Separable Differential Equations – Growth and Decay

Separable or not?

Which of the following differential equations are separable? **(Y/N).** Please put answers on this sheet. If separable, show the separation but do NOT solve!

a) Y $y' = xy$

$\frac{1}{y}dy = x\,dx$

b) N $y' = x+y$

c) N $y' = \ln(xy)$

d) Y $y' = ye^{(\sin x + \cos y)}$

$ye^{\sin x}e^{\cos y}$

$\frac{1}{ye^{\cos y}}dy = e^{\sin x}dx$

e) Y $y' = \ln(x^y)$

$\frac{1}{y}dy = \ln x\,dx$

f) Y $y' = y\sin x + xy$

$y(\sin x + x)$

$\frac{1}{y}dy = (\sin x + x)dx$

g) N $y' = \sin(xy)$

h) Y $y' = \frac{xy+y}{2x-3xy} = \frac{y(x+1)}{x(2-3y)}$

$\frac{2-3y}{y}dy = \frac{x+1}{x}dx$

Solving separable differential equations:

Ex. $y' = \frac{4x}{y^2}$ Find the general solution and the particular solution if (1, 2) on solutions curve.

$\int y^2 dy = \int 4x\,dx$

$\frac{1}{3}y^3 = 2x^2 + c$

$y^3 = 6x^2 + 3c$

$y = \sqrt[3]{6x^2 + 3c}$

$\boxed{y = \sqrt[3]{6x^2 + 2}}$

$\frac{1}{3}(2)^3 = 2(1)^2 + c$

$\frac{8}{3} = 2 + c$

$c = \frac{2}{3}$

Ex. $4xy' = \cos^2 y$ Find the particular solution if (3,0) on solutions curve.

$y' = \frac{\cos^2 y}{4x}$

$\int \sec^2 y\,dy = \int \frac{1}{4x}dx \rightarrow \frac{1}{4}\int \frac{1}{x}dx$

$\tan y = \frac{1}{4}\ln|x| + c$

$y = \arctan\left(\frac{1}{4}\ln|x| + c\right)$

$y = \arctan\left(\frac{1}{4}\ln|x| - \frac{1}{4}\ln 3\right)$

$\boxed{y = \arctan\left(\frac{1}{4}\ln\left|\frac{x}{3}\right|\right)}$

$\cancel{\tan 0} = \frac{1}{4}\ln 3 + c$ (0)

$c = -\frac{1}{4}\ln 3$

Ex. $xy + y' = 100x$ Find the general solution

$y' = 100x - xy = x(100-y)$

$\int \frac{1}{100-y}dy = \int x\,dx$

$-\ln|100-y| = \frac{1}{2}x^2 + c$

$\ln|100-y| = -\frac{1}{2}x^2 - c$

$|100-y| = e^{-\frac{1}{2}x^2 - c} = e^{-\frac{1}{2}x^2}e^{-c}$

$\boxed{|100-y| = Ce^{-\frac{1}{2}x^2}}$

Verifying solutions: Don't actually solve $\Rightarrow$ No integrals!

Ex. Verify whether or not $y = x^2e^x - 4x^2$ is a solution of the differential equation $xy' - 2y = x^3e^x$

$y' = 2xe^x + x^2e^x - 8x$

$x(2xe^x + x^2e^x - 8x) - 2(x^2e^x - 4x^2) \overset{?}{=} x^3e^x$

$\cancel{2x^2e^x} + x^3e^x \cancel{- 8x^2} \cancel{- 2x^2e^x} \cancel{+ 8x^2} = x^3e^x$ ✓

yes it is a solution

Applications:

Ex. The rate of growth of the population in a town is directly proportional to the population. If there were 1000 people in 1980 and 1200 people in 1982, find:

a) Set up and solve the differential equation to determine the population at time t (let t = # of years after 1980)

$\frac{dP}{dt} = kP$

$\int \frac{1}{P} dP = \int k\, dt$

$\ln|P| = kt + c$

$e^{\ln|P|} = e^{kt+c}$

$P = Ce^{kt}$

$1000 = Ce^{k(0)}$

$C = 1000$

$P = 1000e^{kt}$

$1200 = 1000e^{k(2)}$

$1.2 = e^{2k}$

$\ln 1.2 = 2k$

$k = \frac{\ln 1.2}{2}$

$\boxed{P = 1000e^{(\frac{1}{2}\ln 1.2)t}}$

b) The population in 1990.

$P = 1000e^{(\frac{1}{2}\ln 1.2)(10)}$

$= 2488.32 \approx \boxed{2488 \text{ people}}$

Ex. The rate of voltage draining is proportional to the voltage across the terminals. If initially there are 500 volts and after 10 seconds there are 400 volts, find:

a) Set up and solve the differential equation to determine the # of volts, V, after t seconds.

$\frac{dV}{dt} = kV$

$\int \frac{1}{V} dV = \int k\, dt$

$\ln|V| = kt + c$

$e^{\ln|V|} = e^{kt+c}$

$V = Ce^{kt}$

$500 = Ce^{k(0)} \Rightarrow C = 500$

$V = 500e^{kt}$

$400 = 500e^{10k}$

$10k = \ln \frac{4}{5}$

$k = \frac{1}{10}\ln 4/5$

$\boxed{V = 500e^{(\frac{1}{10}\ln\frac{4}{5})t}}$

b) How long it will take the voltage to drop to 10% of its original value.

