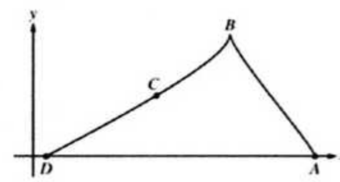
Question 2

A particle starts at point A on the positive x-axis at time $t = 0$ and travels along the curve from A to B to C to D, as shown above. The coordinates of the particle's position $\left(x(t), y(t)\right)$ are differentiable functions of t, where $x'(t) = \frac{dx}{dt} = -9\cos\left(\frac{\pi t}{6}\right)\sin\left(\frac{\pi\sqrt{t+1}}{2}\right)$ and $y'(t) = \frac{dy}{dt}$ is not explicitly given.



At time $t = 9$, the particle reaches its final position at point D on the positive x-axis.

(a) At point C, is $\frac{dy}{dt}$ positive? At point C, is $\frac{dx}{dt}$ positive? Give a reason for each answer.

(b) The slope of the curve is undefined at point B. At what time t is the particle at point B?

(c) The line tangent to the curve at the point $\left(x(8), y(8)\right)$ has equation $y = \frac{5}{9}x - 2$. Find the velocity vector and the speed of the particle at this point.

(d) How far apart are points A and D, the initial and final positions, respectively, of the particle?

(a) At point C, $\frac{dy}{dt}$ is not positive because $y(t)$ is decreasing along the arc BD as t increases.

At point C, $\frac{dx}{dt}$ is not positive because $x(t)$ is decreasing along the arc BD as t increases.

$2 : \begin{cases} 1 : \frac{dy}{dt} \text{ not positive with reason} \\ 1 : \frac{dx}{dt} \text{ not positive with reason} \end{cases}$

(b) $\frac{dx}{dt} = 0$; $\cos\left(\frac{\pi t}{6}\right) = 0$ or $\sin\left(\frac{\pi\sqrt{t+1}}{2}\right) = 0$

$\frac{\pi t}{6} = \frac{\pi}{2}$ or $\frac{\pi\sqrt{t+1}}{2} = \pi$; $t = 3$ for both.

Particle is at point B at $t = 3$.

$2 : \begin{cases} 1 : \text{sets } \frac{dx}{dt} = 0 \\ 1 : t = 3 \end{cases}$

(c) $x'(8) = -9\cos\left(\frac{4\pi}{3}\right)\sin\left(\frac{3\pi}{2}\right) = -\frac{9}{2}$

$\frac{y'(8)}{x'(8)} = \frac{dy}{dx} = \frac{5}{9}$

$y'(8) = \frac{5}{9}x'(8) = -\frac{5}{2}$

The velocity vector is $< -4.5, -2.5 >$.

Speed $= \sqrt{4.5^2 + 2.5^2} = 5.147$ or 5.148

$3 : \begin{cases} 1 : x'(8) \\ 1 : y'(8) \\ 1 : \text{speed} \end{cases}$

(d) $x(9) - x(0) = \int_0^9 x'(t)\,dt$

$= -39.255$

The initial and final positions are 39.255 apart.

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

Question 1

An object moving along a curve in the xy-plane has position $(x(t), y(t))$ at time t with

$$\frac{dx}{dt} = \cos\left(t^3\right) \text{ and } \frac{dy}{dt} = 3\sin\left(t^2\right)$$

for $0 \le t \le 3$. At time $t = 2$, the object is at position (4,5).

(a) Write an equation for the line tangent to the curve at (4,5).

(b) Find the speed of the object at time $t = 2$.

(c) Find the total distance traveled by the object over the time interval $0 \le t \le 1$.

(d) Find the position of the object at time $t = 3$.

(a) $\frac{dy}{dx} = \frac{3\sin\left(t^2\right)}{\cos\left(t^3\right)}$ $\left.\frac{dy}{dx}\right\|_{t=2} = \frac{3\sin\left(2^2\right)}{\cos\left(2^3\right)} = 15.604$ $y - 5 = 15.604(x - 4)$	1 : tangent line
(b) Speed $= \sqrt{\cos^2(8) + 9\sin^2(4)} = 2.275$	1 : answer
(c) Distance $= \int_0^1 \sqrt{\cos^2\left(t^3\right) + 9\sin^2\left(t^2\right)}\, dt$ $= 1.458$	3 : { 2 : distance integral < −1 > each integrand error < −1 > error in limits 1 : answer }
(d) $x(3) = 4 + \int_2^3 \cos\left(t^3\right) dt = 3.953 \text{ or } 3.954$ $y(3) = 5 + \int_2^3 3\sin\left(t^2\right) dt = 4.906$	4 : { 1 : definite integral for x 1 : answer for $x(3)$ 1 : definite integral for y 1 : answer for $y(3)$ }

Question 1

An object moving along a curve in the xy-plane has position $(x(t), y(t))$ at time $t \geq 0$ with

$$\frac{dx}{dt} = 12t - 3t^2 \text{ and } \frac{dy}{dt} = \ln\left(1 + (t-4)^4\right).$$

At time $t = 0$, the object is at position $(-13, 5)$. At time $t = 2$, the object is at point P with x-coordinate 3.

