8.2 → Series (Hwk: p573 ~~1-15 odd, 21-25 odd~~, 33-45 odd, ~~51-61 odd~~, ~~67ab, 73, 74, 99-102~~)

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots + a_n + \dots \quad \underline{\text{infinite series}}$$

a_1, a_2, a_3 are terms of series

To find sum of series, sometimes helps to look at <u>sequence of partial sums</u>

$S_1 = a_1$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3$

$\vdots$

$S_n = a_1 + a_2 + a_3 + \dots + a_n$

For the infinite series $\sum a_n$ the <u>n^{th} partial sum</u> is

$S_n = a_1 + a_2 + a_3 + \dots a_n$.

If the sequence of partial sums converges to S, then the series $\sum a_n$ converges. The limit S is the <u>sum of the series</u>.
If $\{S_n\}$ diverges, then the series diverges.

* There are only a few ways to find the actual sum of the series. This is one of the ways.

Ex. $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \dots + \frac{1}{n(n+1)} + \dots$

a) find S_1, S_2, S_3, S_4

b) find S_n

c) determine if series conv. or div., if conv. find the sum

Ex. a) $S_1 = \frac{1}{2}$

$S_2 = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$

$S_3 = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{3}{4}$

$S_4 = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} = \frac{4}{5}$

* TRY to recognize pattern!!

b) $S_n = \frac{n}{n+1}$

c) $\lim_{n\to\infty} S_n = 1$ so converges & sum = 1

→ Note: Can also look at this w/ partial fractions:

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$= \left(1 - \cancel{\tfrac{1}{2}}\right) + \left(\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}\right) + \left(\cancel{\tfrac{1}{3}} - \tfrac{1}{4}\right) \cdots + \left(\tfrac{1}{n} - \tfrac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}$$

* called <u>telescoping series</u>.

<u>TRY</u>: $\sum_{n=1}^{\infty} \frac{1}{2^n}$

a) $S_1 = \frac{1}{2}$ $\quad S_2 = \frac{3}{4}$ $\quad S_3 = \frac{7}{8}$

b) $S_n = \frac{2^n - 1}{2^n}$

c) $\lim_{n\to\infty} S_n = 1 \Rightarrow$ converges & sum = 1

note: Geometric

$$\sum \frac{1}{2^n} = \sum \left(\frac{1}{2}\right)^n$$

$\uparrow r$

$$S_n = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1$$

2nd 200

$$\sum_{n=1}^{\infty} \frac{-2}{(2n+5)(2n+3)} \qquad \frac{A}{2n+5} + \frac{B}{2n+3} \qquad -2 = A(2n+3) + B(2n+5)$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{2n+5} - \frac{1}{2n+3}\right)$$

$n = -\frac{3}{2}$ $\quad -2 = 2B \quad B = -1$

$n = -\frac{5}{2}$ $\quad -2 = -2A \quad A = 1$

$$\left(\cancel{\tfrac{1}{7}} - \tfrac{1}{5}\right) + \left(\cancel{\tfrac{1}{9}} - \cancel{\tfrac{1}{7}}\right) + \left(\cancel{\tfrac{1}{11}} - \cancel{\tfrac{1}{9}}\right) + \left(\cdots\tfrac{1}{13} - \cancel{\tfrac{1}{11}}\right) = \boxed{-\frac{1}{5}}$$

<u>Special Series:</u>

Harmonic: $\sum_{n=1}^{\infty} \frac{1}{n}$ → diverges $\quad 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$

Telescoping: $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges ⇒ find sum using p. fractions

Geometric $\sum_{n=1}^{\infty} ar^{n-1}$ OR $\sum_{n=0}^{\infty} ar^n$ converges if $|r| < 1 \Rightarrow$ Sum $= \frac{a}{1-r}$

diverges if $|r| \geq 1$

3rd 2004

Ex. $\sum_{n=0}^{\infty} \frac{2}{3^{n-1}}$

$\Rightarrow \sum_{n=\emptyset}^{\infty} 2\left(\frac{1}{3}\right)^{n-1} \Rightarrow r<1$ converges $S = \frac{2}{1-\frac{1}{3}} = \frac{2}{\frac{2}{3}} = 3$

Ex. $\sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^n \Rightarrow r=\frac{1}{5}<1$ conv. $S = \frac{\frac{1}{5}}{1-\frac{1}{5}} = \frac{\frac{1}{5}}{\frac{4}{5}} = \frac{1}{4}$ * make sure figure out what $\frac{a}{2}$ is by plugging in.