$50 = 500e^{(\frac{1}{10}\ln\frac{4}{5})t}$

$\frac{1}{10} = e^{(\frac{1}{10}\ln\frac{4}{5})t}$

$\ln(\frac{1}{10}) = (\frac{1}{10}\ln\frac{4}{5})t$

$t = \frac{10\ln(\frac{1}{10})}{\ln(4/5)}$ seconds

Ex. In processing raw sugar, the rate of change of the amount of raw sugar is proportional to the amount of raw sugar remaining. Set up and solve a differential equation to determine the amount of raw sugar, S, at time t. If 1000 kg of raw sugar reduces to 800 kg of sugar during the first 10 minutes, what is the half life?

$t=0$

time it takes for $\frac{1}{2}$ decay

$\frac{dS}{dt} = kS$

$\int \frac{1}{S} dS = \int k\,dt$

$\ln|S| = kt + c$

$S = Ce^{kt}$

$S = 1000e^{kt}$

$800 = 1000e^{k(10)}$

$k = \frac{1}{10}\ln(.8)$

$S = 1000e^{(\frac{1}{10}\ln .8)t}$

$\frac{1}{2}$ life: $500 = 1000e^{(\frac{1}{10}\ln .8)t}$

$\frac{1}{2} = e^{(\frac{1}{10}\ln .8)t}$

$\ln \frac{1}{2} = (\frac{1}{10}\ln .8)t$

$\boxed{t = \frac{10\ln \frac{1}{2}}{\ln .8} \text{ minutes}}$

half-life $= \frac{\ln \frac{1}{2}}{k}$

Ex. The rate of change of the number of wolves W(t) in a population is directly proportional to 700 – W, where t is the time in years. When t = 0, the population is 400, and when t = 2 the population has increased to 500. Set up and solve the differential equation to determine the population of wolves at time t and then use this to determine the number of wolves at t = 3 years.

$\frac{dW}{dt} = k(700 - W)$

$\int \frac{1}{700 - W} dW = \int k\,dt$

$-\ln|700 - W| = kt + c$

$\ln|700 - W| = -kt - c$

$700 - W = Ce^{kt}$

$700 - 400 = Ce^{k(0)} \Rightarrow C = 300$

$700 - 500 = 300e^{k(2)}$

$200 = 300e^{2k}$

$k = \frac{\ln \frac{2}{3}}{2}$

$W = 700 - 300e^{(\frac{1}{2}\ln \frac{2}{3})t}$

$W(3) =$ ______

Ex. The rate of change of w with respect to t is inversely proportional to the cube of t. Determine an equation for w in terms of t.

$\frac{dw}{dt} = \frac{k}{t^3}$

$\int dw = \int \frac{k}{t^3} dt$

$\boxed{w = -\frac{1}{2}kt^{-2} + c}$

Ex. The rate of change of w with respect to t varies jointly with t and 20 – w. Determine an equation for w in terms of t.

$\frac{dw}{dt} = kt(20 - w)$

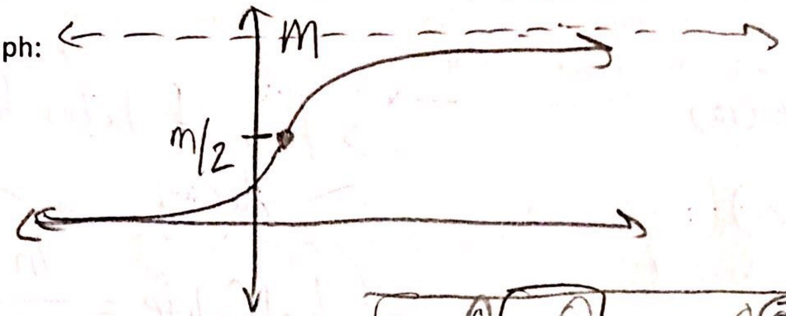
$\int \frac{1}{20 - w} dw = \int kt\,dt$

$-\ln|20 - w| = \frac{1}{2}kt^2 + c$

$\boxed{|20 - w| = Ce^{-\frac{1}{2}kt^2}}$

Logistic differential equations : M = carrying capacity (limit to growth)

Graph:



KNOW logistic when of the form: $\boxed{\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)}$ or $\boxed{\frac{dP}{dt} = \frac{k}{M}P(M - P)}$

$$\boxed{\begin{array}{l} *** \lim_{t\to\infty} P(t) = M \\ *** \text{Max growth rate occurs at } P = \frac{M}{2} \end{array}}$$

$\Rightarrow$ 2 things you will be asked. So, recognize what M is from eqn.

Ex. Given $\frac{dP}{dt} = .02P\left(1 - \frac{P}{50}\right)$.

What is $\lim_{t\to\infty} P(t)$?

50

At what population does the max growth rate occur?

25

Ex. Given $\frac{dP}{dt} = .005P(300 - P)$

What is $\lim_{t\to\infty} P(t)$?

300

At what population does the max growth rate occur?

150

Ex. Given $\frac{dP}{dt} = .08P\left(4 - \frac{P}{50}\right)$.

$\left(1 - \frac{P}{200}\right)$ $(200 - P)$

What is $\lim_{t\to\infty} P(t)$?

200

At what population does the max growth rate occur?

100

Ex. A certain wild animal preserve can support no more than a certain number of gorillas. Thirty gorillas were known to be in the preserve in 1980. Assume that the rate of growth of the population is

$dP/dt = 0.0004P(600\text{-}3P)$, where t is in years. $M = 200$

$M - P$

a) At what population of the gorillas will the growth rate be greatest?

100 gorillas

b) Determine $\lim_{t\to\infty} P(t)$.

200 gorillas

To "solve" a logistic differential equation, you need to use an integration method called Partial Fractions. We will learn this method early in BC . . .