http://calculuslab.deltacollege.edu/ODE/7-1/7-1-0-h.html

A Theoretical Introduction to Slope Fields

Long ago, in a class room not so very far away, you learned about a mathematical idea called the **derivative.** You should have learned a very important property of the derivative:

the *derivative* of a function gives its *slope*

When we work with **differential equations,** we are dealing with expressions in which the derivative appears as a variable. For example, we might be asked to analyze the differential equation:

$$dy/dx = x^2$$

If we simply replace the variable *dy/dx* in the above equation by what we learned it means from our calculus course, we get the following statement:

$$slope = x^2$$

So what? Well, let's remind ourselves of our usual **goal** when we are given a differential equation:

find the function whose derivative appears in the equation ***(we will be doing this over the next few days . . .)***

In our example this means that our goal is to find y.

Now you may be one of those clever students who's always one step ahead of the instructor. If so you're probably already having thoughts about how you could easily solve the current example (i.e. find y). Hold that thought! Unfortunately integration isn't something we will always be able to use. Many (most?) differential equations *can't* be integrated directly. What we're leading into here is a method that can help us on far more differential equations than can be solved using integration.

Anyway, let's get back to our analysis of slope. We've established that our goal is to find the function which satisfies:

$$slope = x^2$$

In other words, we're seeking a function whose slope at any point in the (x,y)-plane is equal to the value of x^2 at that point. Let's amplify that by examining a few selected points.

- At the point (1,2) the slope would be $1^2 = 1$.
- At the point (5,3) the slope would be $5^2 = 25$.
- At the point (-3,11) the slope would be $(-3)^2 = 9$.

(Notice that the y-value of these points doesn't influence the slope in this particular example. This will not always be the case.)

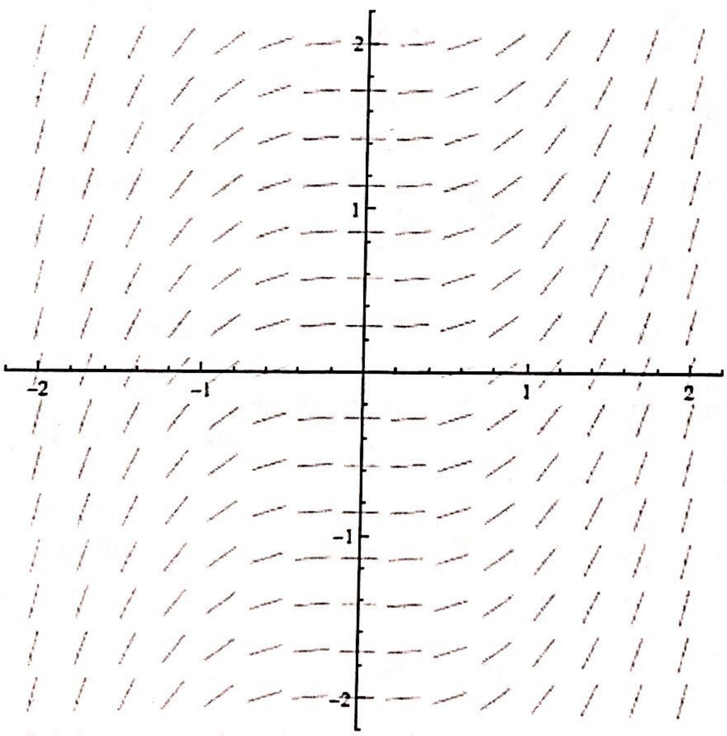
Hmmm...maybe we could use these slopes to get a picture of what the function we seek—the function which has these slopes—looks like? Well, for starters we'd need to be a bit more systematic about how we choose our points. (The choices I made above were somewhat random.)

We could divide the entire plane into a grid, kind of like the squares on a piece of graph paper, and at each grid intersection we could make a slope calculation like we were doing above. Obviously doing this for the entire plane is actually impossible, since it's infinite, so we'll have to be satisfied with some "reasonable" subset of the plane.

