t) = -8$.	3 : { 1 : positions at $t = 3$, $t = 5$, and $t = 6$ 1 : description of motion 1 : conclusion
(c) The speed is decreasing on the interval $2 < t < 3$ since on this interval $v < 0$ and v is increasing.	1 : answer with reason
(d) The acceleration is negative on the intervals $0 < t < 1$ and $4 < t < 6$ since velocity is decreasing on these intervals.	2 : { 1 : answer 1 : justification

Question 5

The derivative of a function f is given by $f'(x) = (x-3)e^x$ for $x > 0$, and $f(1) = 7$.

(a) The function f has a critical point at $x = 3$. At this point, does f have a relative minimum, a relative maximum, or neither? Justify your answer.

(b) On what intervals, if any, is the graph of f both decreasing and concave up? Explain your reasoning.

(c) Find the value of $f(3)$.

(a) $f'(x) < 0$ for $0 < x < 3$ and $f'(x) > 0$ for $x > 3$

Therefore, f has a relative minimum at $x = 3$.

$2 : \begin{cases} 1: \text{minimum at } x = 3 \\ 1: \text{justification} \end{cases}$

(b) $f''(x) = e^x + (x-3)e^x = (x-2)e^x$

$f''(x) > 0$ for $x > 2$

$f'(x) < 0$ for $0 < x < 3$

Therefore, the graph of f is both decreasing and concave up on the interval $2 < x < 3$.

$3 : \begin{cases} 2 : f''(x) \\ 1 : \text{answer with reason} \end{cases}$

(c) $f(3) = f(1) + \int_1^3 f'(x)\,dx = 7 + \int_1^3 (x-3)e^x\,dx$

$u = x - 3 \quad dv = e^x\,dx$

$du = dx \quad v = e^x$

$f(3) = 7 + (x-3)e^x\Big|_1^3 - \int_1^3 e^x\,dx$

$= 7 + \left((x-3)e^x - e^x\right)\Big|_1^3$

$= 7 + 3e - e^3$

$4 : \begin{cases} 1 : \text{uses initial condition} \\ 2 : \text{integration by parts} \\ 1 : \text{answer} \end{cases}$

Question 4

Let f be the function defined for $x > 0$, with $f(e) = 2$ and f', the first derivative of f, given by $f'(x) = x^2$

(a) Write an equation for the line tangent to the graph of f at the point $(e, 2)$.

(b) Is the graph of f concave up or concave down on the interval $1 < x < 3$? Give a reason for your answer.

(c) Use antidifferentiation to find $f(x)$.

(a) $f'(e) = e^2$

An equation for the line tangent to the graph of f at the point $(e, 2)$ is $y - 2 = e^2(x - e)$.

$2 : \begin{cases} 1 : f'(e) \\ 1 : \text{equation of tangent line} \end{cases}$

(b) $f''(x) = x + 2x\ln x$.

For $1 < x < 3$, $x > 0$ and $\ln x > 0$, so $f''(x) > 0$. Thus, the graph of f is concave up on $(1, 3)$.

$3 : \begin{cases} 2 : f''(x) \\ 1 : \text{answer with reason} \end{cases}$

(c) Since $f(x) = \int (x^2 \ln x)\,dx$, we consider integration by parts.

$u = \ln x \quad dv = x^2\,dx$

$du = \frac{1}{x}\,dx \quad v = \int (x^2)\,dx = \frac{1}{3}x^3$

Therefore,

$f(x) = \int (x^2 \ln x)\,dx$

$= \frac{1}{3}x^3 \ln x - \int \left(\frac{1}{3}x^3 \cdot \frac{1}{x}\right) dx$

$= \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C.$

Since $f(e) = 2$, $2 = \frac{e^3}{3} - \frac{e^3}{9} + C$ and $C = 2 - \frac{2}{9}e^3$.

Thus, $f(x) = \frac{x^3}{3}\ln x - \frac{1}{9}x^3 + 2 - \frac{2}{9}e^3$.

$4 : \begin{cases} 2 : \text{antiderivative} \\ 1 : \text{uses } f(e) = 2 \\ 1 : \text{answer} \end{cases}$

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2007 SCORING GUIDELINES

Question 2

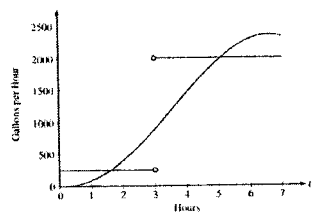
The amount of water in a storage tank, in gallons, is modeled by a continuous function on the time interval $0 \le t \le 7$, where t is measured in hours. In this model, rates are given as follows:

(i) The rate at which water enters the tank is

$f(t) = 100t^2 \sin(\sqrt{t})$ gallons per hour for $0 \le t \le 7$.

(ii) The rate at which water leaves the tank is

$g(t) = \begin{cases} 250 & \text{for } 0 \le t < 3 \\ 2000 & \text{for } 3 < t \le 7 \end{cases}$ gallons per hour.



The graphs of f and g, which intersect at $t = 1.617$ and $t = 5.076$, are shown in the figure above. At time $t = 0$, the amount of water in the tank is 5000 gallons.

(a) How many gallons of water enter the tank during the time interval $0 \le t \le 7$? Round your answer to the nearest gallon.

(b) For $0 \le t \le 7$, find the time intervals during which the amount of water in the tank is decreasing. Give a reason for each answer.

(c) For $0 \le t \le 7$, at what time t is the amount of water in the tank greatest? To the nearest gallon, compute the amount of water at this time. Justify your answer.

(a) $\int_0^7 f(t)\,dt \approx 8264$ gallons

$2: \begin{cases} 1: \text{integral} \\ 1: \text{answer} \end{cases}$

(b) The amount of water in the tank is decreasing on the intervals $0 \le t \le 1.617$ and $3 \le t \le 5.076$ because $f(t) < g(t)$ for $0 \le t < 1.617$ and $3 < t < 5.076$.

$2: \begin{cases} 1: \text{intervals} \\ 1: \text{reason} \end{cases}$

(c) Since $f(t) - g(t)$ changes sign from positive to negative only at $t = 3$, the candidates for the absolute maximum are at $t = 0,\ 3,$ and 7.

t (hours)	gallons of water
0	5000
3	$5000 + \int_0^3 f(t)\,dt - 250(3) = 5126.591$
7	$5126.591 + \int_3^7 f(t)\,dt - 2000(4) = 4513.807$

The amount of water in the tank is greatest at 3 hours. At that time, the amount of water in the tank, rounded to the nearest gallon, is 5127 gallons.

$5: \begin{cases} 1: \text{identifies } t = 3 \text{ as a candidate} \\ 1: \text{integrand} \\ 1: \text{amount of water at } t = 3 \\ 1: \text{amount of water at } t = 7 \\ 1: \text{conclusion} \end{cases}$