Power Series Practice

I Find the interval of convergence.

1. $\sum_{n=0}^{\infty} (-1)^{n+1} n x^n$ 2. $\sum_{n=0}^{\infty} \frac{(3x)^n}{(2n)!}$ 3. $\sum_{n=0}^{\infty} \frac{3^{2n}}{n+1}(x-2)^n$

4. $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n(n+1)}$ 5. $\sum_{n=1}^{\infty} \frac{n!\,x^n}{(2n)!}$

II Find the power series representation for the following

6. $f(x) = \frac{3}{2x-1}$, $c=2$ 7. $f(x) = \frac{4}{4+x^2}$, $c=0$ 8. $f(x) = xe^{-2x}$, c

9. $f(x) = \cos x^2 - 1$, $c=0$ 10. $f(x) = 3x^2 \arctan x$, $c=0$

11. $f(x) = \sin\sqrt{x}$, $c=0$

I . Find the given Taylor polynomial to represent the funct

12. $f(x) = \tan x$, $c = \frac{\pi}{4}$, $P_3(x)$ 13. $f(x) = \sqrt{x}$, $c=4$ $P_3(x)$

14. $f(x) = \cos x$, $c = -\frac{\pi}{4}$, $P_4(x)$ 15. $f(x) = \sin(x^2)$, $c=0$, $P_{10}(x)$

16. $f(x) = \frac{1}{x}$, $c=2$, $P_3(x)$ 17. $f(x) = \cos x$, $c = \frac{\pi}{3}$, $P_3(x)$

IV Use a Taylor polynomial to approximate $f(x)$ with error $\leq$.

Show all work include function & center used.

18. $\sin 95°$ 19. $\ln(1.15)$ 20. $\int_0^1 e^{-x^2}\,dx$

V Given Taylor series to represent $f(x)$, describe the graph of f at c

21. $P_3(x) = -4 - 3(x+2) + 4(x+2)^2 - (x+2)^3$ 22. $P_2(x) = 3 + \frac{5(x-1)^2}{4}$

23. For #21 above, give values for $f(-2)$, $f'(-2)$, $f''(-2)$ & $f'''(-2$

1. $\lim_{n\to\infty}\left|\frac{(n+1)x^{n+1}}{n x^n}\right| = |x|<1$ By ratio test, conv. $(-1,1)$

$x=-1$ $\sum(-1)n$ $\lim_{n\to\infty} n=\infty$ div. by n^{th} term test

$x=1$ $\sum(-1)^{n+1}n$ " " " "

$\boxed{\text{Int. } (-1,1)}$

2. $\lim_{n\to\infty}\left|\frac{\cancel{3^{n+1}} x^{\cancel{n+1}}}{\underset{(2n+2)(2n+1)}{\cancel{(2n+2)!}}}\cdot\frac{\cancel{(2n)!}}{\cancel{3^n}\cancel{x^n}}\right| = \lim_{n\to\infty}\left|\frac{3}{(2n+2)(2n+1)}\cdot x\right| = 0\cdot|x|<1$

$\boxed{\text{Int conv. } (-\infty,\infty)}$ by ratio test.

3. $\lim_{n\to\infty}\left|\frac{3^{2n+2}(x-2)^{n+1}}{n+2}\cdot\frac{n+1}{\cancel{3^{2n}}(x-2)^n}\right| = \lim_{n\to\infty}\left|\frac{9(n+1)}{n+2}(x-2)\right| = 9|x-2|<1$

By ratio test, conv. $|x-2|<\frac{1}{9} \Rightarrow \left(\frac{17}{9},\frac{19}{9}\right)$

$x=\frac{17}{9}$ $\sum\frac{3^{2n=9^n}}{n+1}\cdot\left(-\frac{1}{9}\right)^n = \sum\frac{(-1)^n}{n+1}$ $\lim_{n\to\infty}\frac{1}{n+1}=0$ ✓ $\frac{1}{n+2}<\frac{1}{n+1}$ dec ✓ conv. by AST

$x=\frac{19}{9}$ $\sum\frac{3^{2n}}{n+1}\cdot\left(\frac{1}{9}\right)^n = \sum\frac{1}{n+1}$ $b_n=\frac{1}{n}$ div harmonic $\lim_{n\to\infty}\frac{1}{n+1}\cdot\frac{n}{1}=1>0$ div by LCT

$\boxed{\text{Int: } \left[\frac{17}{9},\frac{19}{9}\right)}$

4. $\lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{(n+1)(n+2)}\cdot\frac{n(n+1)}{(x-2)^n}\right| = |x-2|<1$ By ratio test, conv. $(1,3)$

$x=1$ $\sum\frac{(-1)^n}{n(n+1)}$ conv. b/c $|a_n|$ conv.