This is starting to sound like a lot of work. We may be talking about slope calculations at literally thousands of points, here. Sounds like a job for someone who doesn't mind doing myriads of mind-numbingly repetitive tasks. Someone who can maintain accuracy despite the mountain of (admittedly trivial) calculations involved.

OK, so we can have the computer do the calculations, but there's still something we haven't decided on yet! What do we do with all those thousands of slopes once we've found them? We mentioned earlier that we'd use the slopes to get a picture of what the function y looks like. One way of doing this would be to graphically represent each of the slopes that we find at points all over the plane by a short line segment that is actually as steep as the slope says it should be at that point. We can think of these little line segments as tangent lines to the function y that we've been looking for all this time. We call the resulting picture a ***slope field***, or ***direction field***.

The picture produced by a computer program may look a little like this:



A sample slope field made with Mathematica

A Summary of Making Slope Fields

This procedure may be used to make pictures of slope fields by hand, however, a far less tedious option is to have the computer follow these steps for you. At the moment, we will do this by hand . . .

1. Put the differential equation in the form $dy/dx = g(x,y)$
2. Decide upon what rectangular region of the plane you want to make the picture
3. Impose a grid on this region
4. Calculate the value of the slope, $g(x,y)$, at each grid point, (x,y)
5. Sketch a picture in which at each grid point there is a short line segment having the corresponding slope

Notes: Slope Fields

A slope field is a graph of short line segments constructed using the slope (derivative) equation that is given. It's like if I drew all the little bitty tangent lines to a function and then erased the function. The overall picture of the slope field will give the general shape of the *original function f(x)* plus any arbitrary constants.

Example: Construct a slope field for the following differential equation.

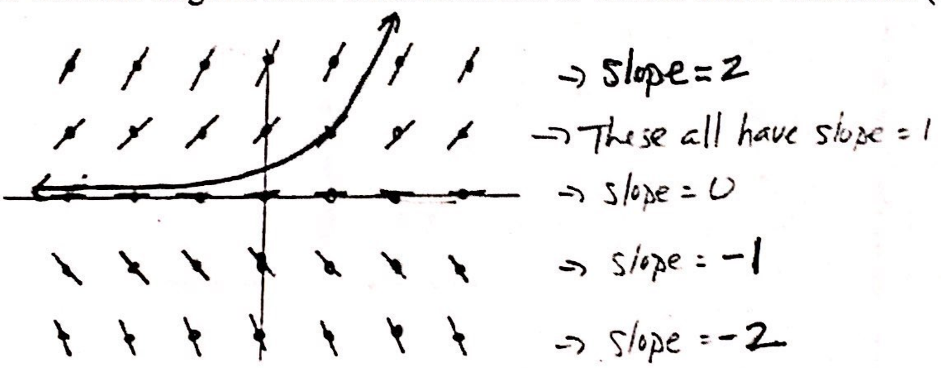
$F'(x,y) = y$ or $\frac{dy}{dx} = y$ or $y' = y$

Working across the coordinate plane, plug in the x and y values into the derivative equation given for each point and determine the slope of the little line segment.

At (-3, -2), $f' = -2$
(-3, -1): $f' = -1$
(-3, 0): $f' = 0$
(-3, 1): $f' = 1$ and so on
(-2, 0): $f' = 0$ and so on
(-1, 2): $f' = 2$ and so on

**Note that I am plugging into F' = y, so with this example, x doesn't matter!

Do this for all points given on coordinate grid. Sketch a little bitty line segment (these are the itty bitty tangent lines) at each of these points that has the slope you determined. And congratulations!! You just constructed a slope field. You should be able to look at this and see the general shape of what the original curve would look like at various initial conditions (think +c's).



You can also *sketch* what the solution curve would be if you were given a specific initial condition by starting at the point given and following the path of the little lines.

Ex. Given slope field for $\frac{dy}{dx} = y$, sketch the solution curve through the point (1,1).

