Taylor SA Practice Key

AP® CALCULUS BC
2006 SCORING GUIDELINES (Form B)

Question 6

The function f is defined by $f(x) = \frac{1}{1+x^3}$. The Maclaurin series for f is given by

$$1 - x^3 + x^6 - x^9 + \cdots + (-1)^n x^{3n} + \cdots,$$

which converges to $f(x)$ for $-1 < x < 1$.

(a) Find the first three nonzero terms and the general term for the Maclaurin series for $f'(x)$.

(b) Use your results from part (a) to find the sum of the infinite series $-\frac{3}{2^2} + \frac{6}{2^5} - \frac{9}{2^8} + \cdots + (-1)^n \frac{3n}{2^{3n-1}} + \cdots$.

(c) Find the first four nonzero terms and the general term for the Maclaurin series representing $\int_0^x f(t)\,dt$.

(d) Use the first three nonzero terms of the infinite series found in part (c) to approximate $\int_0^{1/2} f(t)\,dt$. What are the properties of the terms of the series representing $\int_0^{1/2} f(t)\,dt$ that guarantee that this approximation is within $\frac{1}{10,000}$ of the exact value of the integral?

(a) $f'(x) = -3x^2 + 6x^5 - 9x^8 + \cdots + 3n(-1)^n x^{3n-1} + \cdots$	2 : { 1 : first three terms; 1 : general term }
(b) The given series is the Maclaurin series for $f'(x)$ with $x = \frac{1}{2}$. $f'(x) = -(1+x^3)^{-2}(3x^2)$. Thus, the sum of the series is $f'\left(\frac{1}{2}\right) = -\frac{3\left(\frac{1}{4}\right)}{\left(1+\frac{1}{8}\right)^2} = -\frac{16}{27}$.	2 : { 1 : $f'(x)$; 1 : $f'\left(\frac{1}{2}\right)$ }
(c) $\int_0^x \frac{1}{1+t^3}\,dt = x - \frac{x^4}{4} + \frac{x^7}{7} - \frac{x^{10}}{10} + \cdots + \frac{(-1)^n x^{3n+1}}{3n+1} + \cdots$	2 : { 1 : first four terms; 1 : general term }
(d) $\int_0^{1/2} \frac{1}{1+t^3}\,dt \approx \frac{1}{2} - \frac{\left(\frac{1}{2}\right)^4}{4} + \frac{\left(\frac{1}{2}\right)^7}{7}$. The series in part (c) with $x = \frac{1}{2}$ has terms that alternate, decrease in absolute value, and have limit 0. Hence the error is bounded by the absolute value of the next term. $\left\vert \int_0^{1/2} \frac{1}{1+t^3}\,dt - \left(\frac{1}{2} - \frac{\left(\frac{1}{2}\right)^4}{4} + \frac{\left(\frac{1}{2}\right)^7}{7}\right) \right\vert < \frac{\left(\frac{1}{2}\right)^{10}}{10} = \frac{1}{10240} < 0.0001$	3 : { 1 : approximation; 1 : properties of terms; 1 : absolute value of fourth term < 0.0001 }

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2004 SCORING GUIDELINES

Question 6

Let f be the function given by $f(x) = \sin\left(5x + \frac{\pi}{4}\right)$, and let $P(x)$ be the third-degree Taylor polynomial for f about $x = 0$.

(a) Find $P(x)$.

(b) Find the coefficient of x^{22} in the Taylor series for f about $x = 0$.

(c) Use the Lagrange error bound to show that $\left\vert f\left(\frac{1}{10}\right) - P\left(\frac{1}{10}\right) \right\vert < \frac{1}{100}$.

(d) Let G be the function given by $G(x) = \int_0^x f(t)\,dt$. Write the third-degree Taylor polynomial for G about $x = 0$.