(a) Find the acceleration vector at time $t = 2$ and the speed at time $t = 2$.

(b) Find the y-coordinate of P.

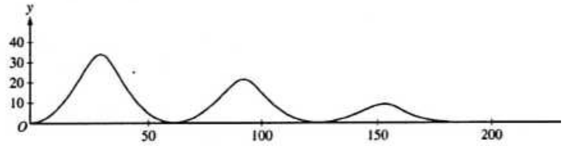
(c) Write an equation for the line tangent to the curve at P.

(d) For what value of t, if any, is the object at rest? Explain your reasoning.

(a) $x''(2) = 0,\ y''(2) = -\frac{32}{17} = -1.882$ $a(2) = \langle 0, -1.882 \rangle$ Speed $= \sqrt{12^2 + (\ln(17))^2} = 12.329$ or 12.330	$2: \begin{cases} 1: \text{acceleration vector} \\ 1: \text{speed} \end{cases}$
(b) $y(t) = y(0) + \int_0^t \ln\left(1 + (u-4)^4\right) du$ $y(2) = 5 + \int_0^2 \ln\left(1 + (u-4)^4\right) du = 13.671$	$3: \begin{cases} 1: \int_0^2 \ln\left(1 + (u-4)^4\right) du \\ 1: \text{handles initial condition} \\ 1: \text{answer} \end{cases}$
(c) At $t = 2$, slope $= \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{\ln(17)}{12} = 0.236$ $y - 13.671 = 0.236(x - 3)$	$2: \begin{cases} 1: \text{slope} \\ 1: \text{equation} \end{cases}$
(d) $x'(t) = 0$ if $t = 0, 4$ $y'(t) = 0$ if $t = 4$ $t = 4$	$2: \begin{cases} 1: \text{reason} \\ 1: \text{answer} \end{cases}$

Question 3

The figure above shows the path traveled by a roller coaster car over the time interval $0 \le t \le 18$ seconds. The position of the car at time t seconds can be modeled parametrically by $x(t) = 10t + 4\sin t$, $y(t) = (20 - t)(1 - \cos t)$, where x and y are measured in meters. The derivatives of these functions are given by



$x'(t) = 10 + 4\cos t$, $y'(t) = (20 - t)\sin t + \cos t - 1$.

(a) Find the slope of the path at time $t = 2$. Show the computations that lead to your answer.

(b) Find the acceleration vector of the car at the time when the car's horizontal position is $x = 140$.

(c) Find the time t at which the car is at its maximum height, and find the speed, in m/sec, of the car at this time.

(d) For $0 < t < 18$, there are two times at which the car is at ground level ($y = 0$). Find these two times and write an expression that gives the average speed, in m/sec, of the car between these two times. Do not evaluate the expression.

(a) Slope $= \left.\dfrac{dy}{dx}\right|_{t=2} = \dfrac{y'(2)}{x'(2)} = \dfrac{18\sin 2 + \cos 2 - 1}{10 + 4\cos 2}$

$= 1.793$ or 1.794

1 : answer using $\dfrac{dy}{dx} = \dfrac{dy}{dt}\Big/\dfrac{dx}{dt}$

(b) $x(t) = 10t + 4\sin t = 140$; $t_0 = 13.647083$

$x''(t_0) = -3.529$, $y''(t_0) = 1.225$ or 1.226

Acceleration vector is $\langle -3.529, 1.225 \rangle$ or $\langle -3.529, 1.226 \rangle$

2 {
1 : identifies acceleration vecto as derivative of velocity vec
1 : computes acceleration vect when $x = 140$

(c) $y'(t) = (20 - t)\sin t + \cos t - 1 = 0$

$t_1 = 3.023$ or 3.024 at maximum height

Speed $= \sqrt{\left(x'(t_1)\right)^2 + \left(y'(t_1)\right)^2} = |x'(t_1)|$

$= 6.027$ or 6.028

3 {
1 : sets $y'(t) = 0$
1 : selects first $t > 0$
1 : speed

(d) $y(t) = 0$ when $t = 2\pi$ and $t = 4\pi$

Average speed $= \dfrac{1}{2\pi}\displaystyle\int_{2\pi}^{4\pi} \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}\,dt$

$= \dfrac{1}{2\pi}\displaystyle\int_{2\pi}^{4\pi} \sqrt{(10 + 4\cos t)^2 + ((20 - t)\sin t + \cos t - 1)^2}\,dt$

3 {
1 : $t = 2\pi, t = 4\pi$
1 : limits and constant
1 : integrand

Question 1

A particle moves in the xy-plane so that its position at any time t, for $-\pi \le t \le \pi$, is given by $x(t) = \sin(3t)$ and $y(t) = 2t$.