$x=3$ $\sum\frac{1}{n(n+1)}$ conv. b/c telescoping

$\boxed{\text{Interval } [1,3]}$

5. $\lim_{n\to\infty}\left|\frac{\overset{(n+1)}{\cancel{(n+1)!}}\, x^{\cancel{n+1}}}{\underset{(2n+2)(2n+1)}{\cancel{(2n+2)!}}}\cdot\frac{\cancel{(2n)!}}{\cancel{n!}\,\cancel{x^n}}\right| = \lim_{n\to\infty}\left|\frac{n+1}{(2n+2)(2n+1)}\cdot x\right| = 0|x|<1$

$\therefore$ By ratio test, conv. $\boxed{(-\infty,\infty)}$

6. $\frac{3}{2x-1} = \frac{3}{3+2(x-2)} = \frac{1}{1+\frac{2}{3}(x-2)} = 1 - \frac{2}{3}(x-2) + \frac{4}{9}(x-2)^2 - \dots = \sum_{n=0}^{\infty} (-1)^n \left(\frac{2}{3}\right)^n (x-2)^n$

7. $\frac{4}{4+x^2} = \frac{1}{1+\frac{1}{4}x^2} = 1 - \frac{1}{4}x^2 + \frac{1}{16}x^4 - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{4^n}$

8. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\quad e^{-2x} = 1 - 2x + \frac{4x^2}{2!} - \frac{8x^3}{3!} + \dots \sum_{n=0}^{\infty} \frac{(-2)^n x^n}{n!}$

$xe^{-2x} = x - 2x^2 + \frac{4x^3}{2!} - \frac{8x^4}{3!} + \dots = \sum_{n=0}^{\infty} \frac{(-2)^n x^{n+1}}{n!}$

9. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

$\cos(x^2) = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!}$

$\cos(x^2) - 1 = -\frac{x^4}{2!} + \frac{x^8}{4!} - \dots = \sum_{n=1}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{4n+4}}{(2n+2)!}$

10. $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$

$3x^2 \arctan x = 3x^3 - \frac{3x^5}{3} + \frac{3x^7}{5} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n (3) x^{2n+3}}{2n+1}$

11. $\sin\sqrt{x} \Rightarrow \sin\sqrt{0} = 0$

Can't do this one!!

$f' = \frac{1}{2\sqrt{x}} \cos\sqrt{x}$ undef. Also, $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!}$

$\sin\sqrt{x} = \sqrt{x} - \frac{x^{3/2}}{3!} + \frac{x^{5/2}}{5!} \rightarrow$ not power series

must be x to integer power

Could do $\cos\sqrt{x}$ though $\Rightarrow$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots = \sum \frac{(-1)^n x^{2n}}{(2n)!}$ $\quad \cos(\sqrt{x}) = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!}$

but int. conv. $(0,\infty) \rightarrow$ domain

12. $\tan x \Rightarrow \tan\frac{\pi}{4}=1$

$f' = \sec^2 x \Rightarrow f'(\frac{\pi}{4}) = 2$

$f'' = 2\sec^2 x \tan x \Rightarrow f''(\frac{\pi}{4}) = 4.$

$f''' = 4\sec^2 x \tan^2 x + 2\sec^4 x \Rightarrow f'''(\frac{\pi}{4}) = 4(2)+8 = 16$

$$P_3(x) = 1 + 2(x-\tfrac{\pi}{4}) + \frac{4}{2!}(x-\tfrac{\pi}{4})^2 + \frac{16}{3!}(x-\tfrac{\pi}{4})^3 \approx \tan x$$

13. $f(x) = \sqrt{x} \quad f(4) = 2$

$f'(x) = \frac{1}{2}x^{-1/2} \quad f'(4) = \frac{1}{4}$

$f''(x) = -\frac{1}{4}x^{-3/2} \quad f''(4) = -\frac{1}{32}$

$f'''(x) = \frac{3}{8}x^{-5/2} \quad f'''(4) = \frac{3}{256}$

$$\boxed{P_3(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{32\cdot 2!}(x-4)^2 + \frac{3}{256\cdot 3!}(x-4)^3}$$

14. $f(x) = \cos x \quad f(-\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$