(a) $f(0) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$; $f'(0) = 5\cos\left(\frac{\pi}{4}\right) = \frac{5\sqrt{2}}{2}$; $f''(0) = -25\sin\left(\frac{\pi}{4}\right) = -\frac{25\sqrt{2}}{2}$; $f'''(0) = -125\cos\left(\frac{\pi}{4}\right) = -\frac{125\sqrt{2}}{2}$; $P(x) = \frac{\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}x - \frac{25\sqrt{2}}{2(2!)}x^2 - \frac{125\sqrt{2}}{2(3!)}x^3$	4 : $P(x)$ ⟨−1⟩ each error or missing term; deduct only once for $\sin\left(\frac{\pi}{4}\right)$ evaluation error; deduct only once for $\cos\left(\frac{\pi}{4}\right)$ evaluation error; ⟨−1⟩ max for all extra terms, $+\cdots$, misuse of equality
(b) $\frac{-5^{22}\sqrt{2}}{2(22!)}$	2 : { 1 : magnitude; 1 : sign }
(c) $\left\vert f\left(\frac{1}{10}\right) - P\left(\frac{1}{10}\right) \right\vert \le \max_{0 \le c \le \frac{1}{10}} \left\vert f^{(4)}(c) \right\vert \left(\frac{1}{4!}\right)\left(\frac{1}{10}\right)^4 \le \frac{625}{4!}\left(\frac{1}{10}\right)^4 = \frac{1}{384} < \frac{1}{100}$	1 : error bound in an appropriate inequality
(d) The third-degree Taylor polynomial for G about $x = 0$ is $\int_0^x \left(\frac{\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}t - \frac{25\sqrt{2}}{4}t^2\right) dt = \frac{\sqrt{2}}{2}x + \frac{5\sqrt{2}}{4}x^2 - \frac{25\sqrt{2}}{12}x^3$	2 : third-degree Taylor polynomial for G about $x = 0$; ⟨−1⟩ each incorrect or missing term; ⟨−1⟩ max for all extra terms, $+\cdots$, misuse of equality

AP® CALCULUS BC
2005 SCORING GUIDELINES (Form B)

Question 3

The Taylor series about $x = 0$ for a certain function f converges to $f(x)$ for all x in the interval of convergence. The nth derivative of f at $x = 0$ is given by

$$f^{(n)}(0) = \frac{(-1)^{n+1}(n+1)!}{5^n(n-1)^2} \text{ for } n \ge 2.$$

The graph of f has a horizontal tangent line at $x = 0$, and $f(0) = 6$.

(a) Determine whether f has a relative maximum, a relative minimum, or neither at $x = 0$. Justify your answer.

(b) Write the third-degree Taylor polynomial for f about $x = 0$.

(c) Find the radius of convergence of the Taylor series for f about $x = 0$. Show the work that leads to your answer.

(a) f has a relative maximum at $x = 0$ because $f'(0) = 0$ and $f''(0) < 0$.

$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{reason} \end{cases}$

(b) $f(0) = 6,\ f'(0) = 0$

$f''(0) = -\frac{3!}{5^2 1^2} = -\frac{6}{25},\ f'''(0) = \frac{4!}{5^3 2^2}$

$P(x) = 6 - \frac{3!x^2}{5^2 2!} + \frac{4!x^3}{5^3 2^2 3!} = 6 - \frac{3}{25}x^2 + \frac{1}{125}x^3$

3 : $P(x)$
⟨−1⟩ each incorrect term
Note: ⟨−1⟩ max for use of extra terms

(c) $u_n = \frac{f^{(n)}(0)}{n!}x^n = \frac{(-1)^{n+1}(n+1)}{5^n(n-1)^2}x^n$

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{\frac{(-1)^{n+2}(n+2)}{5^{n+1}n^2}x^{n+1}}{\frac{(-1)^{n+1}(n+1)}{5^n(n-1)^2}x^n}\right|$$

$$= \left(\frac{n+2}{n+1}\right)\left(\frac{n-1}{n}\right)^2\frac{1}{5}|x|$$

$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \frac{1}{5}|x| < 1$ if $|x| < 5$.

The radius of convergence is 5.