Ex $\sum_{n=0}^{\infty} \frac{3}{2}\left(\frac{4}{3}\right)^n \Rightarrow r=\frac{4}{3}>1$ diverges

Properties: If $\sum a_n = A$, $\sum b_n = B$ + $c \in \mathbb{R}$

① $\sum_{n=1}^{\infty} ca_n = cA$ ② $\sum_{n=1}^{\infty} (a_n \pm b_n) = A \pm B$ * If $\sum a_n$ converges + $\sum b_n$ div. then $\sum(a_n+b_n)$ diverges

If series $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=k}^{\infty} a_n$ converges (can delete terms + still conv.)
likewise w/ diverges

If want sum, can subtract out missing terms:

Ex. $\sum_{n=1}^{\infty} \frac{1}{(n+2)(n+3)} = \frac{1}{3\cdot4} + \frac{1}{4\cdot5} + \frac{1}{5\cdot6} + \dots$

Know that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$

$= \frac{1}{1\cdot2} + \frac{1}{2\cdot3} + \frac{1}{3\cdot4} + \dots$

So $\sum_{n=1}^{\infty} \frac{1}{(n+2)(n+3)} = 1 - \left(\frac{1}{2} + \frac{1}{6}\right) = \frac{1}{3}$

1st 2 terms

Ex. $\sum_{n=1}^{\infty} \left(\frac{3}{n(n+1)} + \frac{1}{4^n}\right)$

$\sum_{n=1}^{\infty} \frac{3}{n(n+1)} = 3\cdot\sum \frac{1}{n(n+1)} = 3(1) = 3$

$\sum_{n=1}^{\infty} \frac{1}{4^n} = \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n = \frac{\frac{1}{4}}{1-\frac{1}{4}} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$

So, $\sum\left(\frac{3}{n(n+1)} + \frac{1}{4^n}\right) = 3 + \frac{1}{3} = 3\frac{1}{3}$

Ex. $\sum_{n=1}^{\infty}\left(\frac{2}{n}-\left(\frac{1}{2}\right)^n\right)$

$\sum_{n=1}^{\infty}\frac{2}{n}=2\sum_{n=1}^{\infty}\frac{1}{n} \Rightarrow$ diverges harmonic $\quad\therefore$ Diverges

<u>Try:</u>

$\sum_{n=1}^{\infty}\left[\frac{7}{n(n+1)}-\frac{2^n}{3^{n-1}}\right] = 13$

$\sum_{n=1}^{\infty}\frac{7}{n(n+1)}=7(1)=7$

$\sum_{n=1}^{\infty}\frac{2^n}{3^{n-1}}=\sum\frac{2^n}{3^n\cdot 3^{-1}}=\sum_{n=1}^{\infty}3\left(\frac{2}{3}\right)^n=\frac{2}{1-\frac{2}{3}}=\frac{2}{\frac{1}{3}}=6$

<u>n^{th} Term test for divergence</u>

If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n=0$

* If $\lim_{n\to\infty} a_n\neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges *

Does <u>NOT</u> mean that if $\lim_{n\to\infty} a_n=0$ then converges.

In fact, if $\lim_{n\to\infty} a_n=0$ <u>must</u> <u>do</u> <u>more</u> <u>tests</u>!!

Ex. $\sum_{n=0}^{\infty}\left(\frac{n}{n+3}\right) \Rightarrow \lim_{n\to\infty}\frac{n}{n+3}=1\neq 0$ so <u>diverges</u>

$\sum_{n=0}^{\infty}\frac{n!}{2n!+1} \Rightarrow \lim_{n\to\infty}\frac{n!}{2n!+1}=\frac{1}{2}\neq 0$ so <u>diverges</u>

$\sum_{n=1}^{\infty}\frac{1}{n} \Rightarrow \lim_{n\to\infty}\frac{1}{n}=0$ so must do more tests $\Rightarrow$ inconclusive!!

(* Know harmonic so diverges)