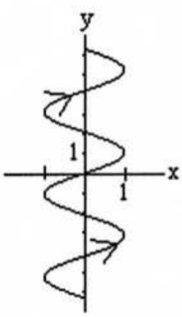
(a) Sketch the path of the particle in the xy-plane provided. Indicate the direction of motion along the path.

(b) Find the range of $x(t)$ and the range of $y(t)$.

(c) Find the smallest positive value of t for which the x-coordinate of the particle is a local maximum. What is the speed of the particle at this time?

(d) Is the distance traveled by the particle from $t = -\pi$ to $t = \pi$ greater than 5π? Justify your answer.

(a)



2 $\begin{cases} 1 : \text{graph} \\ \quad \text{three cycles of sine} \\ \quad x \text{ between } -1 \text{ and } 1 \\ \quad y \text{ between } -2\pi \text{ and } 2\pi \\ 1 : \text{direction} \end{cases}$

(b) $-1 \le x(t) \le 1$

$-2\pi \le y(t) \le 2\pi$

2 $\begin{cases} 1 : \text{closed interval for } x(t) \\ 1 : \text{closed interval for } y(t) \end{cases}$

(c) $x'(t) = 3\cos 3t = 0$

$3t = \frac{\pi}{2}$; $t = \frac{\pi}{6}$

Speed $= \sqrt{9\cos^2(3t) + 4}$

At $t = \frac{\pi}{6}$,

Speed $= \sqrt{9\cos^2\left(\frac{\pi}{2}\right) + 4} = 2$

3 $\begin{cases} 1 : x'(t) = 3\cos 3t = 0 \\ 1 : \text{solves for } t \\ 1 : \text{speed at student's time} \end{cases}$

(d) Distance $= \int_{-\pi}^{\pi} \sqrt{9\cos^2(3t) + 4}\, dt$

$= 17.973 > 5\pi$

2 $\begin{cases} 1 : \text{integral for distance} \\ 1 : \text{conclusion with justification} \end{cases}$

Question 4

A particle moves in the xy-plane so that the position of the particle at any time t is given by

$$x(t) = 2e^{3t} + e^{-7t} \text{ and } y(t) = 3e^{3t} - e^{-2t}.$$

(a) Find the velocity vector for the particle in terms of t, and find the speed of the particle at time $t = 0$.

(b) Find $\frac{dy}{dx}$ in terms of t, and find $\lim_{t\to\infty} \frac{dy}{dx}$.

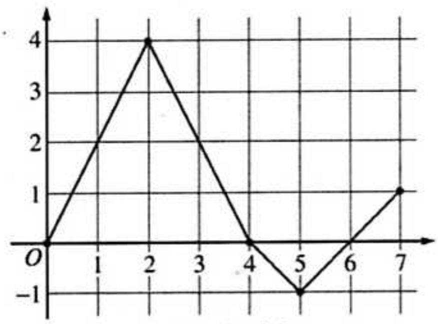
(c) Find each value t at which the line tangent to the path of the particle is horizontal, or explain why none exists.

(d) Find each value t at which the line tangent to the path of the particle is vertical, or explain why none exists.

Solution	Scoring
(a) $x'(t) = 6e^{3t} - 7e^{-7t}$ $y'(t) = 9e^{3t} + 2e^{-2t}$ Velocity vector is $< 6e^{3t} - 7e^{-7t}, 9e^{3t} + 2e^{-2t} >$ Speed $= \sqrt{x'(0)^2 + y'(0)^2} = \sqrt{(-1)^2 + 11^2}$ $= \sqrt{122}$	3 : { 1 : $x'(t)$; 1 : $y'(t)$; 1 : speed }
(b) $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{9e^{3t} + 2e^{-2t}}{6e^{3t} - 7e^{-7t}}$ $\lim_{t\to\infty} \frac{dy}{dx} = \lim_{t\to\infty} \frac{9e^{3t} + 2e^{-2t}}{6e^{3t} - 7e^{-7t}} = \frac{9}{6} = \frac{3}{2}$	2 : { 1 : $\frac{dy}{dx}$ in terms of t; 1 : limit }
(c) Need $y'(t) = 0$, but $9e^{3t} + 2e^{-2t} > 0$ for all t, so none exists.	2 : { 1 : considers $y'(t) = 0$; 1 : explains why none exists }
(d) Need $x'(t) = 0$ and $y'(t) \neq 0$. $6e^{3t} = 7e^{-7t}$ $e^{10t} = \frac{7}{6}$ $t = \frac{1}{10}\ln\left(\frac{7}{6}\right)$	2 : { 1 : considers $x'(t) = 0$; 1 : solution }