See curve graphed above

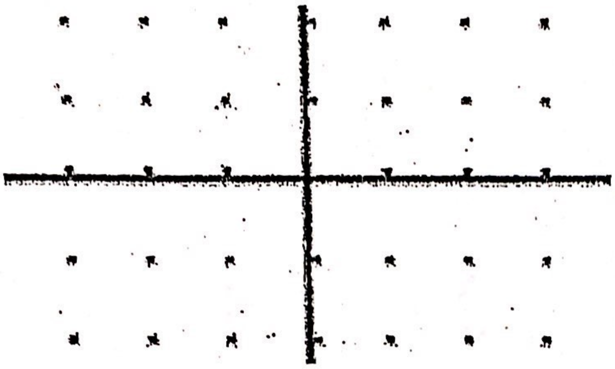
Make sure curve you draw follows the direction of the little tangent lines.

Now, try the in-class practice. I put the answers on the back – Do NOT look until you have tried all 4!!! You'll give away the surprise! After you've got these, do the packet – be careful of what the question is asking!

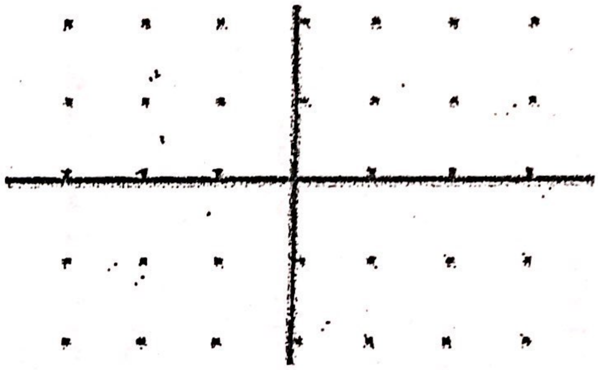
SLOPE FIELDS - In class practice

Draw a slope field for each of the following differential equations.

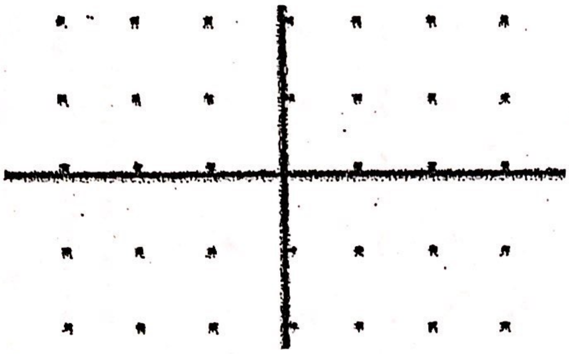
1. $\frac{dy}{dx} = 1 - x$



2. $\frac{dy}{dx} = y + 1$



3. $\frac{dy}{dx} = xy$



4. $\frac{dy}{dx} = x^2$

Need to see a video example of constructing a slope field? Go to
http://www.youtube.com/watch?v=r7VA_4JRPR4

Here are some more slope fields to practice on. In each, match the slope field with its differential equation.

