Parametric

(x, y) defined in terms of independent variable (parameter) like time (t) or angle (θ).

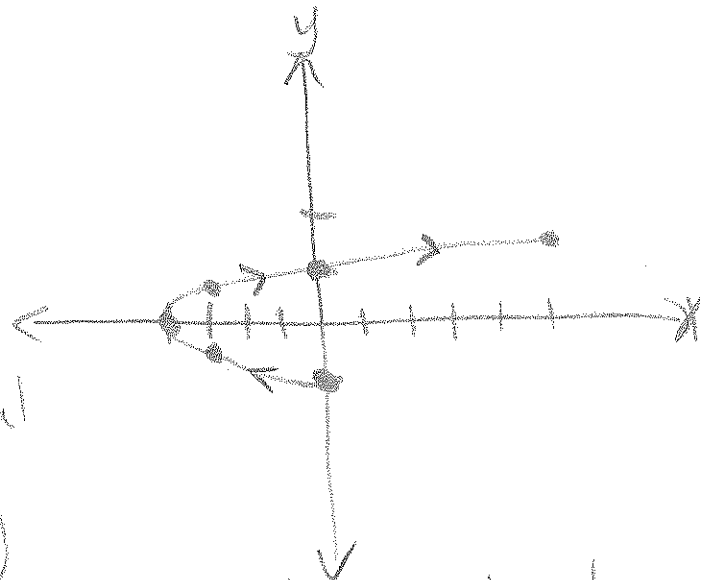
* Define x and y separately.
* On graph still just x & y $\Rightarrow$ won't see parameter.

Sketch the graph of $x(t) = t^2 - 4$, $y(t) = \frac{t}{2}$, $t \in [-2, 3]$

t	-2	-1	0	1	2	3
x	0	-3	-4	-3	0	5
y	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$

initial pt $(0, -1)$

terminal pt $(5, 3/2)$



orientation $\Rightarrow$ direction of travel (arrows on graph)

Graph: $x = 4t^2 - 8t$, $y = 1 - t$, $[-\frac{1}{2}, 2]$
on calc

* same graph but opposite orientation

$x = 4t^2 - 4$, $y = t$, $[-1, 3/2]$

Same graph & same orientation as orig.

Change Parametric to Rectangular
"Eliminate the parameter"

Ex. $x(t) = t^2 - 4$ * Don't want t's ⇒ only x+y

$y(t) = \frac{t}{2}$ ⇒ Solve one eqn. for t + subst.

$t = 2y \Rightarrow x = (2y)^2 - 4 \Rightarrow \boxed{x = 4y^2 - 4}$

Init pt ⇒ $(0, -1)$

term pt ⇒ $(5, \frac{3}{2})$

Ex. $x = 4t^2 - 4$, $y = t$ > $x = 4y^2 - 4$ same eqn

Ex. $x = 4t^2 - 8t$

$y = 1 - t \longrightarrow t = 1 - y \Rightarrow x = 4(1-y)^2 - 8(1-y)$

$x = 4 - \cancel{8y} + 4y^2 - 8 + \cancel{8y}$

$x = 4y^2 - 4 \Rightarrow$ same eqn

Ex. $x = 3(\ln t) - 8$ $\boxed{x = 3y - 8}$

$y = \ln t$

With trig, use identities: $\cos^2 t + \sin^2 t = 1$

Ex. $x = \cos t + 1 \Rightarrow \cos t = x - 1$ $\boxed{(x-1)^2 + \frac{y^2}{16} = 1}$

$y = 4\sin t \Rightarrow \sin t = \frac{y}{4}$

Ellipse

Slope of tangent lines $\Rightarrow \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$

Horiz tangent lines when $\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dt} = 0$

Vert. tangent lines when $\dfrac{dy}{dx} = \text{undef} \Rightarrow \dfrac{dx}{dt} = 0$

If both $\dfrac{dy}{dt}$ & $\dfrac{dx}{dt} = 0$ at same time $\Rightarrow$ cusp!

Ex. $x(t) = t^3 - 3t \quad y(t) = t^2 - 5t - 1$

a) Find $\dfrac{dy}{dx}$. $\quad \dfrac{dy}{dx} = \dfrac{2t-5}{3t^2-3}$

b) Find equation of tangent line @ $t=2$.

Slope $\Rightarrow \dfrac{dy}{dx} = \left.\dfrac{2t-5}{3t^2-3}\right|_{t=2} = -\dfrac{1}{9}$

pt $\Rightarrow (x, y)$

Sub $t=2$ into orig

$x = 2^3 - 3(2) = 2$

$y = 2^2 - 5(2) - 1 = -7$

$(2, -7)$

$$\boxed{y + 7 = -\frac{1}{9}(x - 2)}$$

* Sometimes given pt + have find t.