Question 5

Let f be a function defined on the closed interval $[0,7]$. The graph of f, consisting of four line segments, is shown above. Let g be the function given by $g(x) = \int_2^x f(t)\,dt$.



Graph of f

(a) Find $g(3)$, $g'(3)$, and $g''(3)$.

(b) Find the average rate of change of g on the interval $0 \le x \le 3$.

(c) For how many values c, where $0 < c < 3$, is $g'(c)$ equal to the average rate found in part (b)? Explain your reasoning.

(d) Find the x-coordinate of each point of inflection of the graph of g on the interval $0 < x < 7$. Justify your answer.

(a) $g(3) = \int_2^3 f(t)\,dt = \frac{1}{2}(4+2) = 3$

$g'(3) = f(3) = 2$

$g''(3) = f'(3) = \frac{0-4}{4-2} = -2$

$3 : \begin{cases} 1 : g(3) \\ 1 : g'(3) \\ 1 : g''(3) \end{cases}$

(b) $\frac{g(3)-g(0)}{3} = \frac{1}{3}\int_0^3 f(t)\,dt$

$= \frac{1}{3}\left(\frac{1}{2}(2)(4) + \frac{1}{2}(4+2)\right) = \frac{7}{3}$

$2 : \begin{cases} 1 : g(3) - g(0) = \int_0^3 f(t)\,dt \\ 1 : \text{answer} \end{cases}$

(c) There are two values of c.

We need $\frac{7}{3} = g'(c) = f(c)$

The graph of f intersects the line $y = \frac{7}{3}$ at two places between 0 and 3.

$2 : \begin{cases} 1 : \text{answer of 2} \\ 1 : \text{reason} \end{cases}$

Note: 1/2 if answer is 1 by MVT

(d) $x = 2$ and $x = 5$

because $g' = f$ changes from increasing to decreasing at $x = 2$, and from decreasing to increasing at $x = 5$.

$2 : \begin{cases} 1 : x = 2 \text{ and } x = 5 \text{ only} \\ 1 : \text{justification} \\ \quad (\text{ignore discussion at } x = 4) \end{cases}$

Question 3

An object moving along a curve in the xy-plane has position $(x(t), y(t))$ at time $t \ge 0$ with $\frac{dx}{dt} = 3 + \cos(t^2)$. The derivative $\frac{dy}{dt}$ is not explicitly given. At time $t = 2$, the object is at position $(1, 8)$.

(a) Find the x-coordinate of the position of the object at time $t = 4$.

(b) At time $t = 2$, the value of $\frac{dy}{dt}$ is -7. Write an equation for the line tangent to the curve at the point $(x(2), y(2))$.

(c) Find the speed of the object at time $t = 2$.

(d) For $t \ge 3$, the line tangent to the curve at $(x(t), y(t))$ has a slope of $2t + 1$. Find the acceleration vector of the object at time $t = 4$.

(a) $x(4) = x(2) + \int_2^4 \left(3 + \cos(t^2)\right) dt$

$= 1 + \int_2^4 \left(3 + \cos(t^2)\right) dt = 7.132$ or 7.133

$3 : \begin{cases} 1 : \int_2^4 \left(3 + \cos(t^2)\right) dt \\ 1 : \text{handles initial condition} \\ 1 : \text{answer} \end{cases}$

(b) $\left.\frac{dy}{dx}\right|_{t=2} = \left.\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right|_{t=2} = \frac{-7}{3 + \cos 4} = -2.983$

$y - 8 = -2.983(x - 1)$

$2 : \begin{cases} 1 : \text{finds } \left.\frac{dy}{dx}\right|_{t=2} \\ 1 : \text{equation} \end{cases}$

(c) The speed of the object at time $t = 2$ is $\sqrt{(x'(2))^2 + (y'(2))^2} = 7.382$ or 7.383.

1 : answer

(d) $x''(4) = 2.303$

$y'(t) = \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = (2t + 1)\left(3 + \cos(t^2)\right)$

$y''(4) = 24.813$ or 24.814

The acceleration vector at $t = 4$ is $\langle 2.303, 24.813 \rangle$ or $\langle 2.303, 24.814 \rangle$.