6. Which of the following differential equations has the solution slope field pictured at right?

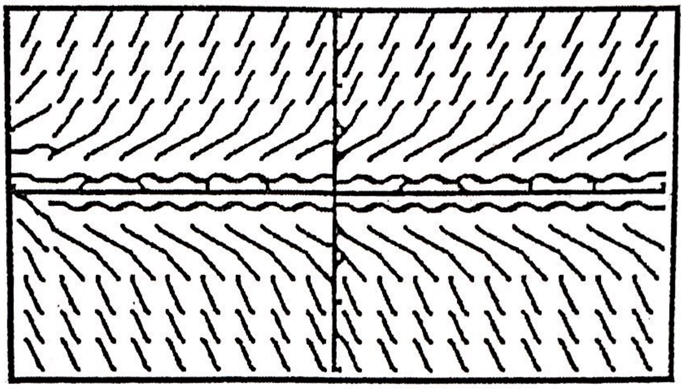
(A) dy/dx = .5y

(B) dy/dx = .2x/y

(C) dy/dx = xy

(D) dy/dx = x + y

(E) dy/dx = 1/x



7. Which of the following differential equations has the solution slope field pictured at right?

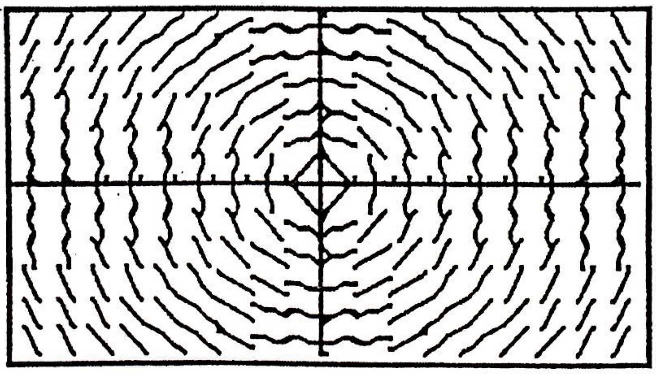
(A) dy/dx = x^2

(B) dy/dx = y/x

(C) dy/dx = -y

(D) dy/dx = -x/y

(E) dy/dx = $x^2 + y^2$



Reading a Slope Field

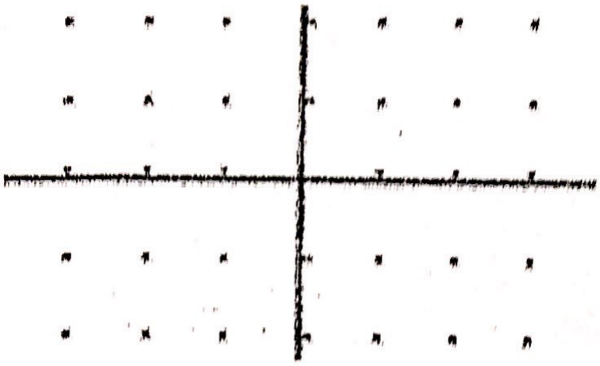
A few features are easy to identify and help sort out most problems:

- Look for the places where the slopes are 0; that is, $\frac{dy}{dx} = 0$.
- Look at the slopes along the x-axis.
- Look at the slopes along the y-axis.
- Look to see if the slopes only depend on x.
- Look to see if the slopes only depend on y.
- Look to see where the slopes are positive and where they are negative.

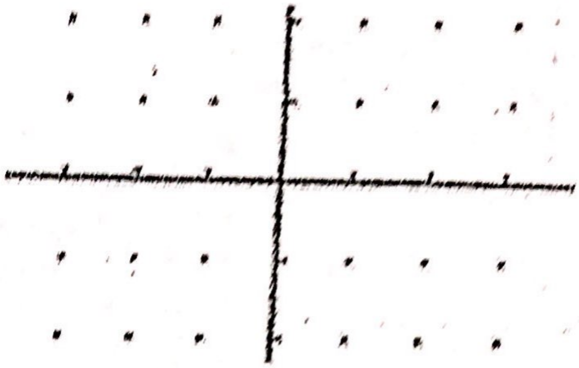
SLOPE FIELDS

Draw a slope field for each of the following differential equations.

