Power Series $\Rightarrow \sum_{n=0}^{\infty} a_n (x-c)^n = f(x)$

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \dots$$
$$+ \dots \frac{f^{(n)}(c)}{n!}(x-c)^n = \sum_{n=0}^{\infty} \underbrace{\boxed{\frac{f^{(n)}(c)}{n!}}}_{a_n}(x-c)^n$$

This is called a Taylor series (it is a power series)

If $c=0 \Rightarrow f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots$

its called a Maclaurin series.

A Taylor polynomial of degree n (order n)

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \dots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

* Can manipulate like power series - deriv, int, sub, etc.
* If center changes, must work out from beginning (see last ex)

Maclaurin series for $f(x) = \sin x$ $c=0$

$f(x) = \sin x \quad f(0) = 0$

$f'(x) = \cos x \quad f'(0) = 1$

$f''(x) = -\sin x \quad f''(0) = 0$

$f'''(x) = -\cos x \quad f'''(0) = -1$

$f^{(4)}(x) = \sin x \quad f^{(4)}(0) = 0$

$$\sin x = 0 + (1)x + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \cdots$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos x = \frac{d}{dx}(\sin x) = 1 - \frac{1}{3!}\cdot 3x^2 + \frac{1}{5!}\cdot 5x^4 - \frac{1}{7!}\cdot 7x^6 \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1) x^{2n}}{(2n+1)!}$$

$$\boxed{\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}}$$

$$\boxed{e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots \sum_{n=0}^{\infty} \frac{x^n}{n!}}$$

$f(x) = e^x \quad f(0) = 1$

$f'(x) = e^x \quad f'(0) = 1$

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$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{x^2} = 1 + x^2 + \frac{1}{2!}\overset{(x^2)^2}{x^4} + \frac{1}{3!}\overset{(x^2)^3}{x^6} + \cdots = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$$

$$e^{x^2} - 1 = x^2 + \frac{1}{2!}x^4 + \frac{1}{3!}x^6 + \cdots = \sum_{n=1}^{\infty} \frac{x^{2n}}{n!}$$

$$\frac{e^{x^2} - 1}{x} = x + \frac{1}{2!}x^3 + \frac{1}{3!}x^5 + \cdots = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(n+1)!}$$

Can only "manipulate" known series if same center. Above, all were centered @ 0. Otherwise, start from scratch

Ex. Write 3rd degree Taylor Poly. for $\cos x$ centered @ $\frac{\pi}{3}$.

* Can't use Maclaurin series !!

stop @ 3rd deriv since need 3rd deg.

$f(x) = \cos x \qquad f(\pi/3) = \frac{1}{2}$

$f'(x) = -\sin x \qquad f'(\pi/3) = -\sqrt{3}/2$

$f''(x) = -\cos x \qquad f''(\pi/3) = -1/2$

$f'''(x) = \sin x \qquad f'''(\pi/3) = \sqrt{3}/2$

$$\cos x \approx P_3(x) = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{\frac{1}{2}}{2!}\left(x - \frac{\pi}{3}\right)^2 + \frac{\sqrt{3}/2}{3!}\left(x - \frac{\pi}{3}\right)^3$$

$$= \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{1}{4}\left(x - \frac{\pi}{3}\right)^2 + \frac{\sqrt{3}}{12}\left(x - \frac{\pi}{3}\right)^3$$

Other types:

1. $f(x) \approx 3 - 2(x-1) + 6(x-1)^2 + \frac{4}{3}(x-1)^3$

   a) Find $f'(1)$ and $f''(1)$

   b) Determine the equation of tangent to $f(x)$ at $x=1$.

   c) Describe the graph of $f(x)$ at $x=1$.

2. $f(x) \approx -4 - 3(x+2)^2 + 9(x+2)^3 - 6(x+2)^4$

   a) Find $f'''(-2)$

   b) Analyze the graph of $f(x)$ @ $x=-2$.

   c) Find the 3rd order Taylor poly for $f'(x)$ centered at -2.

3. $f(x) \approx P_4(x) = 3 - \frac{2}{3!}(x-2)^2 + \frac{6}{5!}(x-2)^4$

   a) Determine $f(2)$, $f'(2)$, $f''(2)$

   b) Describe graph of $f(x)$ at $x=2$.

   c) Find 5th order Taylor poly for $g(x) = \int_2^x f(t)\,dt$.

4. $f(x) \approx P_2(x) = 6 + 2(x-1) - 8(x-1)^2$

   a) $f(1)$, $f'(1)$, $f''(1)$?

   b) Find equation of tangent at $x=1$

   c) Would the tangent line @ $x=1$ be above or below curve & why?

5. Given $f(0)=4$, $f'(0)=-2$, $f''(0)=6$, and $f'''(0)=-1$

a) Write the 3rd order Maclaurin series for $f(x)$ & use it to approximate $f(0.1)$.

b) Write the 2nd order Maclaurin series for $g(x)=f'(x)$

c) Write the 8th order series for $h(x)=\int_0^{x^2} f(t)\,dt$.

d) Does $h(x)$ have extrema at $x=0$? Why or why not? If so, classify and explain.

6. The Taylor series for $x=3$ converges to $f(x)$ for all x in the interval of convergence. Given $f(3)=2$ and the n^{th} derivative can be found by $f^{(n)}(3)=\frac{(-1)^n(n+1)}{n\cdot 2^n}$

$$f^{(n)}(3)=\frac{(-1)^n(n+1)!}{n\cdot 2^n}$$

a) Write the 3rd order Taylor poly about $x=3$.

b) Find the radius of convergence

c) Find the error in approx $f(3.1)$ using the 5th order Taylor poly.