$4 : \begin{cases} 1 : \text{general term} \\ 1 : \text{sets up ratio} \\ 1 : \text{computes limit} \\ 1 : \text{applies ratio test to get radius of convergence} \end{cases}$

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2003 SCORING GUIDELINES

Question 6

The function f is defined by the power series

$$f(x) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n+1)!} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots + \frac{(-1)^n x^{2n}}{(2n+1)!} + \cdots$$

for all real numbers x.

(a) Find $f'(0)$ and $f''(0)$. Determine whether f has a local maximum, a local minimum, or neither at $x = 0$. Give a reason for your answer.

(b) Show that $1 - \frac{1}{3!}$ approximates $f(1)$ with error less than $\frac{1}{100}$.

(c) Show that $y = f(x)$ is a solution to the differential equation $xy' + y = \cos x$.

(a) $f'(0) =$ coefficient of x term $= 0$

$f''(0) = 2$ (coefficient of x^2 term) $= 2\left(-\frac{1}{3!}\right) = -\frac{1}{3}$

f has a local maximum at $x = 0$ because $f'(0) = 0$ and $f''(0) < 0$.

$4 : \begin{cases} 1 : f'(0) \\ 1 : f''(0) \\ 1 : \text{critical point answer} \\ 1 : \text{reason} \end{cases}$

(b) $f(1) = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots + \frac{(-1)^n}{(2n+1)!} + \cdots$

This is an alternating series whose terms decrease in absolute value with limit 0. Thus, the error is less than the first omitted term, so $\left|f(1) - \left(1 - \frac{1}{3!}\right)\right| \le \frac{1}{5!} = \frac{1}{120} < \frac{1}{100}$.

1 : error bound $< \frac{1}{100}$

(c) $y' = -\frac{2x}{3!} + \frac{4x^3}{5!} - \frac{6x^5}{7!} + \cdots + \frac{(-1)^n 2nx^{2n-1}}{(2n+1)!} + \cdots$

$xy' = -\frac{2x^2}{3!} + \frac{4x^4}{5!} - \frac{6x^6}{7!} + \cdots + \frac{(-1)^n 2nx^{2n}}{(2n+1)!} + \cdots$

$xy' + y = 1 - \left(\frac{2}{3!} + \frac{1}{3!}\right)x^2 + \left(\frac{4}{5!} + \frac{1}{5!}\right)x^4 - \left(\frac{6}{7!} + \frac{1}{7!}\right)x^6 + \cdots + (-1)^n\left(\frac{2n}{(2n+1)!} + \frac{1}{(2n+1)!}\right)x^{2n} + \cdots$

$= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots + \frac{(-1)^n}{(2n)!}x^{2n} + \cdots$

$= \cos x$

$4 : \begin{cases} 1 : \text{series for } y' \\ 1 : \text{series for } xy' \\ 1 : \text{series for } xy' + y \\ 1 : \text{identifies series as c} \end{cases}$

OR

$xy = xf(x) = x - \frac{x^3}{3!} + \cdots + (-1)^n\frac{1}{(2n+1)!}x^{2n+1} + \cdots$

$= \sin x$

$xy' + y = (xy)' = (\sin x)' = \cos x$

OR

$4 : \begin{cases} 1 : \text{series for } xf(x) \\ 1 : \text{identifies series as s} \\ 1 : \text{handles } xy' + y \\ 1 : \text{makes connection} \end{cases}$

AP® CALCULUS BC
2008 SCORING GUIDELINES (Form B)

Question 6

Let f be the function given by $f(x) = \frac{2x}{1+x^2}$.

(a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 0$.

(b) Does the series found in part (a), when evaluated at $x = 1$, converge to $f(1)$? Explain why or why not.

(c) The derivative of $\ln(1+x^2)$ is $\frac{2x}{1+x^2}$. Write the first four nonzero terms of the Taylor series for $\ln(1+x^2)$ about $x = 0$.

(d) Use the series found in part (c) to find a rational number A such that $\left|A - \ln\left(\frac{5}{4}\right)\right| < \frac{1}{100}$. Justify your answer.