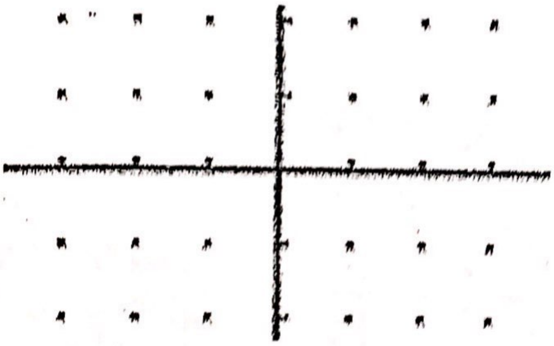
1. $\frac{dy}{dx} = x+1$



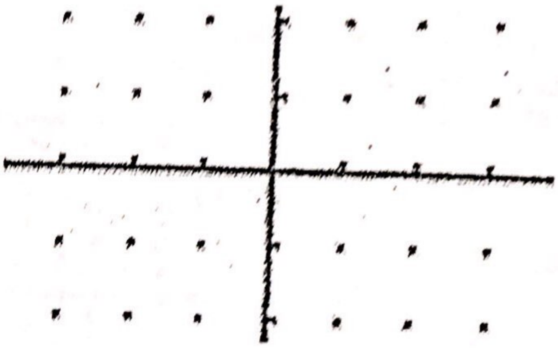
2. $\frac{dy}{dx} = 2y$



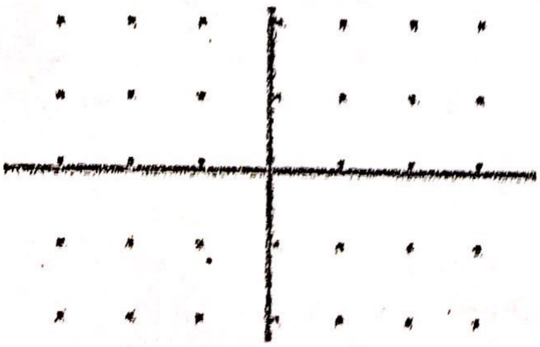
3. $\frac{dy}{dx} = x+y$



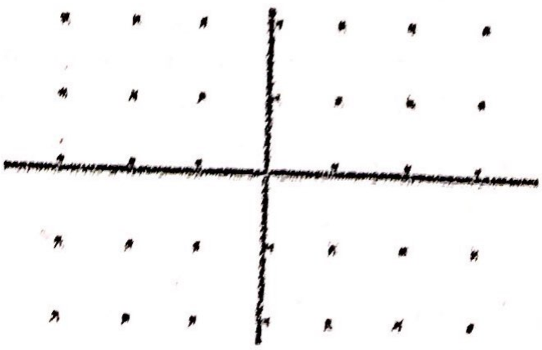
4. $\frac{dy}{dx} = 2x$



5. $\frac{dy}{dx} = y-1$

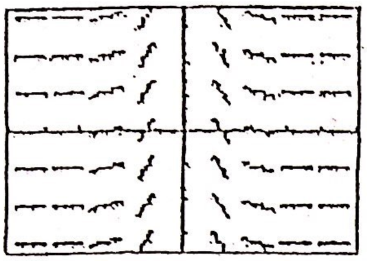


6. $\frac{dy}{dx} = -\frac{y}{x}$

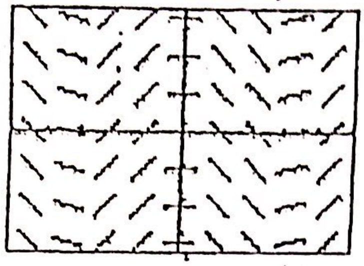


Match each slope field with the equation that the slope field could represent.

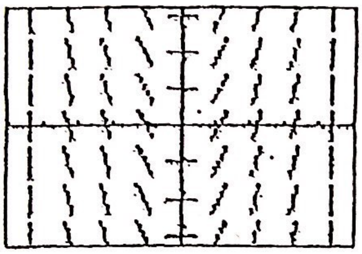
(A)



(B)

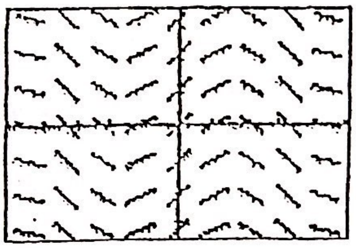


(C)

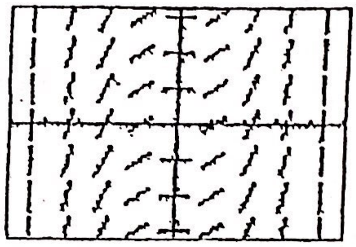


(D)

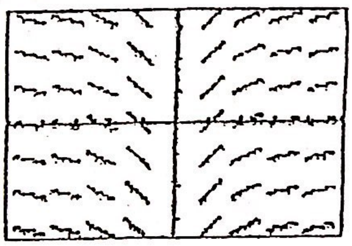
(E)



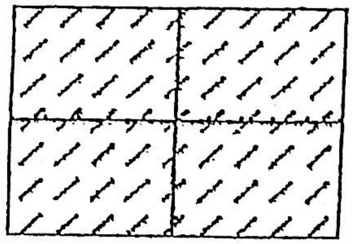
(F)



(G)



(H)



