Power Series ⇒ series whose terms contain powers of x.

Most basic one →

$$1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

Only sum when Converges ⇒ $|x| < 1$

Interval of Convergence ⟶ $(-1, 1)$

Radius of convergence → 1

Center → 0

Power series - let x be a variable. A power series centered at 0 has form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

where a_k is a real #.

When centered at c, $\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots$

explicit seq. formula like have seen all week ☺

Where does power series converge & have sum?

If geo, do $|r|<1$ like in last example, $|x|<1$
$(-1,1)$

Ex 1. $\frac{1}{5}x + \frac{2}{5^2}x^2 + \dots = \sum_{n=1}^{\infty} \frac{n}{5^n}x^n$ Center? $x=0$

* Not geometric! Use ratio test.

$$\lim_{n\to\infty} \left| \frac{(n+1)\cancel{x^{n+1}}}{\cancel{5^{n+1}}} \cdot \frac{5^n}{n x^n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)x}{5n} \right| = \lim_{n\to\infty} \left| \frac{n+1}{5n} \right| \cdot |x|$$

($x^{n+1} = x^n \cdot x$, $5^{n+1} = 5^n \cdot 5$)

* Ratio test converges when limit < 1

$= \frac{1}{5}|x| < 1 \Rightarrow$ Converges when $\boxed{|x|<5}$ ← radius of conv.
$(-5,5)$

Problem with ratio test, inconclusive when limit $=1$ (at endpts). So, need to test endpts separately to see if included.

$x=-5 \Rightarrow \sum_{n=1}^{\infty} \frac{n}{5^n}(-5)^n = \sum_{n=1}^{\infty} (-1)^n n \Rightarrow \lim_{n\to\infty} n = \infty \neq 0$
diverges by n^{th} term

$x=5 \Rightarrow \sum_{n=1}^{\infty} \frac{n}{\cancel{5^n}}\cancel{(5)^n} = \sum_{n=1}^{\infty} n \Rightarrow \lim_{n\to\infty} n = \infty \neq 0$ div by n^{th} term

Interval of convergence (IOC) $= (-5,5)$
Center $=0$ Radius of conv $= 5$

③ Ex 2. $1+\frac{1}{1!}(x-1)+\frac{1}{2!}(x-1)^2+\frac{1}{3!}(x-1)^3+\dots=\sum_{n=0}^{\infty}\frac{1}{n!}(x-1)^n$

Find IOC. Use ratio test!

Center = 1

$$\lim_{n\to\infty}\left|\frac{\overset{(x-1)^n(x-1)}{(x-1)^{n+1}}}{\underset{(n+1)\cancel{n!}}{(n+1)!}}\cdot\frac{\cancel{n!}}{\cancel{(x-1)^n}}\right|=\lim_{n\to\infty}\left|\frac{1}{n+1}\right||x-1|=0\cdot|x-1|<1$$

This always <1

so IOC $(-\infty,\infty)$

Radius $=\infty$

Ex. 3 $\sum_{n=0}^{\infty} n!\,(x+2)^n = 1+(x+2)+2(x+2)^2+6(x+2)^3+\dots$

IOC? Ratio Test!

$$\lim_{n\to\infty}\left|\frac{\overset{(n+1)\cancel{n!}}{(n+1)!}\,\overset{(x+2)}{(x+2)^{n+1}}}{\cancel{n!}\,\cancel{(x+2)^n}}\right|=\lim_{n\to\infty}|n+1|\cdot|x+2|=\infty\cdot|x+2|<1$$

Only converges when $x=-2$.

Radius? 0

* Only converges at center.

TRY:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}(x-3)^n$$ Find center, radius of conv, & IOC.

$$\lim_{n\to\infty} \left| \frac{\cancel{(x-3)^{n+1}}^{(x-3)}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{\cancel{(x-3)^n}} \right| = \lim_{n\to\infty} \left| \frac{\sqrt{n}}{\sqrt{n+1}} \right| \cdot |x-3| = 1 \cdot |x-3| < 1$$

Converges $|x-3| < 1 \Rightarrow (2, 4)$

Center ↗ Radius ↖

$\boxed{\text{* Center} = 3 \quad \text{Radius} = 1}$

* don't need to check endpts to get radius, but do need to check for IOC.

$x = 2$ $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}(2-3)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ AST $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ ✓ ✓ $\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$ so decr. conv. by AST

$x = 4$ $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}(4-3)^n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}\underbrace{(1)^n}_{\text{just }=1} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ p series $p = \frac{1}{2} < 1$ so diverges

$\boxed{\text{IOC}: [2, 4)}$

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$ * IOC for $f(x)$ is same for any deriv. or integral but endpt conv. might change

$$f'(x) = \sum_{n=1}^{\infty} n \cdot a_n (x-c)^{n-1}$$ * Will never have $x^{\text{neg \#}}$

$n=1$ ← depends on series

$$\int_c^x f(x)\,dx = \sum_{n=0}^{\infty} \frac{1}{n+1} a_n (x-c)^{n+1}$$ * Just like any power function $a_n \Rightarrow \#$

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}(x-3)^n = (x-3) + \cdots$$

Find $f'(x) = \sum_{n=1}^{\infty} n \cdot \frac{1}{\sqrt{n}}(x-3)^{n-1} = \boxed{\sum_{n=1}^{\infty} \sqrt{n}\,(x-3)^{n-1}}$

Find $\int_3^x f(x)\,dx = \sum_{n=1}^{\infty} \frac{1}{(n+1)\sqrt{n}}(x-3)^{n+1} = \boxed{\sum_{n=1}^{\infty} \frac{1}{(n+1)\sqrt{n}}\,(x-3)^{n+1}}$

If want IOC for $f'(x)$ use $f(x)$ has IOC $[2,4)$

Don't need to redo ratio test, just check endpts w/ new series

$$f'(x) = \sum_{n=1}^{\infty} \sqrt{n}\,(x-3)^{n-1}$$

Check endpts

$x=2$ $\quad \sum_{n=1}^{\infty} \sqrt{n}\,(-1)^{n-1}$

$x=4$ $\quad \sum_{n=1}^{\infty} \sqrt{n}$

$\lim_{n\to\infty} \sqrt{n} = \infty \neq 0$ both div. by n^{th} term

$\boxed{\text{So IOC } (2,4)}$

HWK p.623 #1-29 odd