$3 : \begin{cases} 1 : x''(4) \\ 1 : \frac{dy}{dt} \\ 1 : \text{answer} \end{cases}$

Question 1

A particle moving along a curve in the plane has position $(x(t), y(t))$ at time t, where

$$\frac{dx}{dt} = \sqrt{t^4 + 9} \text{ and } \frac{dy}{dt} = 2e^t + 5e^{-t}$$

for all real values of t. At time $t = 0$, the particle is at the point $(4, 1)$.

(a) Find the speed of the particle and its acceleration vector at time $t = 0$.
(b) Find an equation of the line tangent to the path of the particle at time $t = 0$.
(c) Find the total distance traveled by the particle over the time interval $0 \le t \le 3$.
(d) Find the x-coordinate of the position of the particle at time $t = 3$.

(a) At time $t = 0$:

Speed $= \sqrt{x'(0)^2 + y'(0)^2} = \sqrt{3^2 + 7^2} = \sqrt{58}$

Acceleration vector $= \langle x''(0), y''(0) \rangle = \langle 0, -3 \rangle$

$2 : \begin{cases} 1 : \text{speed} \\ 1 : \text{acceleration vector} \end{cases}$

(b) $\dfrac{dy}{dx} = \dfrac{y'(0)}{x'(0)} = \dfrac{7}{3}$

Tangent line is $y = \frac{7}{3}(x - 4) + 1$

$2 : \begin{cases} 1 : \text{slope} \\ 1 : \text{tangent line} \end{cases}$

(c) Distance $= \displaystyle\int_0^3 \sqrt{\left(\sqrt{t^4 + 9}\right)^2 + \left(2e^t + 5e^{-t}\right)^2}\, dt$
$= 45.226$ or 45.227

$3 : \begin{cases} 2 : \text{distance integral} \\ \quad \langle -1 \rangle \text{ each integrand error} \\ \quad \langle -1 \rangle \text{ error in limits} \\ 1 : \text{answer} \end{cases}$

(d) $x(3) = 4 + \displaystyle\int_0^3 \sqrt{t^4 + 9}\, dt$
$= 17.930$ or 17.931

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

Question 2

Let f be a function having derivatives of all orders for all real numbers. The third-degree Taylor polynomial for f about $x = 2$ is given by $T(x) = 7 - 9(x-2)^2 - 3(x-2)^3$.

(a) Find $f(2)$ and $f''(2)$.

(b) Is there enough information given to determine whether f has a critical point at $x = 2$? If not, explain why not. If so, determine whether $f(2)$ is a relative maximum, a relative minimum, or neither, and justify your answer.

(c) Use $T(x)$ to find an approximation for $f(0)$. Is there enough information given to determine whether f has a critical point at $x = 0$? If not, explain why not. If so, determine whether $f(0)$ is a relative maximum, a relative minimum, or neither, and justify your answer.

(d) The fourth derivative of f satisfies the inequality $\left|f^{(4)}(x)\right| \le 6$ for all x in the closed interval $[0, 2]$. Use the Lagrange error bound on the approximation to $f(0)$ found in part (c) to explain why $f(0)$ is negative.

Solution	Scoring
(a) $f(2) = T(2) = 7$ $\dfrac{f''(2)}{2!} = -9$ so $f''(2) = -18$	$2: \begin{cases} 1: f(2) = 7 \\ 1: f''(2) = -18 \end{cases}$
(b) Yes, since $f'(2) = T'(2) = 0$, f does have a critical point at $x = 2$. Since $f''(2) = -18 < 0$, $f(2)$ is a relative maximum value.	$2: \begin{cases} 1: \text{states } f'(2) = 0 \\ 1: \text{declares } f(2) \text{ as a relative} \\ \quad \text{maximum because } f''(2) < 0 \end{cases}$
(c) $f(0) \approx T(0) = -5$ It is not possible to determine if f has a critical point at $x = 0$ because $T(x)$ gives exact information only at $x = 2$.	$3: \begin{cases} 1: f(0) \approx T(0) = -5 \\ 1: \text{declares that it is not} \\ \quad \text{possible to determine} \\ 1: \text{reason} \end{cases}$
(d) Lagrange error bound $= \dfrac{6}{4!}\lvert 0-2\rvert^4 = 4$ $f(0) \le T(0) + 4 = -1$ Therefore, $f(0)$ is negative.	$2: \begin{cases} 1: \text{value of Lagrange error} \\ \quad \text{bound} \\ 1: \text{explanation} \end{cases}$