7. $y = 1$

8. $y = x$

9. $y = x^2$

10. $y = \frac{1}{6}x^3$

11. $y = \frac{1}{x^2}$

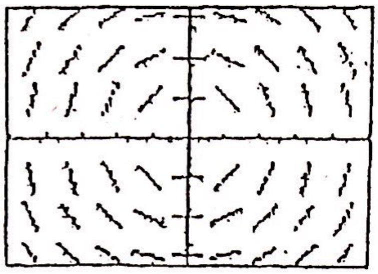
12. $y = \sin x$

13. $y = \cos x$

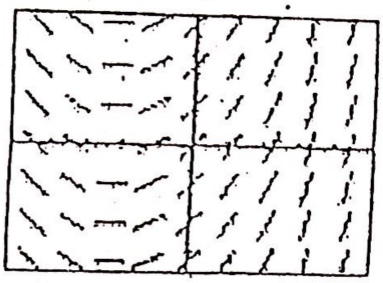
14. $y = \ln|x|$

Match the slope fields with their differential equations.

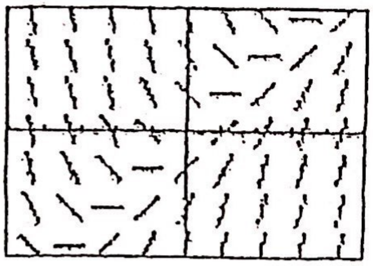
(A)



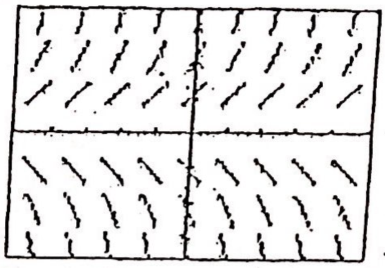
(B)



(C)



(D)



15. $\frac{dy}{dx}=\frac{1}{2}x+1$

16. $\frac{dy}{dx}=y$

17. $\frac{dy}{dx}=x-y$

18. $\frac{dy}{dx}=-\frac{x}{y}$

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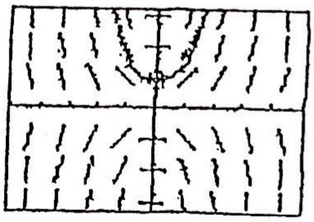
19. The calculator drawn slope field for the differential equation $\frac{dy}{dx}=xy$ is shown in the figure below. The solution curve passing through the point (0, 1) is also shown.

(a) Sketch the solution curve through the point (0, 2).

(b) Sketch the solution curve through the point $(0, -1)$.



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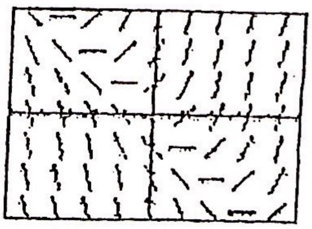
20. The calculator drawn slope field for the differential equation $\frac{dy}{dx}=x+y$ is shown in the figure below.

(a) Sketch the solution curve through the point (0, 1).

(b) Sketch the solution curve through the point $(-3, 0)$.



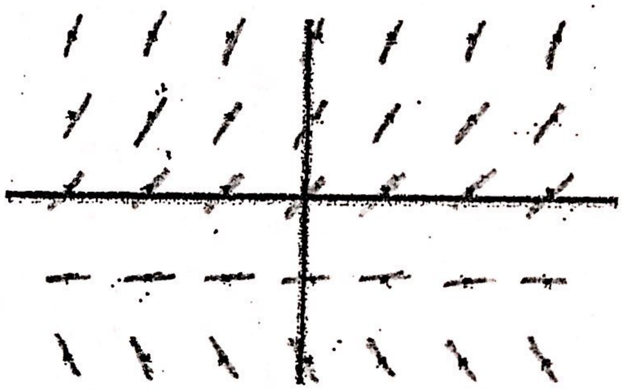
Key

SLOPE FIELDS - In class practice

Draw a slope field for each of the following differential equations.

1. $\frac{dy}{dx} = 1 - x$

2. $\frac{dy}{dx} = y + 1$



3. $\frac{dy}{dx} = xy$

4. $\frac{dy}{dx} = x^2$