(a) $\frac{1}{1-u} = 1 + u + u^2 + \cdots + u^n + \cdots$ $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots + \left(-x^2\right)^n + \cdots$ $\frac{2x}{1+x^2} = 2x - 2x^3 + 2x^5 - 2x^7 + \cdots + (-1)^n 2x^{2n+1} + \cdots$	3 : { 1 : two of the first four terms 1 : remaining terms 1 : general term
(b) No, the series does not converge when $x = 1$ because when $x = 1$, the terms of the series do not converge to 0.	1 : answer with reason
(c) $\ln\left(1+x^2\right) = \int_0^x \frac{2t}{1+t^2}\,dt$ $= \int_0^x \left(2t - 2t^3 + 2t^5 - 2t^7 + \cdots\right) dt$ $= x^2 - \frac{1}{2}x^4 + \frac{1}{3}x^6 - \frac{1}{4}x^8 + \cdots$	2 : { 1 : two of the first four terms 1 : remaining terms
(d) $\ln\left(\frac{5}{4}\right) = \ln\left(1 + \frac{1}{4}\right) = \left(\frac{1}{2}\right)^2 - \frac{1}{2}\left(\frac{1}{2}\right)^4 + \frac{1}{3}\left(\frac{1}{2}\right)^6 - \frac{1}{4}\left(\frac{1}{2}\right)^8 + \cdots$ Let $A = \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^4 = \frac{7}{32}$. Since the series is a converging alternating series and the absolute values of the individual terms decrease to 0, $\left|A - \ln\left(\frac{5}{4}\right)\right| < \left|\frac{1}{3}\left(\frac{1}{2}\right)^6\right| = \frac{1}{3}\cdot\frac{1}{64} < \frac{1}{100}$.	3 : { 1 : uses $x = \frac{1}{2}$ 1 : value of A 1 : justification

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2008 SCORING GUIDELINES

Question 3

x	$h(x)$	$h'(x)$	$h''(x)$	$h'''(x)$	$h^{(4)}(x)$
1	11	30	42	99	18
2	80	128	$\frac{488}{3}$	$\frac{448}{3}$	$\frac{584}{9}$
3	317	$\frac{753}{2}$	$\frac{1383}{4}$	$\frac{3483}{16}$	$\frac{1125}{16}$

Let h be a function having derivatives of all orders for $x > 0$. Selected values of h and its first four derivatives are indicated in the table above. The function h and these four derivatives are increasing on the interval $1 \le x \le 3$.

(a) Write the first-degree Taylor polynomial for h about $x = 2$ and use it to approximate $h(1.9)$. Is this approximation greater than or less than $h(1.9)$? Explain your reasoning.

(b) Write the third-degree Taylor polynomial for h about $x = 2$ and use it to approximate $h(1.9)$.

(c) Use the Lagrange error bound to show that the third-degree Taylor polynomial for h about $x = 2$ approximates $h(1.9)$ with error less than 3×10^{-4}.

(a) $P_1(x) = 80 + 128(x-2)$, so $h(1.9) \approx P_1(1.9) = 67.2$ $P_1(1.9) < h(1.9)$ since h' is increasing on the interval $1 \le x \le 3$.	4 : { 2 : $P_1(x)$ 1 : $P_1(1.9)$ 1 : $P_1(1.9) < h(1.9)$ with reason
(b) $P_3(x) = 80 + 128(x-2) + \frac{488}{6}(x-2)^2 + \frac{448}{18}(x-2)^3$ $h(1.9) \approx P_3(1.9) = 67.988$	3 : { 2 : $P_3(x)$ 1 : $P_3(1.9)$
(c) The fourth derivative of h is increasing on the interval $1 \le x \le 3$, so $\max\limits_{1.9 \le x \le 2}\left|h^{(4)}(x)\right| = \frac{584}{9}$. Therefore, $\left|h(1.9) - P_3(1.9)\right| \le \frac{584}{9}\frac{\left|1.9-2\right|^4}{4!}$ $= 2.7037 \times 10^{-4}$ $< 3 \times 10^{-4}$.	2 : { 1 : form of Lagrange error estimate 1 : reasoning