Ex. Given $(2, -7)$ + not t.

$2 = t^3 - 3t \qquad -7 = t^2 - 5t - 1$

$t^2 - 5t + 6 = 0$

$(t-3)(t-2) = 0$

$\cancel{t=3}$ $\quad t = 2$

+ solve for t! Can use either, but pick t works in both

c) For what t values is the tangent horiz?
vert?

Horiz when $\frac{dy}{dt}=0$ $\quad 2t-5=0 \quad t=\frac{5}{2}$

Vert. when $\frac{dx}{dt}=0$ $\quad 3t^2-3=0 \quad t=\pm 1$

Finding 2nd deriv : $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{dx/dt}$

$$\frac{dy}{dx} = \frac{2t-5}{3t^2-3}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{2(3t^2-3)-(2t-5)(6t)}{(3t^2-3)^2}}{3t^2-3} = \boxed{\frac{-6t^2+30t-6}{(3t^2-3)^3}}$$

$\frac{dx}{dt} \longrightarrow 3t^2-3$

Describe concavity @ $t=2 \Rightarrow$ sub into $\frac{d^2y}{dx^2}$

$$\left.\frac{d^2y}{dx^2}\right|_{t=2} = \frac{30}{729} > 0 \text{ so concave up @ } t=2.$$

Arclength: $L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$

$t \in [a, b]$

Ex. Find length of one arch of cycloid

$x(t) = t - \sin t \quad y(t) = 1 - \cos t \quad [0, 2\pi]$

$$L = \int_0^{2\pi} \sqrt{(1-\cos t)^2 + (\sin t)^2}\, dt = 8$$

Motion $\Rightarrow$ Vectors $\langle \vec{x}(t), \vec{y}(t) \rangle$

$\vec{x}(t) \Rightarrow$ horiz position $\quad \vec{y}(t) \Rightarrow$ vert. position

<u>Velocity</u> $\Rightarrow \langle \vec{x}'(t), \vec{y}'(t) \rangle$

<u>acceleration</u> $\Rightarrow \langle \vec{x}''(t), \vec{y}''(t) \rangle$

<u>displacement</u> $\Rightarrow \langle \int_a^b \vec{x}'(t)\,dt, \int_a^b \vec{y}'(t)\,dt \rangle$

Scalar quantities $\Rightarrow$ only has magnitude not direction

<u>Speed</u> $\Rightarrow$ magnitude of $\vec{v}(t) = \sqrt{(x')^2 + (y')^2}$

<u>distance</u> $\Rightarrow \int_a^b \sqrt{(x')^2 + (y')^2}\, dt$

arclength formula

Ex. An object is moving so that its position at time t is $\langle 3t^2-4t+1, \sqrt{t}+6 \rangle$

a) Find $\vec{v}(2)$ & $\vec{a}(2)$

$$\vec{v}(t) = \langle x', y' \rangle = \langle 6t-4, \frac{1}{2\sqrt{t}} \rangle \Big|_{t=2} \Rightarrow \langle 8, \frac{1}{2\sqrt{2}} \rangle$$

$$\vec{a}(t) = \langle x'', y'' \rangle = \langle 6, -\frac{1}{4t^{3/2}} \rangle \Big|_{t=2} \Rightarrow \langle 6, -\frac{1}{8\sqrt{2}} \rangle$$

Direction @ $t=2$. Look @ signs of x' & y'

$x'(2) > 0 \Rightarrow$ moving right

$y'(2) > 0 \Rightarrow$ moving up

Speed incr / decr $\Rightarrow$ compare signs of x' & x'', y' & y''

$x''(2) > 0$ speed incr (since $x' > 0$ too)

$y''(2) < 0$ speed decr (since $y' > 0$)

b) Speed @ $t=2$ $\quad \sqrt{(8)^2 + \left(\frac{1}{2\sqrt{2}}\right)^2} \approx 8.0078$

c) Find displacement & distance traveled $[0,4]$

$$\text{disp} = \langle \int_0^4 6t-4\,dt, \int_0^4 \frac{1}{2\sqrt{t}}\,dt \rangle = \langle 32, 2 \rangle$$

$$\text{dist} = \int_0^4 \sqrt{(6t-4)^2 + \left(\frac{1}{2\sqrt{t}}\right)^2}\,dt = 35.041 \text{ ft}$$