$f'(x) = -\sin x \quad f'(-\frac{\pi}{4}) = \frac{\sqrt{2}}{2}.$

$f''(x) = -\cos x \quad f''(-\frac{\pi}{4}) = -\frac{\sqrt{2}}{2}$

$f'''(x) = \sin x \quad f'''(-\frac{\pi}{4}) = -\frac{\sqrt{2}}{2}$

$f^{iv}(x) = \cos x \quad f^{iv}(-\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$

$$P_4(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(x+\tfrac{\pi}{4}) - \frac{\sqrt{2}}{2\cdot 2!}(x+\tfrac{\pi}{4})^2 - \frac{\sqrt{2}}{2\cdot 3!}(x+\tfrac{\pi}{4})^3 + \frac{\sqrt{2}}{2\cdot 4!}(x+\tfrac{\pi}{4})^4$$

15. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \qquad \sin(x^2) \approx P_{10}(x) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!}$

16. $f(x) = \frac{1}{x} \qquad f(2) = \frac{1}{2}$

$f'(x) = \frac{-1}{x^2} \quad f'(2) = -\frac{1}{4} \qquad P_3(x) = \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{4\cdot 2!}(x-2)^2 - \frac{3}{8\cdot 3!}(x-2)^3$

$f''(x) = \frac{2}{x^3} \quad f''(2) = \frac{1}{4}$

$f''' = -\frac{6}{x^4} \quad f'''(2) = \frac{-3}{8}$

17. $f(x)=\cos x \quad f(\pi/3)=\frac{1}{2}$

$f'(x)=-\sin x \quad f'(\pi/3)=-\sqrt{3}/2$

$f''(x)=-\cos x \quad f''(\pi/3)=-\frac{1}{2}$

$f'''(x)=\sin x \quad f'''(\pi/3)=\sqrt{3}/2$

$P_3(x)=\frac{1}{2}-\frac{\sqrt{3}}{2}(x-\frac{\pi}{3})-\frac{1}{2\cdot 2!}(x-\frac{\pi}{3})^2+\frac{\sqrt{3}}{2\cdot 3!}(x-\frac{\pi}{3})$

18. $f(x)=\sin x \quad c=\frac{\pi}{2}$ approx $\sin(\frac{\pi}{2}+\underbrace{\frac{\pi}{36}}_{\to 95^\circ})$

$\sin x \approx 1-\frac{(x-\frac{\pi}{2})^2}{2!}+\frac{(x-\frac{\pi}{2})^4}{4!}- \dots$

$\sin 95^\circ = \sin(\frac{\pi}{2}+\frac{\pi}{36}) \approx 1-\frac{(\pi/36)^2}{2!}+\boxed{\frac{(\pi/36)^4}{4!}}$ → error term $=2.416\times10^{-6} \le .001$

$\approx .99619$

$|\text{error}| \le \frac{(\pi/36)^4}{4!} \approx 2.416\times10^{-6} \le .001$

19. $f(x)=\ln x \quad c=1$

$\ln x \approx (x-1)-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\frac{(x-1)^4}{4}$

$\ln(1.15) \approx .15-\frac{(.15)^2}{2}+\frac{(.15)^3}{3}\boxed{-\frac{(.15)^4}{4}}$ → error term

$\approx P_3(x) \approx .139875 \qquad |\text{error}| \le \left|\frac{(.15)^4}{4}\right| \approx 1.2656\times10^{-4} \le .001$

20. $e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\dots \qquad e^{-x^2}=1-x^2+\frac{x^4}{2!}-\frac{x^6}{3!}+\frac{x^8}{4!}+\dots$

$\int_0^1 e^{-x^2}dx = x-\frac{x^3}{3}+\frac{x^5}{5\cdot 2!}-\frac{x^7}{7\cdot 3!}+\frac{x^9}{9\cdot 4!}+\frac{x^{11}}{11\cdot 5!}\Big|_0^1$

$\approx 1-\frac{1}{3}+\frac{1}{5\cdot 2!}-\frac{1}{7\cdot 3!}+\frac{1}{9\cdot 4!}\boxed{-\frac{1}{11\cdot 5!}}$ → error

$\approx P_9(x) \approx .743783 \qquad |\text{error}| \le \left|-\frac{1}{11\cdot 5!}\right| \approx 7.5758\times10^{-4} \le .00$

21. $f(-2)=-4$

decr. b/c $f'<0$

c.up b/c $f''>0$

22. $f(1)=3$

min b/c $f'(1)=0$ and $f''(1)>0$

23. $f(-2)=-4$

$f'(-2)=-3 \quad f'''(-2)=-6$

$f''(-2)=8$